Nonlinear Spectral Theory

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Walter de Gruyter

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Walter de Gruyter · Berlin · New York

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Für Kristina in Liebe und Dankbarkeit
Jürgen

A mia moglie Adele e ai miei figli Luigi, Egidio, Giovanna, Emanuele, Maria Vittoria e Stefano
Espedito

Alfonso desea dedicar este libro a su adorada esposa Lucilla

Preface

This monograph is the outcome of the authors' work at spectral theory for nonlinear operators (or nonlinear spectral theory, for short) during the last 10 years. It could not have been realized without several mutual visits and invitations. We gratefully acknowledge generous financial support of several visits of the first author in Italy by the *Consiglio Nazionale delle Ricerche* (CNR), the *Istituto Nazionale di Alta Matematica* (INdAM), and the *Ministero dell'Istruzione, dell'Università e della Ricerca* (MIUR, fondi PRIN del programma "Analisi Reale ed Analisi Funzionale"). Visiting professorships of the second and third author in Germany have been supported by the *Deutsche Forschungsgemeinschaft* (DFG) and *Deutscher Akademischer Austauschdienst* (DAAD). We are particularly indebted to the DFG for supporting the project "Nonlinear Spectral and Eigenvalue Theory" (grant # Ap 40/15-1) between 1999 and 2001, and to the *VolkswagenStiftung* (grant # I/77 562) for supporting a sabbatical semester of the first author at the University of Rome.

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Würzburg, Arcavacata, Rome — The authors

Contents

Introduction

How should we define a spectrum for nonlinear operators which attempts to preserve the useful properties of the linear case, but admits applications to a possibly large variety of nonlinear problems? This is the question that will provide the main focus of this book. In fact, spectral theory for nonlinear operators is a rather new field in nonlinear analysis, and, of course, is far from complete even in its most rudimentary aspects. We do not believe that this should be an obstacle for describing the "state-of-the-art" in a monograph like this: in fact, a book should open a field rather than close it.

It is only a slight exaggeration to say that *spectral theory for linear operators* is one of the most important topics of functional analysis and operator theory. In fact, much information about a linear operator is "hidden" in its spectrum, and thus knowing the spectrum means knowing a large part of the properties of the operator. The range of applications of linear spectral theory is vast: apart from extremely important applications of spectra of differential operators, say, to the modern theory of elliptic boundary value problems, spectral theory is also at the heart of classical quantum mechanics.

To put things in the right framework, let us recall some of the remarkable properties of the spectrum of linear operators. Given a Banach space X over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and a bounded linear operator $L\colon X \to X$, the *spectrum* of L is

$$\sigma(L) = \{\lambda \in \mathbb{K} : \lambda I - L \text{ is not a bijection}\}, \tag{1}$$

where I denotes the identity operator. As a simple consequence of the closed graph theorem, the *resolvent operator* $(\lambda I - L)^{-1}$ is then automatically bounded for each $\lambda \in \mathbb{K} \setminus \sigma(L) =: \rho(L)$, the *resolvent set* of L. Now, the resolvent set, spectrum, and resolvent operator have the following well-known properties.

First of all, the *Neumann series*

$$(\lambda I - L)^{-1} = \frac{1}{\lambda}\left(I - \frac{L}{\lambda}\right)^{-1} = \frac{1}{\lambda}\sum_{k=0}^{\infty}\left(\frac{L}{\lambda}\right)^{k} = \frac{1}{\lambda}I + \frac{1}{\lambda^2}L + \frac{1}{\lambda^3}L^2 + \cdots \tag{2}$$

converges (in the operator norm) for each $\lambda \in \mathbb{K}$ with $|\lambda| > r(L)$, where

$$r(L) := \sup\{|\lambda| : \lambda \in \sigma(L)\} \tag{3}$$

is the *spectral radius* of L. The conditions $\lambda \in \rho(L)$ and $|\mu - \lambda| < \|(\lambda I - L)^{-1}\|^{-1}$ imply that $\mu \in \rho(L)$; consequently, $\rho(L)$ is *open* in $\mathbb{K}$.

By adapting methods from complex analysis one may show that the map $\lambda \mapsto (\lambda I - L)^{-1}$ is *analytic* whenever it is defined; consequently, $\rho(L)$ is a proper subset of the complex plane in case $\mathbb{K} = \mathbb{C}$, and so $\sigma(L)$ is *nonempty*.

The spectrum $\sigma(L)$ is closed and bounded, hence *compact*. Conversely, given any nonempty compact set $\Sigma \subset \mathbb{K}$ one can find a Banach space X and a bounded linear operator L such that $\sigma(L) = \Sigma$. A more precise statement is possible for a compact linear operator L: in this case $\sigma(L)$ is at most countable, and so has no interior points.

For any polynomial $p\colon \mathbb{K} \to \mathbb{K}$, the *spectral mapping formula*

$$\sigma(p(L)) = p(\sigma(L)) \tag{4}$$

holds. So if one knows the spectrum of L, one also knows that of all powers L^n of L and all linear combinations of such powers.

Finally, the multivalued map σ which associates to each bounded linear operator L its spectrum $\sigma(L)$ as a subset of $\mathbb{K}$ is *upper semicontinuous*, and so the spectrum $\sigma(L)$ cannot "collapse" when L changes continuously.

In view of the importance of spectral theory for linear operators, it is not surprising at all that various attempts have been made to define and study spectra also for *nonlinear operators*. In the very beginning, the term *spectrum* was used for nonlinear operators just in the sense of *point spectrum* (i.e., the set of eigenvalues) e.g. by Nemytskij [202], [203] or Krasnosel'skij [160]. Here by *eigenvalue* of a nonlinear operator $F\colon X \to X$ we mean, of course, any scalar λ for which the equation $F(x) = \lambda x$ has a nontrivial solution.

Later it became clear that a more complete description requires, as in the linear case, other (i.e., "non-discrete") spectral sets. Starting from the late sixties, this led to a number of definitions of nonlinear spectra which are all different. In this connection, it was tacitly assumed that a reasonable definition of a spectrum of a continuous nonlinear operator should satisfy some *minimal requirements*, namely:

- It should reduce to the familiar spectrum in case of linear operators.
- It should share some of the usual properties with the linear spectrum (e.g., compactness).
- It should contain the eigenvalues of the operator involved.
- It should have nontrivial applications, i.e. those which may not be obtained by other known means.

From the viewpoint of these four requirements, any definition of a spectrum should focus on its analytical and topological properties and, of course, on applications.

Unfortunately, it turned out that, when building a nonlinear spectral theory, one encounters several "unpleasant" phenomena. First of all, in contrast to the linear case, the spectrum of a nonlinear operator contains practically no information on this operator. Moreover, such familiar properties as boundedness, closedness, or nonemptiness fail, in general, for all the spectra proposed so far in the literature.

We shall return to this several times in the first chapters of this monograph. For the time being, let us just give a series of quite simple examples which show what may happen if we try to study spectral properties of rather harmless nonlinear maps.

Example 1. Let $F: \mathbb{K} \to \mathbb{K}$ be any continuous operator with $F(0) = 0$. Clearly, the set

$$M := \left\{ \frac{F(x)}{x} : x \in \mathbb{K},\ x \neq 0 \right\} \tag{5}$$

is then nothing but the set of eigenvalues of F as defined above. Now, if we want the spectrum of F to be bounded and contain the eigenvalues, then the set M must be bounded, and so *F must be of sublinear growth*, i.e. $|F(x)| \leq c|x|$ for some $c > 0$. ♡

Example 2. Let $F: \mathbb{K} \to \mathbb{K}$ again be any continuous operator. Then the set

$$N := \left\{ \frac{F(x) - F(y)}{x - y} : x, y \in \mathbb{K},\ x \neq y \right\} \tag{6}$$

contains precisely all scalars λ such that $\lambda I - F$ is not injective, and so $M = N$ if F is linear. Now, if F is nonlinear with $F(0) = 0$ we have of course only $M \subseteq N$, where the inclusion may be strict (see Example 3). On the other hand, in case $F(0) \neq 0$ there is no relation between M and N. For instance, for the affine function $F(x) = ax + b$ with $b \neq 0$ we have $M = \mathbb{R} \setminus \{a\}$ and $N = \{a\}$. Generally, if X is any Banach space and $F(x) = Lx + b$, with L being linear and b nonzero, then every $\lambda \in \mathbb{K}$ satisfying $|\lambda| > r(L)$ is an eigenvalue of F, while $N = \sigma(L)$.

Now, if we want the spectrum of F to be bounded and contain the set N, then this set must be bounded, and so *F must be Lipschitz continuous* i.e. $|F(x) - F(y)| \leq L|x - y|$ for some $L > 0$. ♡

Example 3. Let $F: \mathbb{R} \to \mathbb{R}$ be defined by

$$F(x) = \sqrt{|x|}. \tag{7}$$

For this function (which we call the "seagull" in what follows) the sets (5) and (6) are of course $M = \mathbb{R} \setminus \{0\}$ and $N = \mathbb{R}$. In particular, *every nonzero real number is an eigenvalue of F*. This remains even true if we replace (7), for any $\varepsilon > 0$, by the "mutilated seagull"

$$F_\varepsilon(x) = \begin{cases} \sqrt{|x|} & \text{if } |x| \leq \varepsilon^2, \\ \varepsilon & \text{if } |x| > \varepsilon^2, \end{cases} \tag{8}$$

which has arbitrarily small range $[0, \varepsilon]$, but still every nonzero real number is an eigenvalue. Observe that F_ϵ tends to the zero operator, uniformly on the real line, as $\varepsilon \to 0$. This shows that the set of eigenvalues may drastically "blow up" if the operator changes continuously. On the other hand, in Chapter 6 we will see that one of the most common spectra of the function (7) (or (8)) is $\{0\}$, i.e. *disjoint* from the set of eigenvalues. This is of course in sharp contrast to the linear case, where the eigenvalues constitute an important part of the spectrum.

Observe that the operator $\lambda I - F$, with F given by (7) or (8), is not bijective for *any* $\lambda \in \mathbb{R}$. So the analogue of the spectrum (1) for F would be the whole real axis, and thus unbounded. ♡

Example 4. Let $F\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$F(x) = \begin{cases} x & \text{if } |x| > 1, \\ x^2 & \text{if } 0 \le x \le 1, \\ -x^2 & \text{if } -1 \le x \le 0. \end{cases} \tag{9}$$

In this case the function $\lambda I - F$ is a bijection precisely for $\lambda \le 0$ or $\lambda > 1$. So the analogue of the spectrum (1) for F would be the interval $(0, 1]$, and thus not closed. ♡

Example 5. Let $F\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$F(x) = \begin{cases} 0 & \text{if } x \le 1, \\ x - 1 & \text{if } 1 < x < 2, \\ 1 & \text{if } x \ge 2. \end{cases} \tag{10}$$

It is easy to see that $\lambda I - F$ is a bijection for $\lambda < 0$ or $\lambda > 1$, and so one can expect that any "reasonable" spectrum contains the interval $[0, 1]$. On the other hand, $F^2(x) \equiv 0$, and so the spectral mapping formula (4) certainly has no analogue for nonlinear operators. ♡

Example 6. Let $F\colon \mathbb{R} \to \mathbb{R}$ be defined as in Example 4. For $0 < \delta < \frac{1}{2}$, denote by p_δ the quadratic polynomial $p_\delta(x) := (x - \delta)^2 + 2\delta - \delta^2$ which satisfies $p_\delta(0) = 2\delta$, $p_\delta(\delta) = 2\delta - \delta^2$, and $p(-\delta) = 2\delta + 3\delta^2$. Put

$$F_\delta(x) = \begin{cases} x & \text{if } x \le -1, \\ -p_\delta(x) & \text{if } -1 < x < -\delta, \\ 0 & \text{if } -\delta \le x \le \delta, \\ p_\delta(x) & \text{if } \delta < x < 1, \\ x & \text{if } x \ge 1. \end{cases} \tag{11}$$

Then $F - F_\delta$ is Lipschitz continuous with Lipschitz constant 2δ. Moreover, all $\lambda \in \mathbb{R}$ for which $\lambda I - F$ is not a bijection are contained in the open set $G := \mathbb{R} \setminus \{0\}$, as we have seen in Example 4. On the other hand, F_δ is not injective, and so the set of all $\lambda \in \mathbb{R}$ for which $\lambda I - F_\delta$ is not a bijection is *not* contained in G. This shows that the dependence of the "spectrum" on a nonlinear operator need not be upper semicontinuous as in the linear case. ♡

Example 7. Let $F\colon \mathbb{C} \to \mathbb{C}$ be defined by

$$F(z) = \min\{|z|, 1\}e^z. \tag{12}$$

Then F is surjective, but for each $\varepsilon > 0$ one may find a continuous perturbation $G\colon \mathbb{C} \to \mathbb{C}$ of F such that $|G(z)| \le \varepsilon$ and $F + G$ is not surjective. (For example, one may choose $G(z) := \max\{\varepsilon(1 - |z|), 0\}$.) This shows that, in contrast to the linear case, surjectivity is an unstable property. ♡

Example 8. Let $F\colon \mathbb{C}^2 \to \mathbb{C}^2$ be defined by

$$F(z, w) = (\overline{w}, i\overline{z}). \tag{13}$$

Obviously, F is a Lipschitz continuous bijection with Lipschitz continuous inverse $F^{-1}(z, w) = (i\overline{w}, \overline{z})$. More generally, for *every* $\lambda \in \mathbb{C}$ the operator $\lambda I - F$ has the Lipschitz continuous inverse

$$(\lambda I - F)^{-1}(z, w) = \left(\frac{\overline{\lambda} z + \overline{w}}{i + |\lambda|^2}, -\frac{\overline{\lambda} w + i\overline{z}}{i - |\lambda|^2} \right).$$

This means that the operator $\lambda I - F$ is extremely regular, and so every "reasonable" spectrum for F should be *empty*.

Observe that $F^2(z, w) = (-iz, iw)$ is a *linear* operator in this example, and so $\sigma(F^2) = \{\pm i\}$ is of course nonempty. ♡

At this point, however, one should pose the following critical question: *why should the spectrum of a nonlinear operator have the same properties as that of a linear operator?* After all, it is not the *intrinsic structure* of the spectrum itself which leads to interesting applications, but its property of being a *useful tool for solving nonlinear equations*. For instance, the fact that the spectrum of a nonlinear operator in a complex space may be empty should be regarded as an advantage, not a drawback: as a matter of fact, it means that a certain nonlinear equation involving a complex parameter is solvable for *all* possible values of this parameter.

So, it is only for notational convenience that we retain the name "spectrum" in the nonlinear case; actually we are more interested in studying specific nonlinear equations rather than abstract spectra. In particular, we will show in the last chapter that, when adopting the language of nonlinear spectral theory, one may get not only a deeper insight into new classes of nonlinear problems, but also sometimes obtain a re-interpretation of classical results from a different viewpoint.

In this book we discuss spectra for various classes of nonlinear operators, compare their properties, point out their advantages and drawbacks, and indicate some possible applications. Unless otherwise specified, *all operators considered in this book are supposed to be continuous*. In most cases, we will also tacitly assume that the operator in consideration is *bounded*, i.e., maps bounded sets into bounded sets. We point out that, in contrast to linear operators, for nonlinear operators these two requirements are independent; in fact, it is easy to find continuous operators which are unbounded, or bounded operators which are discontinuous.

Important special classes of continuous operators we are interested in are, roughly speaking, *Fréchet differentiable operators, Lipschitz continuous operators, quasibounded operators*, and *linearly bounded operators*. These are in fact operators for which many important results have been proved in nonlinear spectral theory. When discussing such abstract results, we will illustrate them throughout by numerous examples. Some selected applications to bifurcation theory, integral equations, and boundary value problems, and partial differential equations will be given in the last chapter.

The present book consists of 12 chapters. In Chapter 1 we recall the basic facts on the *spectrum of a bounded linear operator* in a Banach space including its subdivision in several subspectra. These facts will be needed for a comparison with many new notions and concepts introduced in the following chapters. In Chapter 2 we introduce and study some *numerical characteristics* which provide a "quantitative" description of certain mapping properties of nonlinear operators, such as boundedness, quasiboundedness, or Lipschitz continuity, and compare these characteristics with the classical (Kuratowski) measure of noncompactness. These operator characteristics are not only needed to define the various spectra considered in the sequel, but seem to be also of independent interest. For example, they allow us to give a certain *subdivision of spectra* which in the case of a bounded linear operator leads to the approximate point spectrum, the approximate defect spectrum, and the continuous spectrum.

Since spectra have something to do with the "lack of invertibility" of operators, in Chapter 3 we study *general invertibility results*. In particular, we will be interested in conditions under which the local invertibility of a nonlinear operator implies its global invertibility. For example, a classical condition of this type is provided by the *Banach–Mazur lemma* which states that a continuous map is a global homeomorphism if and only if it is both a local homeomorphism and proper.

Chapter 4 is devoted to a *spectrum for continuous operators* due to Rhodius, and a *spectrum for C^1 operators* which goes back to Neuberger. The *Rhodius spectrum* may be noncompact or empty, while the *Neuberger spectrum* is always nonempty (in the complex case), but it need be neither closed nor bounded.

In Chapter 5 we discuss a *spectrum for Lipschitz continuous operators* which was apparently first proposed by Kachurovskij in 1969, and a *spectrum for linearly bounded operators* introduced recently by Dörfner. In contrast to the Neuberger spectrum, the *Kachurovskij spectrum* is compact, but it may be empty even in (complex) dimension 2. The *Dörfner spectrum* in turn is always closed, but it may be unbounded or empty. All four spectra considered in this and the preceding chapter, however, reduce to the familiar spectrum in the linear case, and they all contain the eigenvalues of the operator involved. Interestingly, these spectra always contain 0 in case of a *compact* operator in an infinite dimensional Banach space; this is of course completely analogous to the linear case.

Chapter 6 is concerned with a *spectrum for certain special continuous operators* which was introduced by Furi, Martelli, and Vignoli in 1978. This *Furi–Martelli–Vignoli spectrum* is always closed, sometimes even compact, and it has many interesting applications. It again coincides with the usual spectrum in the linear case, but it need not contain the point spectrum in the nonlinear case. We also consider a certain modification of this spectrum in Chapter 6 which was recently introduced by Giorgieri, Väth, and the first author.

The flaw of the Furi–Martelli–Vignoli spectrum of not containing the eigenvalues is "repaired" in a certain sense by another spectrum which is some kind of "interpolation" between the Furi–Martelli–Vignoli spectrum and the Dörfner spectrum. This spectrum was introduced by Feng in 1997; it has similar topological properties as the Furi–Martelli–Vignoli spectrum, but contains the eigenvalues. We will discuss the *Feng spectrum* in detail in Chapter 7.

Roughly speaking, one may say that the Furi–Martelli–Vignoli spectrum takes into account the "asymptotic" properties of an operator, while the Feng spectrum reflects its "global" properties. This is also one reason why the latter contains the eigenvalues, but the former does not. A very interesting approach to some kind of "local" spectrum is due to Väth; since his construction is rather far from what one usually calls a spectrum, we have decided to call it "phantom". More precisely, there are two kinds of *Väth phantoms* which we will study in detail in Chapter 8.

There are several other spectra for nonlinear operators in the literature which do not seem to be as important as the Furi–Martelli–Vignoli spectrum, the Feng spectrum, and the Väth phantoms, but deserve being treated in this book. In Chapter 9 we discuss a "semilinear" Feng spectrum proposed by Feng and Webb, the analogous variant of the Furi–Martelli–Vignoli spectrum, and another two spectra introduced by Singhof–Weyer and Weber. Moreover, we briefly mention a "strange spectrum" for Lipschitz continuous operators which is defined through some kind of "nonlinear adjoint". Finally, we discuss spectra and phantoms for *homogeneous operators* which seem to be particularly useful in view of applications.

Chapter 10 is concerned with *nonlinear eigenvalue problems*. This is one of the historical roots of nonlinear spectral theory which goes back to, among others, M. A. Krasnosel'skij, V. V. Nemytskij, and M. M. Vajnberg. Now the literature on eigenvalues of nonlinear operators is so large that taking into account all results and applications would have required another book. We therefore restrict ourselves to those problems which are closely related to spectral theory.

A deeper analysis of such problems leads to a somewhat paradoxical statement: the fact that some of the spectra mentioned above, though being very important in applications, do not contain the point spectrum, is actually not a flaw of these spectra, but a consequence of a "wrong" definition of the term *eigenvalue*. Here we are again led to the same critical question as above: *Why should an eigenvalue of a nonlinear operator be defined in the same way as in the linear case?* In fact, the problem of finding the "appropriate" definition of the term *eigenvalue* in the nonlinear case is quite subtle and will be discussed in detail in Chapter 10. This definition is equivalent to, but formally completely different from, the familiar definition in the linear case. This again illustrates the fact that *it is misleading, if not dangerous, to "borrow" notions naively from the linear theory.*

At this stage we cannot help mentioning that this was already observed seven centuries ago, not by a mathematician, but by the genious of Dante Alighieri who writes at the beginning of his *Divina Commedia*

Nel mezzo del cammin di nostra vita
Mi ritrovai per una selva oscura,
Chè la diritta via era smarrita.

We consider Dante's intuition so important that we think the reader should spend some time concentrating on these remarkable verses. To facilitate the task we shall give below the English translation of these verses of Dante's immortal masterpiece.

Midway upon the journey of our life,
I found myself within a forest dark,
For the straight pathway had been lost.

Our crucial problem could not be described in a more plastic and suggestive way: in the brushwood of the nonlinear theory (Dante's "selva oscura"), the linear methods (the "diritta via") fail completely, and we have to find something new.

Dante in the "dark forest"

In fact, this may be achieved if one adopts a completely new approach, as we do in this book. Surprisingly, this was also noted by Dante who writes some lines afterwards:

Ma per trattar del ben ch'i' vi trovai,
Dirò de l'altre cose ch'io v'ho scorte,

which reads in English as follows:

But of the good to treat, which there I found
Speak will I of the other things I saw there.

The remaining two chapters of the monograph are more application-oriented. An interesting tool in spectral theory, both linear and nonlinear, is based on the notion of *numerical range*. To the best of our knowledge, the first numerical range for nonlinear operators in Hilbert spaces was introduced by Zarantonello in 1964. Subsequently, numerical ranges for larger classes of operators in Banach spaces have been studied by Rhodius and Pietschmann, Verma, Dörfner, and others. Interestingly, a new feature which comes in here is the *geometry of the underlying Banach space*. We will show in Chapter 11 how numerical ranges may be used to "localize" the spectrum of a nonlinear operator on the real line or in the complex plane.

The final Chapter 12 is concerned with selected *applications*. In the first section we derive some *general solvability results* for nonlinear equations which may be directly obtained by means of spectral methods. Afterwards we discuss some applications to *nonlinear integral equations* and *boundary value problems* for differential equations, as well as applications to *bifurcation theory*. In the last section we show how spectral theory for homogeneous nonlinear operators may be used to derive a certain *nonlinear Fredholm alternative* which applies to existence and perturbation results for the *p-Laplace equation*. We believe that this list of applications is by no means exhaustive, and we hope that spectral methods will be a fruitful source for further research, both theoretical and application-oriented.

Our exposition of nonlinear spectral theory is basically self-contained. We prove all major statements, and we try to illustrate each definition and result by some examples. By □ we denote the end of a proof and by ♡ the end of an example. Each chapter closes with a section called *Notes, remarks and references*, where we collect some additional information and refer to the bibliography at the end of the book.

The notation is also standard: given a subset M of a metric space X, by $\overline{M}$ we denote the closure, by M^o the interior, and by ∂M the boundary of M. Furthermore,

$$\operatorname{diam} M := \sup\{d(x, y) : x, y \in M\} \tag{14}$$

denotes the diameter of M, and

$$\operatorname{dist}(x, M) := \inf\{d(x, y) : y \in M\} \tag{15}$$

denotes the distance of a point x from M. If X is in addition a linear space, $\operatorname{co} M$ is the convex hull, $\overline{\operatorname{co}}\, M$ the closed convex hull, and $\operatorname{span} M$ is the linear hull of M.

Since spectra are subsets of the complex plane, the special notation

$$\mathbb{D}_r := \{z \in \mathbb{C} : |z| < r\}, \quad \mathbb{D}_1 =: \mathbb{D} \tag{16}$$

for the open complex disc of radius $r > 0$, and

$$\mathbb{S}_r := \{z \in \mathbb{C} : |z| = r\}, \quad \mathbb{S}_1 =: \mathbb{S} \tag{17}$$

for its boundary will be useful. All other notation will be explained when it occurs for the first time in the text.

Understanding this monograph requires only a modest background of nonlinear analysis and operator theory. Consequently, this book is addressed to non-specialists who want to get a first idea of the development of the theory, methods, and applications of this fascinating field during the last 30 years. Hopefully this monograph will give some appreciation of nonlinear spectra, as well as a glimpse of the diversity of the directions in which current research is moving.

As we have pointed out at the beginning, the theory treated in this monograph is by no means complete, and we are far from the last word on the subject. Even worse, one could be somewhat pessimistic by stating that *we do not yet have a reasonable definition of the terms "spectrum" and "eigenvalue" for nonlinear operators*. All we can do, when applying spectral theory to a specific nonlinear problem, is to choose carefully a spectrum which has at least some of the needed features. Even a giant like Sir Isaac Newton should have had a similar feeling when he wrote about his mathematical achievements in the last years of his life:

I do not know what I may appear to the world; but to myself I seem to have been only like a boy playing on the seashore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me.

Chapter 1

Spectra of Bounded Linear Operators

In this chapter we recall the basic properties of the spectrum of a bounded linear operator in a Banach space. The main purpose is to compare these properties with those of spectra of nonlinear operators which will be defined in subsequent chapters. In particular, we discuss different subdivisions of the spectrum which sometimes will have natural analogues in the nonlinear case.

1.1 The spectrum of a bounded linear operator

Given two Banach spaces X and Y over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, by $\mathfrak{L}(X, Y)$ we denote the Banach space of all bounded linear operators $L : X \to Y$ with the usual operator norm

$$\|L\| = \sup\{\|Lx\| : \|x\| \leq 1\}. \tag{1.1}$$

As usual, we write $\mathfrak{L}(X, X) =: \mathfrak{L}(X)$ for brevity. Throughout this monograph, we denote by θ the zero element of a vector space, and by Θ the zero operator $\Theta x \equiv \theta$. Moreover, we write

$$N(L) := \{x \in X : Lx = \theta\} \tag{1.2}$$

for the *nullspace* and

$$R(L) := \{Lx : x \in X\} \tag{1.3}$$

for the *range* of a linear operator L. Thus, L is injective (1-1) if and only if $N(L) = \{\theta\}$, and surjective (onto) if and only if $R(L) = Y$.

A Banach space which we will use very often in the numerous examples is the space l_p of all sequences $x = (x_1, x_2, x_3, \dots)$ with finite norm

$$\|x\| = \begin{cases} \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} & \text{if } 1 \leq p < \infty, \\ \sup\{|x_n| : n = 1, 2, \dots\} & \text{if } p = \infty, \end{cases}$$

as well as the subspace $c \subset l_\infty$ of all convergent sequences and the subspace $c_0 \subset l_\infty$ of all null sequences. Moreover, several examples will be constructed in the *Chebyshev space* $C = C[0, 1]$ of all continuous functions on $[0, 1]$ with finite norm

$$\|x\| = \max\{|x(t)| : 0 \leq t \leq 1\},$$

or in the *Lebesgue space* $L_p = L_p[0,1]$ of all (equivalence classes of) measurable functions on $[0,1]$ with finite norm

$$\|x\| = \begin{cases} \left(\int_0^1 |x(t)|^p \, dt\right)^{1/p} & \text{if } 1 \leq p < \infty, \\ \operatorname{ess\,sup}\{|x(t)| : 0 \leq t \leq 1\} & \text{if } p = \infty. \end{cases}$$

Here the essential supremum in L_∞ is defined as usual by

$$\operatorname{ess\,sup}\{|x(t)| : 0 \leq t \leq 1\} = \inf_{\operatorname{mes} D = 0} \sup\{|x(t)| : t \in [0,1] \setminus D\},$$

where $\operatorname{mes} D$ denotes the Lebesgue measure of $D \subseteq \mathbb{R}$.

Given an operator $L \in \mathfrak{L}(X)$, the set

$$\rho(L) := \{\lambda \in \mathbb{K} : \lambda I - L \text{ is a bijection}\} \tag{1.4}$$

is called the *resolvent set* of L, its complement

$$\sigma(L) := \mathbb{K} \setminus \rho(L) \tag{1.5}$$

the *spectrum* of L. By the closed graph theorem, the inverse operator

$$R(\lambda; L) := (\lambda I - L)^{-1} \quad (\lambda \in \rho(L)) \tag{1.6}$$

is always bounded; this operator is usually called *resolvent operator* of L at λ. For every λ with $|\lambda| > \|L\|$ we have $\lambda \in \rho(L)$ and

$$\|R(\lambda; L)\| \leq \frac{1}{|\lambda| - \|L\|}. \tag{1.7}$$

A certain improvement of this will be given in Theorem 1.1 (b) below. Finally, the number

$$r(L) := \sup\{|\lambda| : \lambda \in \sigma(L)\} \tag{1.8}$$

is called the *spectral radius* of L. This number may be calculated in case of a complex Banach space by the *Gel'fand formula*

$$r(L) = \lim_{n \to \infty} \sqrt[n]{\|L^n\|} = \inf_{n \in \mathbb{N}} \sqrt[n]{\|L^n\|}. \tag{1.9}$$

For further reference, we collect some important facts about the spectrum, resolvent operator, and spectral radius in the following theorem.

Theorem 1.1. *The sets* (1.4) *and* (1.5)*, the operator* (1.6)*, and the number* (1.8) *have the following properties:*

(a) *The resolvent identities*

$$R(\lambda; L) - R(\mu; L) = (\mu - \lambda) R(\lambda; L) R(\mu; L) \quad (\lambda, \mu \in \rho(L)) \tag{1.10}$$

and

$$R(\lambda; L) - R(\lambda; K) = R(\lambda; L)(L-K)R(\lambda; K) \quad (\lambda \in \rho(L) \cap \rho(K)) \tag{1.11}$$

are true for $L, K \in \mathfrak{L}(X)$*; moreover,* $R(\lambda; L)$ *and* $R(\mu; L)$ *commute for* $\lambda, \mu \in \rho(L)$.

(b) *The Neumann series*

$$R(\lambda; L) = \frac{1}{\lambda}\left(I - \frac{L}{\lambda}\right)^{-1} = \frac{1}{\lambda}\sum_{k=0}^{\infty}\left(\frac{L}{\lambda}\right)^{k} = \frac{1}{\lambda}I + \frac{1}{\lambda^2}L + \frac{1}{\lambda^3}L^2 + \cdots \tag{1.12}$$

converges in $\mathfrak{L}(X)$ *for each* $\lambda \in \mathbb{K}$ *with* $|\lambda| > r(L)$.

(c) $\lambda \in \rho(L)$ *and* $|\mu - \lambda| < \|R(\lambda; L)\|^{-1}$ *imply that also* $\mu \in \rho(L)$*; consequently,* $\rho(L)$ *is open in* $\mathbb{K}$.

(d) *The map* $\rho(L) \ni \lambda \mapsto R(\lambda; L) \in \mathfrak{L}(X)$ *is analytic; consequently,* $\rho(L)$ *is a proper subset of the complex plane in case* $\mathbb{K} = \mathbb{C}$*, hence* $\sigma(L) \neq \emptyset$.

(e) *The estimate*

$$r(L) \leq \|L\| \tag{1.13}$$

is true; equality holds in (1.13)*, for example, if* X *is a Hilbert space and* L *is normal, i.e. commutes with its Hilbert space adjoint.*

(f) $\sigma(L)$ *is closed and bounded, hence compact.*

(g) *Conversely, given a nonempty compact set* $\Sigma \subset \mathbb{K}$ *one can find a Banach space* X *and an operator* $L \in \mathfrak{L}(X)$ *such that* $\sigma(L) = \Sigma$.

(h) *For any polynomial* $p\colon \mathbb{K} \to \mathbb{K}$, $p(\lambda) = a_n\lambda^n + \cdots + a_1\lambda + a_0$*, the identity*

$$\sigma(p(L)) = p(\sigma(L)) \tag{1.14}$$

holds; here $p(L) = a_nL^n + \cdots + a_1L + a_0I$ *and* $p(\sigma(L)) = \{p(\lambda) : \lambda \in \sigma(L)\}$.

(i) *For fixed* $L \in \mathfrak{L}(X)$ *and any open set* $G \supseteq \sigma(L)$ *there exists a* $\delta > 0$ *such that* $\sigma(K) \subseteq G$ *for every* $K \in \mathfrak{L}(X)$ *with* $\|K - L\| < \delta$.

Proof. To prove (a), note first that $(\mu - \lambda)I = (\mu I - L) - (\lambda I - L)$ and hence

$$\begin{aligned} R(\lambda; L) - R(\mu; L) &= R(\lambda; L)(\mu I - L)R(\mu; L) - R(\lambda; L)(\lambda I - L)R(\mu; L) \\ &= R(\lambda; L)[(\mu I - L) - (\lambda I - L)]R(\mu; L) \\ &= (\mu - \lambda)R(\lambda; L)R(\mu; L) \end{aligned}$$

for $\lambda, \mu \in \rho(L)$, which proves (1.10). To prove (1.11), we use the trivial equality $L - K = (\lambda I - K) - (\lambda I - L)$ and obtain

$$(\lambda I - L)^{-1}(L - K)(\lambda I - K)^{-1} = R(\lambda; L) - R(\lambda; K)$$

for $\lambda \in \rho(L) \cap \rho(K)$. From (1.10) we get, in particular,

$$(\mu - \lambda)R(\lambda; L)R(\mu; L) = -[R(\mu; L) - R(\lambda; L)] = -(\lambda - \mu)R(\mu; L)R(\lambda; L)$$

which shows that $R(\lambda; L)$ and $R(\mu; L)$ commute.

The property (b) follows from the fact that the Neumann series for L/λ converges to $(I - L/\lambda)^{-1} = \lambda R(\lambda; L)$, since $r(L/\lambda) = r(L)/|\lambda| < 1$. The assumption $|\mu - \lambda| < \|R(\lambda; L)\|^{-1}$ in (c) implies that

$$\begin{aligned}\|I - (\lambda I - L)^{-1}(\mu I - L)\| &= \|R(\lambda; L)[(\lambda I - L) - (\mu I - L)]\| \\ &\leq \|R(\lambda; L)\| \, |\mu - \lambda| < 1,\end{aligned}$$

and so $(\lambda I - L)^{-1}(\mu I - L)$ is invertible. Consequently, also $\mu I - L$ is invertible, and thus $\mu \in \rho(L)$.

To see that (d) is true, observe that the resolvent identity (1.10) implies the differentiability of the resolvent operator (1.6) with

$$\frac{d}{d\lambda}R(\lambda; L) = \lim_{\mu \to \lambda} \frac{R(\mu; L) - R(\lambda; L)}{\mu - \lambda} = -\lim_{\mu \to \lambda} R(\lambda; L)R(\mu; L) = -R(\lambda; L)^2$$

and, more generally,

$$\frac{d^k}{d\lambda^k}R(\lambda; L) = (-1)^k k! R(\lambda; L)^{k+1}$$

for all $k \in \mathbb{N}$. Moreover, we have $\mu \in \rho(L)$ for $|\mu - \lambda| < \|R(\lambda; L)\|^{-1}$, by (c), and the series with operator coefficients

$$R(\mu; L) = \sum_{k=0}^{\infty} (-1)^k R(\lambda; L)^{k+1} (\mu - \lambda)^k \tag{1.15}$$

converges in the space $\mathfrak{L}(X)$ with norm (1.1). Consequently, $\rho(L) = \mathbb{C}$ would imply that $\lambda \mapsto R(\lambda; L)$ is a bounded entire function, hence constant, by Liouville's theorem. This contradiction shows that $\rho(L) \neq \mathbb{C}$, hence $\sigma(L) \neq \emptyset$.

The assertion (e) is an immediate consequence of the Gel'fand formula (1.9) and the fact that $\|L^n\| \leq \|L\|^n$ in any space, while $\|L^n\| = \|L\|^n$ for normal operators in Hilbert spaces. The assertion (f) follows directly from (c) and (e)

To illustrate (g), we may choose $X = l_p$ $(1 \leq p \leq \infty)$ and

$$L(x_1, x_2, x_3, \dots) := (a_1 x_1, a_2 x_2, a_3 x_3, \dots), \tag{1.16}$$

where $(a_n)_n$ is a dense set in Σ. It is clear that $a_n I - L$ is not invertible for any $n \in \mathbb{N}$, which together with the closedness of $\sigma(L)$ implies that $\Sigma \subseteq \sigma(L)$. On the other hand, any point $\lambda \in \mathbb{K} \setminus \Sigma$ has a positive distance from Σ, and so the operator $\lambda I - L$ is an isomorphism with inverse

$$R(\lambda; L)(x_1, x_2, x_3, \dots) := ((\lambda - a_1)^{-1} x_1, (\lambda - a_2)^{-1} x_2, (\lambda - a_3)^{-1} x_3, \dots).$$

Consequently, $\lambda \in \rho(L)$ for these λ, and hence $\sigma(L) \subseteq \Sigma$.

Let us now prove (h). We write p as product of linear functions

$$p(z) = a(\lambda_1 - z)(\lambda_2 - z) \cdots (\lambda_n - z),$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the complex roots (counting multiplicities) of the polynomial p and $a \in \mathbb{C}$. Obviously, the operator

$$p(L) = a(\lambda_1 I - L)(\lambda_2 I - L) \cdots (\lambda_n I - L)$$

is invertible if and only if each factor $\lambda_j I - L$ $(j = 1, 2, \ldots, n)$ is invertible. Now, $\lambda \in \sigma(p(L))$ implies that $0 \in \sigma(\tilde{p}(L))$ with $\tilde{p}(z) := \lambda - p(z)$, and so $\tilde{p}(\mu) = 0$ for some $\mu \in \sigma(L)$. But then $p(\mu) = \lambda$ for this μ, and hence $\lambda \in p(\sigma(L))$. Since all these implications may be inverted, we have shown that $\lambda \in \sigma(p(L))$ if and only if $\lambda \in p(\sigma(L))$.

It remains to prove (i). If λ belongs to the closed set $F := \mathbb{K} \setminus G$, then $\lambda \in \rho(L)$, hence $R(\lambda; L) \in \mathfrak{L}(X)$. By what we have proved before, for any $K \in \mathfrak{L}(X)$ satisfying

$$\|K - L\| = \|(\lambda I - L) - (\lambda I - K)\| < \frac{1}{\|R(\lambda; L)\|}$$

we have then $\lambda \in \rho(K)$ as well. But the fact that $\|R(\lambda; L)\| \to 0$ as $|\lambda| \to \infty$ implies that actually

$$\delta := \min_{\lambda \in F} \frac{1}{\|R(\lambda; L)\|} > 0,$$

and so the assertion follows. □

We make some comments on the last two assertions of Theorem 1.1. The statement in (h) is usually called the *spectral mapping theorem*; it is true also for more general maps than polynomials. The statement in (i) shows that the (multivalued) map which associates to each bounded linear operator L its spectrum $\sigma(L)$ is *upper semicontinuous* from $\mathfrak{L}(X)$ into $\mathbb{K}$. The following example shows that this map is, in general, *not* lower semicontinuous; this means that the spectrum $\sigma(L)$ cannot "expand suddenly" when L is changed continuously, but it may well "shrink suddenly".

Example 1.1. Let $X = l_1(\mathbb{Z})$ be the space of all summable complex sequences

$$x = (x_n)_n = (\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots),$$

indexed by the integers, with the usual norm. For any $\varepsilon \in \mathbb{R}$, let $L_\varepsilon \in \mathfrak{L}(X)$ be defined by $L_\varepsilon x = y$, where $x = (x_n)_n$ and $y = (y_n)_n$ are related by

$$y_k = \begin{cases} x_{k-1} & \text{if } k \neq 0, \\ \varepsilon x_{-1} & \text{if } k = 0. \end{cases}$$

Then we have

$$\sigma(L_0) = \overline{\mathbb{D}},$$

where

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\} \tag{1.17}$$

denotes the open complex unit disc. On the other hand,

$$\sigma(L_\varepsilon) = \mathbb{S} := \partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\} \quad (\varepsilon \neq 0).$$

So, here the spectrum "collapses" when ε changes from zero to a nonzero value. ♡

1.2 Compact and α-contractive linear operators

If we pass from the whole operator algebra $\mathfrak{L}(X)$ to some subalgebra of operators, much more can be said about the spectrum. Let us first consider the ideal $\mathfrak{KL}(X, Y) \subseteq \mathfrak{L}(X, Y)$ of compact linear operators; as before, we write $\mathfrak{KL}(X, X) =: \mathfrak{KL}(X)$.

Recall that a linear operator $L : X \to Y$ is called *compact* if it maps every bounded set $M \subset X$ into a precompact set $L(M) \subset Y$ (i.e. $\overline{L(M)}$ is compact). Compactness may be defined equivalently with sequences: every bounded sequence $(x_n)_n$ in X contains a subsequence $(x_{n_k})_k$ such that $(Lx_{n_k})_k$ converges. Obviously, every compact linear operator is bounded; the converse is true only in finite dimensional spaces. For example, the operator (1.16) is bounded in the sequence space l_p $(1 \leq p \leq \infty)$ if and only if $(a_n)_n$ is bounded, and compact in l_p if and only if $a_n \to 0$. Similarly, the multiplication operator

$$Lx(t) = a(t)x(t) \tag{1.18}$$

is bounded in the Lebesgue space $L_p[0, 1]$ $(1 \leq p \leq \infty)$ if and only if $a \in L_\infty[0, 1]$, and compact in $L_p[0, 1]$ if and only if $a(t) = 0$ almost everywhere on $[0, 1]$. An analogous result is true in the space $C[0, 1]$ in case of a continuous multiplier a.

Let X be a Banach space over $\mathbb{K}$ and $L \in \mathfrak{L}(X)$. Recall that a number $\lambda \in \mathbb{K}$ is called *eigenvalue* of L if the equation

$$Lx = \lambda x \tag{1.19}$$

has a *nontrivial* solution $x \in X$. Any such x is then called *eigenvector*, and the set of all eigenvectors is a subspace of X called *eigenspace*. The (not necessarily finite) number

$$n(\lambda; L) = \dim \bigcup_{k=1}^{\infty} N((\lambda I - L)^k) \tag{1.20}$$

is called *algebraic multiplicity* of the eigenvalue λ; even in finite dimensions this number may be strictly larger than the *geometric multiplicity* $\dim N(\lambda I - L)$ of λ. Throughout the following, we will call the set of eigenvalues

$$\sigma_{\mathrm{p}}(L) := \{\lambda \in \mathbb{K} : Lx = \lambda x \text{ for some } x \neq \theta\} \tag{1.21}$$

the *point spectrum* of L. The point spectrum is, of course, a subset of $\sigma(L)$, because $\lambda I - L$ is not injective for $\lambda \in \sigma_p(L)$. For example, in case $X = \mathbb{C}^n$ every operator (matrix) $L \in \mathfrak{L}(X)$ has a pure point spectrum $\sigma(L) = \sigma_p(L) = \{\lambda_1, \dots, \lambda_k\} \subset \mathbb{C}$ satisfying $n(\lambda_1; L) + \dots + n(\lambda_k; L) = n$.

The following theorem shows that the situation is similar for the spectrum of a compact operator, even in infinite dimensions.

Theorem 1.2. *For $L \in \mathfrak{K}\mathfrak{L}(X)$, the following is true*:

(a) *For any $\varepsilon > 0$, the set $\{\lambda \in \sigma_p(L) : |\lambda| \geq \varepsilon\}$ is finite.*

(b) *If $\lambda \neq 0$ is not an eigenvalue of L, then $\lambda I - L\colon X \to X$ is an isomorphism.*

(c) *Every point $\lambda \in \sigma(L) \setminus \{0\}$ is an eigenvalue of L.*

(d) *$0 \in \sigma(L)$ if X is infinite dimensional.*

We omit the proof of the statements (a)–(c), since we will prove below a more general result (Theorem 1.3). To see that (d) is true, suppose that $0 \in \rho(L)$. Then L is a compact bijection on X with bounded inverse L^{-1}, and hence $I = LL^{-1}$ is also compact. This implies that X is finite dimensional.

Both Theorem 1.2 and the more general Theorem 1.3 will be illustrated in the next section by a series of examples.

There is an important generalization of the class of compact operators which we will discuss now. In this section we restrict ourselves to linear operators, while parallel results for nonlinear operators will be considered in Section 2.3.

To this end, we have to recall a "topological" characteristic which is extremely useful in the theory and applications of both linear and nonlinear analysis. Let X be a Banach space and $M \subset X$ a bounded subset. The (Hausdorff) *measure of noncompactness* of M is defined by

$$\alpha(M) = \inf\{\varepsilon : \varepsilon > 0,\ M \text{ has a finite } \varepsilon\text{-net in } X\}, \tag{1.22}$$

where by a finite ε-net for M we mean, as usual, a finite set $\{z_1, \dots, z_m\} \subset X$ with the property that

$$M \subseteq [z_1 + B_\varepsilon(X)] \cup \dots \cup [z_m + B_\varepsilon(X)].$$

Here and throughout the following we use the notation

$$B_r(X) := \{x \in X : \|x\| \leq r\} \tag{1.23}$$

for the closed ball with centre θ and radius $r > 0$ in X, and

$$S_r(X) := \partial B_r(X) = \{x \in X : \|x\| = r\} \tag{1.24}$$

for the corresponding sphere. In case $r = 1$ we simply write $B_1(X) =: B(X)$ and $S_1(X) =: S(X)$. The corresponding open balls will be denoted by

$$B_r^o(X) = \{x \in X : \|x\| < r\}$$

and, in particular, $B^o_1(X) =: B^o(X)$.

In the following Proposition 1.1 we recall some properties of the measure of noncompactness (1.22).

Proposition 1.1. *The measure of noncompactness* (1.22) *has the following properties* ($M, N \subset X, \lambda \in \mathbb{K}, z \in X$):

(a) $\alpha(M) = 0$ *if and only if* M *is precompact, i.e. has compact closure.*

(b) $|\alpha(M) - \alpha(N)| \leq \alpha(M + N) \leq \alpha(M) + \alpha(N)$.

(c) $\alpha(\lambda M) = |\lambda|\alpha(M)$.

(d) $\alpha(M + \{z\}) = \alpha(M)$.

(e) $\alpha(\overline{\text{co}}\, M) = \alpha(M)$.

(f) $\alpha(M \cup N) = \max\{\alpha(M), \alpha(N)\}$.

(g) $\alpha(B_r(X)) = \alpha(B^o_r(X)) = \alpha(S_r(X)) = 0$ *if* $\dim X < \infty$ *and* $= r$ *if* $\dim X = \infty$.

(h) *If* $M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n \supseteq \ldots$ *is a decreasing sequence of closed sets in* X *with* $\alpha(M_n) \to 0$ *as* $n \to \infty$, *then* $M_\infty := \bigcap_{n=1}^{\infty} M_n$ *is nonempty and compact.*

Proof. Property (a) rephrases a well-known characterization of precompactness by finite ε-nets in complete normed (even metric) spaces. To prove (b), it suffices to observe that, if $\{z_1, \ldots, z_m\}$ is a finite ε-net for M, and $\{w_1, \ldots, w_n\}$ is a finite η-net for N, then $\{z_i + w_j : i = 1, \ldots, m;\ j = 1, \ldots, n\}$ is a finite $(\varepsilon + \eta)$-net for $M + N$. Similarly, if $\{z_1, \ldots, z_m\}$ is a finite ε-net for M, then certainly $\{\lambda z_1, \ldots, \lambda z_m\}$ is a finite $|\lambda|\varepsilon$-net for λM, which proves (c). Finally, (d) follows from the trivial observation that $\{z_1, \ldots, z_M\}$ is a finite ε-net for M if and only if $\{z_1 + z, \ldots, z_m + z\}$ is a finite ε-net for $M + z$.

In assertion (e) we have to show only that $\alpha(\overline{\text{co}}\, M) \leq \alpha(M)$, by (f). Fix $\eta > \alpha(M)$, choose a finite η-net $\{z_1, \ldots, z_m\}$ for M, and let $\varepsilon > 0$ be arbitrary. Then $N := \overline{\text{co}}\{z_1, \ldots, z_m\}$ is a compact set satisfying $\text{dist}(x, N) \leq \eta$ for all $x \in \overline{\text{co}}\, M$. But since N is compact, we certainly find a finite ε-net $\{w_1, \ldots, w_n\}$ for N, and this is then a finite $(\eta + \varepsilon)$-net for $\overline{\text{co}}\, M$.

To see that (f) holds it suffices again to observe that, if $\{z_1, \ldots, z_m\}$ is a finite ε-net for M, and $\{w_1, \ldots, w_n\}$ is a finite η-net for N, then $\{z_1, \ldots, z_m\} \cup \{w_1, \ldots, w_n\}$ is a finite δ-net for $M \cup N$, where $\delta := \max\{\varepsilon, \eta\}$.

Let us now prove (g); by (c) and (e), we may restrict ourselves to the closed unit ball $B(X)$. If X is finite dimensional, then $B(X)$ is compact, and hence $\alpha(B(X)) = 0$, by (a). Suppose that X is infinite dimensional. Since $B(X)$ may be covered by itself, we have the trivial estimate $\alpha(B(X)) \leq 1$. Suppose that $\alpha(B(X)) < 1$, and fix $\varepsilon \in (\alpha(B(X)), 1)$. By definition, we find then a finite ε-net for $B(X)$. This ε-net gives rise to finitely many closed balls of radius ε, each of which may in turn be covered by finitely many balls of radius ε^2, by the homogeneity property (c) and the translation invariance (d). Continuing this way we may cover $B(X)$ by an increasing, though finite, number of balls of arbitrarily small radius ε^n. We conclude that $\alpha(B(X)) = 0$, contradicting the well-known fact that the unit ball of an infinite dimensional Banach space is never compact.

It remains to prove (h). Let $(x_n)_n$ be a sequence with $x_n \in M_n$ for all n, and put $N_n = \{x_m : m \geq n\}$. Obviously, $N_n \subseteq M_n$, and hence

$$\alpha(N_1) = \alpha(N_2) = \cdots = \alpha(N_n) \leq \alpha(M_n) \to 0 \quad (n \to \infty).$$

Consequently, the set $N_1 = \{x_1, x_2, x_3 \dots\}$ is precompact, and hence $x_{n_k} \to x_*$ $(k \to \infty)$ for some subsequence $(x_{n_k})_k$. But the limit x_* belongs then to the intersection M_∞, and so $M_\infty \neq \emptyset$. The compactness of M_∞ follows from the assumption that $\alpha(M_n) \to 0$ as $n \to \infty$, and the fact that M_∞ is closed. □

Now let $L\colon X \to Y$ be a bounded linear operator. From the definition (1.22) of the measure of noncompactness it follows then that

$$\alpha(L(M)) \leq k\alpha(M) \tag{1.25}$$

for any bounded subset $M \subset X$, where at least $k = \|L\|$. In fact, if $\{z_1, \dots, z_m\}$ is a finite ε-net for M, then obviously $\{Lz_1, \dots, Lz_m\}$ is a finite $\|L\|\varepsilon$-net for $L(M)$. More generally, we put

$$[L]_{\mathrm{A}} = \inf\{k : k > 0,\ (1.25) \text{ holds}\} \tag{1.26}$$

and call the characteristic $[L]_{\mathrm{A}}$ the *measure of noncompactness* (or *α-norm*) of L. In particular, in case $[L]_{\mathrm{A}} < 1$ the operator L is called *α-contractive* (or a *ball contraction*). Intuitively speaking, the condition $[L]_{\mathrm{A}} < 1$ means that the image $L(M)$ of any bounded set $M \subset X$ is "more compact" than M itself.

Parallel to (1.25) and (1.26), we will also be interested in the "reverse" condition

$$\alpha(L(M)) \geq k\alpha(M) \tag{1.27}$$

for any bounded subset $M \subset X$, and in the "lower" characteristic

$$[L]_{\mathrm{a}} = \sup\{k : k > 0,\ (1.27) \text{ holds}\}. \tag{1.28}$$

Observe that we may write the characteristics (1.26) and (1.28) in the equivalent form

$$[L]_{\mathrm{A}} = \sup_{\alpha(M)>0} \frac{\alpha(L(M))}{\alpha(M)}, \quad [L]_{\mathrm{a}} = \inf_{\alpha(M)>0} \frac{\alpha(L(M))}{\alpha(M)} \tag{1.29}$$

if the space X is infinite dimensional. In finite dimensional spaces this does not make sense, since all bounded sets are precompact, and so there are no sets M satisfying $0 < \alpha(M) < \infty$.

We collect again some basic properties of the characteristics (1.26) and (1.28) in the following Proposition 1.2. Recall that a bounded linear operator is called *left semi-Fredholm* if it has closed range and finite dimensional nullspace, and *right semi-Fredholm* if it has a closed range of finite codimension. Given an operator $L \in \mathfrak{L}(X, Y)$, throughout the following we denote by $L^* \in \mathfrak{L}(Y^*, X^*)$ its Banach space adjoint defined by the equality

$$(L^*\ell)x := \ell(Lx) \quad (\ell \in Y^*,\ x \in X). \tag{1.30}$$

Proposition 1.2. *The characteristics* (1.26) *and* (1.28) *have the following properties* ($L, K \in \mathfrak{L}(X, Y), \lambda \in \mathbb{K}$):

(a) $[L]_{\mathrm{A}} = 0$ *if and only if L is compact.*

(b) $[L + K]_{\mathrm{A}} \le [L]_{\mathrm{A}} + [K]_{\mathrm{A}}$.

(c) $[\lambda L]_{\mathrm{A}} = |\lambda|[L]_{\mathrm{A}}$.

(d) $[K]_{\mathrm{a}}\,[L]_{\mathrm{a}} \le [KL]_{\mathrm{A}} \le [K]_{\mathrm{a}}\,[L]_{\mathrm{A}}$.

(e) $[L]_{\mathrm{a}} - [K]_{\mathrm{A}} \le [L + K]_{\mathrm{a}} \le [L]_{\mathrm{a}} + [K]_{\mathrm{A}}$.

(f) $|[L]_{\mathrm{a}} - [K]_{\mathrm{a}}| \le [L - K]_{\mathrm{A}}$*; in particular,* $[L - K]_{\mathrm{A}} = 0$ *implies* $[L]_{\mathrm{a}} = [K]_{\mathrm{a}}$.

(g) $[L]_{\mathrm{a}}\,[L^{-1}]_{\mathrm{A}} = 1$ *if* $\dim X = \infty$ *and L is a linear isomorphism.*

(h) $[L]_{\mathrm{A}} \le \|L\|$.

(i) $[L]_{\mathrm{a}} \le [L]_{\mathrm{A}}$ *if* $\dim X = \infty$.

(j) $[\lambda I - L]_{\mathrm{A}} = [\lambda I - L]_{\mathrm{a}} = |\lambda|$ *if* $\dim X = \infty$ *and L is compact.*

(k) $[L]_{\mathrm{a}} > 0$ *if and only if L is left semi-Fredholm.*

(l) $[L^*]_{\mathrm{a}} > 0$ *if and only if L is right semi-Fredholm.*

Proof. The properties (a)–(f) are immediate consequences of the definitions (1.26) and (1.28) and the properties of the measure of noncompactness (1.22) proved in Proposition 1.1.

Property (g) follows from the chain of equalities

$$[L^{-1}]_{\mathrm{A}} = \sup_{\alpha(N)>0} \frac{\alpha(L^{-1}(N))}{\alpha(N)} = \sup_{\alpha(M)>0} \frac{\alpha(M)}{\alpha(L(M))} = \left(\inf_{\alpha(M)>0} \frac{\alpha(L(M))}{\alpha(M)} \right)^{-1} = \frac{1}{[L]_{\mathrm{a}}},$$

while property (h) has already been proved after Proposition 1.1.

The estimate (i) is a trivial consequence of (1.29), while (j) follows from the fact that $[I]_{\mathrm{A}} = [I]_{\mathrm{a}} = 1$ in every infinite dimensional space X.

To prove (k), suppose first that $[L]_{\mathrm{a}} > 0$, and fix $k \in (0, [L]_{\mathrm{a}})$. Since the set $M := N(L) \cap B(X)$ is mapped into $F(M) = \{\theta\}$, we get

$$\alpha(M) \le \frac{1}{k}\alpha(F(M)) = 0,$$

which shows that M is precompact, and hence $N(L)$ is finite dimensional. We prove now that the range $R(L)$ of L is closed. Since $\dim N(L) < \infty$, there exists a closed subspace $X_0 \subseteq X$ such that $X = X_0 \oplus N(L)$. Let $(y_n)_n$ be a sequence in $R(L)$ converging to some $y_* \in Y$, and choose $(x_n)_n$ in X with $Lx_n = y_n$. Now we distinguish two cases. First, suppose that $(x_n)_n$ is bounded. With $k > 0$ as before we get then

$$\alpha(\{x_1, x_2, x_3, \dots\}) \leq \frac{1}{k}\alpha(\{y_1, y_2, y_3, \dots\}) = 0,$$

and hence $x_{n_k} \to x_*$ for some subsequence $(x_{n_k})_k$ of $(x_n)_n$ and suitable $x_* \in X$. By continuity we see that $L(x_*) = y_*$, and so $y_* \in R(L)$. On the other hand, suppose that $\|x_n\| \to \infty$. Set $e_n := x_n/\|x_n\|$ and $E := \{e_1, e_2, e_3, \dots\}$. Then clearly $E \subset S(X)$ and

$$Le_n = \frac{Lx_n}{\|x_n\|} = \frac{y_n}{\|x_n\|} \to \theta \quad (n \to \infty),$$

hence $\alpha(L(E)) = 0$. On the other hand, $\alpha(L(E)) \geq k\alpha(E)$, by definition (1.28), and thus $\alpha(E) = 0$. Without loss of generality we may assume that the sequence $(e_n)_n$ converges to some element $e \in S(X_0)$. So $Le = \theta$, contradicting the fact that $X_0 \cap N(L) = \{\theta\}$.

Now we prove that the closedness of $R(L)$ and the fact that $N(L)$ is finite dimensional imply that $[L]_{\mathrm{a}} > 0$. Since $\dim N(L) < \infty$ we may find a closed subspace X_0 of X with $X = X_0 \oplus N(L)$. The projection $P: X \to X_0$ satisfies $[P]_{\mathrm{a}} = 1$, since $I - P$ is compact. Consider the canonical isomorphism $\hat{L}: X_0 \to R(L)$. Since $L = \hat{L}P$ and $[\hat{L}]_{\mathrm{a}} > 0$, we conclude that also $[L]_{\mathrm{a}} \geq [\hat{L}]_{\mathrm{a}}[P]_{\mathrm{a}} > 0$.

It remains to prove (l). To this end, suppose first that $[L^*]_{\mathrm{a}} > 0$. By what we have proved in (k), this implies that $R(L^*)$ is closed, and hence $R(L)$ is closed. Moreover, the nullspace $N(L^*)$ of L^* is finite dimensional, so we may find a basis $\{g_1, \dots, , g_n\}$ for $N(L^*)$, i.e. $N(L^*) = \operatorname{span}\{g_1, \dots, , g_n\}$. But the fact that $R(L) = N(g_1) \cap \dots \cap N(g_n)$ shows that $N(L)$ has finite codimension.

Finally let us show that the closedness of $R(L)$ and the fact that $R(L)$ has finite codimension imply that $[L^*]_{\mathrm{a}} > 0$. Again, the closedness of $R(L)$ implies the closedness of $R(L^*)$. On the other hand, we have $R(L) = N(g_1) \cap \dots \cap N(g_n)$, where $\{g_1, \dots, g_n\}$ is a basis of $N(L^*)$. Thus $\dim N(L^*) < \infty$, and the result proved in the second part of (k) implies that $[L^*]_{\mathrm{a}} > 0$. The proof is complete. □

Proposition 1.2 (k) and (l) give a characterization of semi-Fredholm operators in terms of the lower measure of noncompactness (1.28). In particular, $L \in \mathfrak{L}(X, Y)$ is Fredholm if and only if both $[L]_{\mathrm{a}} > 0$ and $[L^*]_{\mathrm{a}} > 0$.

Theorem 1.2 provides a precise description of the spectrum of a compact linear operator. An important generalization to α-contractive linear operators reads as follows.

Theorem 1.3. *For $L \in \mathfrak{L}(X)$ with $[L]_{\mathrm{A}} < 1$, the following is true:*

(a) *For any $\varepsilon > 0$, the set $\{\lambda \in \sigma_{\mathrm{p}}(L) : |\lambda| \geq [L]_{\mathrm{A}} + \varepsilon\}$ is finite.*

(b) *If λ with $|\lambda| > [L]_{\mathrm{A}}$ is not an eigenvalue of L, then $\lambda I - L: X \to X$ is an isomorphism.*

(c) *Every point* $\lambda \in \sigma(L)$ *with* $|\lambda| > [L]_A$ *is an eigenvalue of* L.

Proof. To prove (a), suppose that there exists a sequence $(\lambda_n)_n$ of distinct eigenvalues of L with $|\lambda_n| \geq [L]_A + \varepsilon$, and let $(x_n)_n$ be a corresponding sequence of eigenvectors. Since L is α-contractive, we may find $k \in \mathbb{N}$ such that $[L]_A^k < \frac{1}{2}$. Since all eigenvalues are distinct, the sequence of spaces X_n spanned by $\{x_1, \ldots, x_n\}$ is strictly increasing. By the well-known Riesz lemma we may find a sequence $(e_n)_n$ in $S(X_n)$ such that $\|x - e_n\| \geq \frac{1}{2}$ for all $x \in X_{n-1}$. From the fact that e_n lies in the linear hull of $\{x_1, \ldots, x_n\}$ it follows that

$$L^k e_n - \lambda_n^k e_n \in X_{n-1}$$

and

$$z_{n,m} := e_n - \frac{L^k e_n}{\lambda_n^k} + \frac{L^k e_m}{\lambda_m^k} \in X_{n-1}$$

for $n > m$. Consequently,

$$\left\| L^k\left(\frac{e_n}{\lambda_n^k}\right) - L^k\left(\frac{e_m}{\lambda_m^k}\right)\right\| = \|e_n - z_{n,m}\| \geq \frac{1}{2} \quad (n > m). \tag{1.31}$$

Now, since the set $M := \{\lambda_1^{-k} e_1, \lambda_2^{-k} e_2, \lambda_3^{-k} e_3, \ldots\}$ is included in the closed ball $B_r(X)$ of radius $r = ([L]_A + \varepsilon)^{-1}$, we conclude that

$$\alpha(L^k(M)) \leq [L]_A^k \alpha(M) < \frac{1}{2}.$$

On the other hand, (1.31) shows that $\alpha(L^k(M)) \geq \frac{1}{2}$, a contradiction.

To prove (b) we show first that the range $R(\lambda I - L)$ of $\lambda I - L$ is closed in X. Let $(y_n)_n$ be a sequence in $R(\lambda I - L)$ which converges to some $y \in Y$, and choose $x_n \in X$ with $\lambda x_n - Lx_n = y_n$. We claim that the sequence $(x_n)_n$ is bounded. In fact, suppose that $\|x_n\| \to \infty$ and put $e_n := x_n/\|x_n\|$. Then $e_n \in S(X)$ and $\lambda e_n - Le_n = y_n/\|x_n\|$, hence

$$\|\lambda e_n - Le_n\| = \frac{\|y_n\|}{\|x_n\|} \to 0 \quad (n \to \infty).$$

Moreover,

$$\begin{aligned}\alpha(\{e_1, e_2, \ldots\}) &\leq \frac{[L]_A}{|\lambda|}\alpha(\{e_1, e_2, \ldots\}) + \frac{1}{|\lambda|}\alpha(\{y_1, y_2, \ldots\}) \\ &= \frac{[L]_A}{|\lambda|}\alpha(\{e_1, e_2, \ldots\}),\end{aligned}$$

which implies that $\alpha(\{e_1, e_2, e_3, \ldots\}) = 0$, by our assumption $|\lambda| > [L]_A$. So $(e_n)_n$ admits a convergent subsequence, say $e_{n_k} \to e \in S(X)$ as $k \to \infty$. The continuity of L implies that $\lambda e - Le = \theta$, contradicting the fact that $\lambda \notin \sigma_p(L)$.

So we have proved that the sequence $(x_n)_n$ is bounded. Repeating the same reasoning as before for $(x_n)_n$ instead of $(e_n)_n$ we obtain a convergent subsequence

$x_{n_k} \to x$ as $k \to \infty$, and again by continuity we get $y = \lambda x - Lx \in R(\lambda I - L)$. The boundedness of the resolvent operator $R(\lambda; L)$ on $R(\lambda I - L)$ follows as usual from the closed graph theorem.

Now we show that $\lambda I - L$ is onto, so actually $R(\lambda I - L) = X$. Denoting $R_n := R((\lambda I - L)^n)$ for $n = 0, 1, 2, 3, \ldots$, we see that $(R_n)_n$ is a decreasing (with respect to inclusion) sequence of closed subsets of X. We claim that this sequence becomes stationary, i.e. there exists $n \in \mathbb{N}$ such that $R_n = R_m$ for all $m \geq n$. In fact, suppose that $R_{n+1} \subset R_n$ for all n, and fix $\delta > 0$ with $[L]_{\mathrm{A}} < |\lambda|\delta$. By Riesz' lemma we find a sequence $(y_n)_n$ with $y_n \in R_n$, $\|y_n\| = 1$, and $\mathrm{dist}(y_n, R_m) \geq \delta$ for every $m > n$. In particular, $\|y_n - y_m\| \geq \delta$ for $m > n$, and so $(y_n)_n$ contains no convergent subsequence. So putting $M := \{y_1, y_2, y_3, \ldots\}$ we see that

$$0 < \alpha(M) \leq 1,$$

since $M \subset S(X)$. Moreover, for $m > n$ we have

$$\|Ly_m - Ly_n\| = \|(\lambda I - L)y_n - (\lambda I - L)y_m + \lambda y_m - \lambda y_n\| \geq |\lambda|\delta,$$

because $(\lambda I - L)y_n \in R_{n+1}$, $(\lambda I - L)y_m \in R_{m+1}$, and $\lambda y_m \in R_m$. Consequently,

$$\alpha(L(M)) \geq |\lambda|\delta > [L]_{\mathrm{A}}.$$

On the other hand, from $\alpha(M) \leq 1$ it follows that $\alpha(L(M)) \leq [L]_{\mathrm{A}}$, a contradiction.

We have proved that $R_n = R_{n+1}$ for some n. Now, any $y \in R_{n-1}$ satisfies $(\lambda I - L)y \in R_n = R((\lambda I - L)^n)$, and so we find $x \in R_{n+1} = R_n$ such that $(\lambda I - L)y = (\lambda I - L)x$. Since $\lambda I - L$ is injective, as already proved, we conclude that $y = x$, hence $y \in R_n$. This shows that $R_{n-1} \subseteq R_n$, hence $R_{n-1} = R_n$. Continuing in this fashion we see that $X = R_0 = R_1 = R(\lambda I - L)$ which means that $\lambda I - L$ is indeed surjective.

The assertion (c) is only a reformulation of (b), and so the proof is complete. □

The simple example $Lx = \mu x$ with $0 < |\mu| < 1$ shows that an analogue of Theorem 1.2 (d) for α-contractive operators is not true.

1.3 Subdivision of the spectrum

There are many different ways to subdivide the spectrum of a bounded linear operator; some of them are motivated by applications to physics (in particular, quantum mechanics).

Let X be a Banach space and $L \in \mathfrak{L}(X)$. We say that $\lambda \in \mathbb{K}$ belongs to the *continuous spectrum* $\sigma_{\mathrm{c}}(L)$ of L if the resolvent operator (1.6) is defined on a dense subspace of X and is *unbounded*. Furthermore, we say that $\lambda \in \mathbb{K}$ belongs to the *residual spectrum* $\sigma_{\mathrm{r}}(L)$ of L if the resolvent operator (1.6) exists, but its domain of definition (i.e. the range $R(\lambda I - L)$ of $\lambda I - L$) is *not* dense in X; in this case $R(\lambda; L)$

may be bounded or unbounded. Together with the point spectrum (1.21), these two subspectra form a disjoint subdivision

$$\sigma(L) = \sigma_{\mathrm{p}}(L) \cup \sigma_{\mathrm{c}}(L) \cup \sigma_{\mathrm{r}}(L) \tag{1.32}$$

of the spectrum of L. Loosely speaking, the elements λ in the subspectrum $\sigma_{\mathrm{p}}(L)$ characterize some lack of injectivity, those in $\sigma_{\mathrm{r}}(L)$ some lack of surjectivity, and those in $\sigma_{\mathrm{c}}(L)$ some lack of stability of the operator $\lambda I - L$. We illustrate the subdivision (1.32) in the following table and then give some examples.

Table 1.1

	$R(\lambda; L)$ exists and is bounded	$R(\lambda; L)$ exists and is unbounded	$R(\lambda; L)$ does not exist
$R(\lambda I - L) = X$	$\lambda \in \rho(L)$	———	$\lambda \in \sigma_{\mathrm{p}}(L)$
$\overline{R(\lambda I - L)} = X$	$\lambda \in \rho(L)$	$\lambda \in \sigma_{\mathrm{c}}(L)$	$\lambda \in \sigma_{\mathrm{p}}(L)$
$\overline{R(\lambda I - L)} \neq X$	$\lambda \in \sigma_{\mathrm{r}}(L)$	$\lambda \in \sigma_{\mathrm{r}}(L)$	$\lambda \in \sigma_{\mathrm{p}}(L)$

Observe that the case in the first row and second column cannot occur in a Banach space X, by the closed graph theorem. If we are not in the third column, i.e., if λ is not an eigenvalue of L, we may always consider the resolvent operator (1.6) (on a possibly "thin" domain of definition) as "algebraic" inverse of $\lambda I - L$. This will be done in several examples in the sequel.

Example 1.2. Let $X = l_p$ $(1 \leq p \leq \infty)$, $(a_n)_n$ a bounded sequence in $\mathbb{K}$, and $L \in \mathfrak{L}(l_p)$ defined by (1.16). Let first $p < \infty$. Denoting by $A = \{a_1, a_2, a_3, \ldots\}$ the set of all elements of the sequence $(a_n)_n$ we get

$$\sigma(L) = \overline{A}, \quad \sigma_{\mathrm{p}}(L) = A, \quad \sigma_{\mathrm{c}}(L) = \overline{A} \setminus A, \quad \sigma_{\mathrm{r}}(L) = \emptyset. \tag{1.33}$$

In particular, $\sigma_{\mathrm{c}}(L) = \{0\}$ if $a_n \to 0$, i.e. $L \in \mathfrak{K}\mathfrak{L}(l_p)$. In case $p = \infty$ the continuous and residual spectrum change their role, i.e. we get

$$\sigma(L) = \overline{A}, \quad \sigma_{\mathrm{p}}(L) = A, \quad \sigma_{\mathrm{c}}(L) = \emptyset, \quad \sigma_{\mathrm{r}}(L) = \overline{A} \setminus A. \tag{1.34}$$

We have used this example already for proving Theorem 1.1 (g). ♡

Example 1.3. Let $X = l_p$ $(1 \leq p \leq \infty)$ over $\mathbb{K} = \mathbb{C}$, and let L be the *left shift operator*

$$L(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots). \tag{1.35}$$

It is easy to see that $L \in \mathfrak{L}(l_p)$ with $r(L) = \|L\| = 1$. Consequently, $\sigma(L)$ is contained in the closure of the complex unit disc (1.17). Moreover, L is surjective but not injective, since its nullspace is

$$N(L) = \{(x_n)_n \in l_p : x_2 = x_3 = x_4 = \cdots = 0\}.$$

This implies, in particular, that $0 \in \sigma_p(L)$. To analyze the spectrum of L, we have to distinguish the two cases $1 \leq p < \infty$ and $p = \infty$. In fact, a careful analysis shows that

$$\sigma(L) = \overline{\mathbb{D}}, \quad \sigma_p(L) = \mathbb{D}, \quad \sigma_c(L) = \mathbb{S}, \quad \sigma_r(L) = \emptyset \tag{1.36}$$

for $1 \leq p < \infty$, and

$$\sigma(L) = \sigma_p(L) = \overline{\mathbb{D}}, \quad \sigma_c(L) = \sigma_r(L) = \emptyset \tag{1.37}$$

for $p = \infty$. Indeed, the eigenspace to the eigenvalue λ is generated by the vector $x_\lambda := (1, \lambda, \lambda^2, \dots)$ which belongs to every l_p for $|\lambda| < 1$, but only to l_∞ for $|\lambda| = 1$. ♡

Example 1.4. Let $X = l_p$ $(1 \leq p \leq \infty)$ over $\mathbb{K} = \mathbb{C}$, and let L be the *right shift operator*

$$L(x_1, x_2, x_3, \dots) = (0, x_1, x_2, \dots). \tag{1.38}$$

It is easy to see that $L \in \mathfrak{L}(l_p)$ with $r(L) = \|L\| = 1$. Consequently, $\sigma(L)$ is again contained in the closure of the complex unit disc (1.17). In this example, L is injective but not surjective, since its range

$$R(L) = \{(y_n)_n \in l_p : y_1 = 0\}$$

is even not dense in l_p. This implies, in particular, that $0 \in \sigma_r(L)$. A scrutiny of the spectrum of L shows that now we have to distinguish the three cases $p = 1$, $1 < p < \infty$ and $p = \infty$. In fact, we have

$$\sigma(L) = \sigma_r(L) = \overline{\mathbb{D}}, \quad \sigma_p(L) = \sigma_c(L) = \emptyset \tag{1.39}$$

for $p = 1$ or $p = \infty$, and

$$\sigma(L) = \overline{\mathbb{D}}, \quad \sigma_p(L) = \emptyset, \quad \sigma_c(L) = \mathbb{S}, \quad \sigma_r(L) = \mathbb{D} \tag{1.40}$$

for $1 < p < \infty$. In particular, the operator (1.38) has no eigenvalues in any of the spaces l_p. ♡

In Theorem 1.2 (d) we have seen that 0 always belongs to the spectrum of a compact operator if the underlying space is infinite dimensional. We show now by three examples that a more precise classification of 0 as spectral point is not possible.

Example 1.5. Let $X = l_p$ $(1 \leq p \leq \infty)$ over $\mathbb{K} = \mathbb{C}$. If we define $L \in \mathfrak{KL}(l_p)$ by

$$L(x_1, x_2, x_3, x_4, \dots) = (x_1, \tfrac{1}{2}x_2, \tfrac{1}{3}x_3, \tfrac{1}{4}x_4, \dots) \tag{1.41}$$

we see that $0 \in \sigma_c(L)$, since the inverse operator

$$L^{-1}(y_1, y_2, y_3, y_4, \dots) = (y_1, 2y_2, 3y_3, 4y_4, \dots)$$

is defined on a dense subspace of l_p and is not bounded on l_p. Actually, we have

$$\sigma_p(L) = \{1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \dots\}, \quad \sigma_c(L) = \{0\}, \quad \sigma_r(L) = \emptyset$$

in this case. On the other hand, if we define $L \in \mathfrak{KL}(l_p)$ by

$$L(x_1, x_2, x_3, x_4, \dots) = (x_2, \tfrac{1}{2}x_3, \tfrac{1}{3}x_4, \tfrac{1}{4}x_5, \dots) \tag{1.42}$$

we see that $0 \in \sigma_p(L)$, since $e_1 = (1, 0, 0, 0, \dots)$ is an eigenvector to the eigenvalue $\lambda = 0$. The inverse operator L^{-1} is not defined here. Actually, we have

$$\sigma(L) = \sigma_p(L) = \{0\}, \quad \sigma_c(L) = \sigma_r(L) = \emptyset \tag{1.43}$$

in this case. Finally, if we define $L \in \mathfrak{KL}(l_p)$ by

$$L(x_1, x_2, x_3, x_4, \dots) = (0, x_1, \tfrac{1}{2}x_2, \tfrac{1}{3}x_3, \dots) \tag{1.44}$$

we see that $0 \in \sigma_r(L)$, since the range $R(L)$ of L is not dense in l_p. Here the inverse operator

$$L^{-1}(y_1, y_2, y_3, y_4, \dots) = (y_2, 2y_3, 3y_4, 4y_5, \dots)$$

is defined, but not bounded on $R(L)$. Actually, we have

$$\sigma(L) = \sigma_r(L) = \{0\}, \quad \sigma_p(L) = \sigma_c(L) = \emptyset \tag{1.45}$$

in this case. ♡

Now we pass to an example which may be considered either in the space $C = C[0, 1]$ of all continuous functions $x\colon [0, 1] \to \mathbb{R}$, or in the space $L_p = L_p[0, 1]$ of all p-integrable $(1 \leq p < \infty)$ or essentially bounded $(p = \infty)$ functions on $[0, 1]$.

Example 1.6. Let $a \in C[0, 1]$ be fixed, and define L by (1.18). Obviously, L is bounded on $X = C[0, 1]$. Moreover, one may easily show (for example, by means of the Arzelà–Ascoli criterion) that L is compact if and only if $a(t) \equiv 0$ on $[0, 1]$. More generally, the equality

$$[L]_A = \|L\| = r(L) = \|a\|$$

is true. In case $a(t) \equiv 0$ we have of course $\sigma(L) = \sigma_p(L) = \{0\}$. In general, the range $R(a) = \{a(t) : 0 \leq t \leq 1\}$ of the function a is compact, by the continuity of a, and

$$\sigma(L) = R(a), \quad \sigma_c(L) = \emptyset, \quad \sigma_p(L) = R_t(a), \quad \sigma_r(L) = R(a) \setminus R_t(a), \tag{1.46}$$

where $R_t(a)$ is the *topological range* of a which consists, by definition, of all $\lambda \in \mathbb{R}$ such that the set $\{t : a(t) = \lambda\}$ has nonempty interior. In particular, if the multiplier a is strictly monotone, then $R_t(a) = \emptyset$, and so the operator (1.18) has no eigenvalues at all and a purely residual spectrum.

If the multiplier a is not continuous, but only essentially bounded, we may consider the corresponding operator (1.18) in the space $L_p[0, 1]$ $(1 \le p \le \infty)$. In this case one has to replace the topological range $R_t(a)$ of a by the *essential range* $R_e(a)$ which consists, by definition, of all $\lambda \in \mathbb{R}$ such that, for every neighbourhood I of λ, the set $\{t : a(t) \in I\}$ has positive measure. The relations (1.46) then become

$$\begin{gathered} \sigma(L) = R_e(a), \quad \sigma_c(L) = R_e(L) \setminus \sigma_p(L), \\ \sigma_p(L) = \{\lambda \in R_e(a), \quad \text{mes}\{t : a(t) = \lambda\} > 0\}, \quad \sigma_r(L) = \emptyset \end{gathered} \tag{1.47}$$

in case $p < \infty$, and

$$\begin{gathered} \sigma(L) = R_e(a), \quad \sigma_c(L) = \emptyset, \\ \sigma_p(L) = \{\lambda \in R_e(a), \quad \text{mes}\{t : a(t) = \lambda\} > 0\}, \quad \sigma_r(L) = R_e(L) \setminus \sigma_p(L) \end{gathered} \tag{1.48}$$

in case $p = \infty$. ♡

The following example contains a noncompact operator with compact square.

Example 1.7. Let $X = l_p$ $(1 \le p \le \infty)$ over $\mathbb{K} = \mathbb{C}$, and let $L \in \mathfrak{L}(l_p)$ be defined by

$$L(x_1, x_2, x_3, x_4, \dots) = (x_2, 0, x_4, 0, \dots). \tag{1.49}$$

Trivially, we have $L^2(x_1, x_2, x_3, x_4, \dots) \equiv (0, 0, 0, 0, \dots)$ which is a compact operator. However, the operator L itself is not compact, since the image $\{Le_k : k \in \mathbb{N}\}$ of the basis vectors $e_k = (\delta_{k,n})_n$ is not precompact in l_p. The spectral mapping formula for polynomials (Theorem 1.1 (h)) implies that $\sigma(L) = \sigma_p(L) = \{0\}$. ♡

Given a bounded linear operator L in a Banach space X, we call a sequence $(x_k)_k$ in X a *Weyl sequence* for L if $\|x_k\| = 1$ and $\|Lx_k\| \to 0$ as $k \to \infty$. For example, the basis elements $e_k = (\delta_{k,n})_n$ in l_p form a Weyl sequence for the operator (1.41).

In what follows, we call the set

$$\sigma_q(L) := \{\lambda \in \mathbb{K} : \text{there exists a Weyl sequence for } \lambda I - L\} \tag{1.50}$$

the *approximate point spectrum* of L; this subspectrum will be defined also for nonlinear operators in the next chapter and will play a very important role in nonlinear spectral theory. Moreover, the subspectrum

$$\sigma_\delta(L) := \{\lambda \in \mathbb{K} : \lambda I - L \text{ is not surjective}\} \tag{1.51}$$

is called *defect spectrum* of L; its nonlinear analogue will be important in Chapter 6.

By definition, we have then $\|\lambda x - Lx\| \geq c\|x\|$ for all $x \in X$ if $\lambda \notin \sigma_{\mathrm{q}}(L)$; equivalently, this may be stated as

$$\inf\{\|\lambda e - Le\| : e \in S(X)\} > 0 \quad (\lambda \notin \sigma_{\mathrm{q}}(L)). \tag{1.52}$$

The two subspectra (1.50) and (1.51) form a (not necessarily disjoint) subdivision

$$\sigma(L) = \sigma_{\mathrm{q}}(L) \cup \sigma_{\delta}(L) \tag{1.53}$$

of the spectrum. There is another subspectrum,

$$\sigma_{\mathrm{co}}(L) = \{\lambda \in \mathbb{K} : \overline{R(\lambda I - L)} \neq X\} \tag{1.54}$$

which is often called *compression spectrum* in the literature and which gives rise to another (not necessarily disjoint) decomposition

$$\sigma(L) = \sigma_{\mathrm{q}}(L) \cup \sigma_{\mathrm{co}}(L) \tag{1.55}$$

of the spectrum. Clearly, $\sigma_{\mathrm{p}}(L) \subseteq \sigma_{\mathrm{q}}(L)$ and $\sigma_{\mathrm{co}}(L) \subseteq \sigma_{\delta}(L)$. Moreover, comparing these subspectra with those in (1.32) we note that

$$\sigma_{\mathrm{r}}(L) = \sigma_{\mathrm{co}}(L) \setminus \sigma_{\mathrm{p}}(L) \tag{1.56}$$

and

$$\sigma_{\mathrm{c}}(L) = \sigma(L) \setminus [\sigma_{\mathrm{p}}(L) \cup \sigma_{\mathrm{co}}(L)]. \tag{1.57}$$

Sometimes it is useful to relate the spectrum of a bounded linear operator to that of its adjoint. Building on classical existence and uniqueness results for linear operator equations in Banach spaces and their adjoints, one may prove the following

Proposition 1.3. *The spectra and subspectra of an operator* $L \in \mathfrak{L}(X)$ *and its adjoint* $L^* \in \mathfrak{L}(X^*)$ *are related by the following relations*:

(a) $\sigma(L^*) = \sigma(L)$.

(b) $\sigma_{\mathrm{c}}(L^*) \subseteq \sigma_{\mathrm{q}}(L)$.

(c) $\sigma_{\mathrm{q}}(L^*) = \sigma_{\delta}(L)$.

(d) $\sigma_{\delta}(L^*) = \sigma_{\mathrm{q}}(L)$.

(e) $\sigma_{\mathrm{p}}(L^*) = \sigma_{\mathrm{co}}(L)$.

(f) $\sigma_{\mathrm{co}}(L^*) \supseteq \sigma_{\mathrm{p}}(L)$.

(g) $\sigma(L) = \sigma_{\mathrm{q}}(L) \cup \sigma_{\mathrm{p}}(L^*) = \sigma_{\mathrm{p}}(L) \cup \sigma_{\mathrm{q}}(L^*)$.

The relations (c)–(f) show that the approximate point spectrum is in a certain sense dual to defect spectrum, and the point spectrum dual to the compression spectrum. The last equation(g) implies, in particular, that $\sigma(L) = \sigma_{\mathrm{q}}(L)$ if X is a Hilbert space and L is normal. Roughly speaking, this shows that normal (in particular, self-adjoint) operators on Hilbert spaces are most similar to matrices in finite dimensional spaces.

Let us briefly illustrate the above decompositions by means of the multiplication operators (1.16) and (1.18). Let L be the operator (1.16) in the sequence space $X = l_p$ ($1 \leq p \leq \infty$) over $\mathbb{K} = \mathbb{R}$, and let again $A = \{a_n : n \in \mathbb{N}\}$ denote the range of the multiplier sequence $(a_n)_n$. Then

$$\sigma_{\mathrm{q}}(L) = \sigma_{\delta}(L) = \sigma_{\mathrm{co}}(L) = \overline{A} \tag{1.58}$$

in case $p < \infty$, and

$$\sigma_{\mathrm{q}}(L) = \overline{A}, \quad \sigma_{\delta}(L) = \sigma_{\mathrm{co}}(L) = A \tag{1.59}$$

in case $p = \infty$. Similarly, if L is the operator (1.18) in the function space $X = C[0, 1]$ over $\mathbb{K} = \mathbb{R}$, then

$$\sigma_{\mathrm{q}}(L) = \sigma_{\delta}(L) = \sigma_{\mathrm{co}}(L) = R(a), \tag{1.60}$$

where $R(a) = \{a(t) : 0 \leq t \leq 1\}$ is the range of the multiplier function a. On the other hand, if we consider the same operator L in the function space $X = L_p[0, 1]$ and denote again by $R_{\mathrm{e}}(a)$ the essential range of a, see Example 1.6, then

$$\sigma_{\mathrm{q}}(L) = \sigma_{\delta}(L) = R_{\mathrm{e}}(a), \quad \sigma_{\mathrm{co}}(L) = \sigma_{\mathrm{p}}(L) \tag{1.61}$$

in case $p < \infty$, and

$$\sigma_{\mathrm{q}}(L) = \sigma_{\delta}(L) = \sigma_{\mathrm{co}}(L) = R_{\mathrm{e}}(a) \tag{1.62}$$

in case $p = \infty$.

We close this section with an important topological property of the approximate point spectrum (1.50). In Chapters 6 and 7 we will prove a certain analogue for nonlinear spectra.

Proposition 1.4. *The following is true*:

(a) $\partial\sigma(L) \subseteq \sigma_{\mathrm{q}}(L)$ *for* $L \in \mathfrak{L}(X)$.

(b) $\partial\sigma(L) = \sigma_{\mathrm{q}}(L)$ *for* $L \in \mathfrak{K}\mathfrak{L}(X)$.

Proof. Fix $\lambda \in \partial\sigma(L)$, and suppose that $\lambda \notin \sigma_{\mathrm{q}}(L)$. Then we find $\delta > 0$ such that $\|\lambda x - Lx\| \geq \delta$ for all $x \in B(X)$. For $|\mu - \lambda| < \delta/2$ we have then

$$\|\mu x - Lx\| \geq \|\lambda x - Lx\| - |\mu - \lambda| \, \|x\| \geq \frac{\delta}{2}\|x\| \quad (x \in X),$$

which implies that $\mu \notin \sigma_{\mathrm{q}}(L)$. Choose a sequence $(\lambda_n)_n$ in $\rho(L)$ such that $|\lambda_n - \lambda| < \delta/(n+1)$, which is possible since $\lambda \in \partial\sigma(L)$. We show that the range $R(\lambda I - L)$ of $\lambda I - L$ is dense in X, and so $\lambda \in \rho(L)$, a contradiction.

Suppose that $\overline{R(\lambda I - L)}$ is a proper subspace of X. By Riesz' lemma, we find then $y \in S(X)$ such that $\|y - \lambda x + Lx\| > 1/2$ for all $x \in X$. On the other hand, since $\lambda_n \in \rho(L)$, every range $R(\lambda_n I - L)$ is dense in X, and so we find a sequence $(x_n)_n$ in X such that $\|y - \lambda_n x_n + Lx_n\| < 1/n$ for all $n \in \mathbb{N}$. Combining these two estimates yields

$$\|y - \lambda x_n + Lx_n\| \leq \|y - \lambda_n x_n + Lx_n\| + |\lambda_n - \lambda|\, \|x_n\| \leq \frac{1}{n} + \frac{\delta}{2}|\lambda_n - \lambda|,$$

and so $\|y - \lambda x_n + Lx_n\| \leq 1/2$ for sufficiently large n. This is a contradiction, and so the inclusion (a) is proved.

If L is compact, then the spectrum $\sigma(L)$ has no interior points, by Theorem 1.2, and so $\sigma_q(L) \subseteq \sigma(L) = \partial\sigma(L) \subseteq \sigma_q(L)$, by (a). □

1.4 Essential spectra of bounded linear operators

A particularly important subdivision of the spectrum of a bounded linear operator is that into a "discrete" and an "essential" part. There are various definitions of essential spectra; they all coincide in case of an underlying Hilbert space, but may be different in the Banach space setting.

Given $L \in \mathfrak{L}(X)$ as before, we set

$$\rho_+(L) := \{\lambda \in \mathbb{K} : R(\lambda I - L) \text{ closed and } \dim N(\lambda I - L) < \infty\}, \tag{1.63}$$

$$\rho_-(L) := \{\lambda \in \mathbb{K} : R(\lambda I - L) \text{ closed and } \operatorname{codim} R(\lambda I - L) < \infty\}, \tag{1.64}$$

and

$$\sigma_\pm(L) := \mathbb{K} \setminus \rho_\pm(L). \tag{1.65}$$

Thus, $\lambda \in \rho_+(L)$ if and only if $\lambda I - L$ is left-semi Fredholm, $\lambda \in \rho_-(L)$ if and only if $\lambda I - L$ is right-semi Fredholm, and $\lambda \in \rho_+(L) \cap \rho_-(L)$ if and only if $\lambda I - L$ is a Fredholm operator.

In what follows, we call the sets

$$\rho_{ek}(L) := \rho_+(L) \cup \rho_-(L), \quad \sigma_{ek}(L) := \sigma_+(L) \cap \sigma_-(L) \tag{1.66}$$

essential resolvent set and *essential spectrum in Kato's sense*, respectively, and the sets

$$\rho_{ew}(L) := \rho_+(L) \cap \rho_-(L), \quad \sigma_{ew}(L) := \sigma_+(L) \cup \sigma_-(L) \tag{1.67}$$

essential resolvent set and *essential spectrum in Wolf's sense*, respectively, of the operator L. So $\lambda \in \rho_{ek}(L)$ if and only if $\lambda I - L$ is semi-Fredholm, and $\lambda \in \rho_{ew}(L)$ if and only if $\lambda I - L$ is Fredholm. Denoting by

$$\operatorname{ind} L := \dim N(L) - \operatorname{codim} R(L)$$

the *index* of a Fredholm operator L, we still recall the sets

$$\rho_{es}(L) := \{\lambda \in \rho_{ew}(L) : \operatorname{ind}(\lambda I - L) = 0\}, \quad \sigma_{es}(L) := \mathbb{K} \setminus \rho_{es}(L) \tag{1.68}$$

called the *essential resolvent set* and *essential spectrum in Schechter's sense*, respectively. So $\lambda \in \rho_{\mathrm{es}}(L)$ if and only if $\lambda I - L$ is Fredholm of index zero. The inclusions

$$\sigma_{\mathrm{ek}}(L) \subseteq \sigma_{\mathrm{ew}}(L) \subseteq \sigma_{\mathrm{es}}(L) \tag{1.69}$$

follow immediately from the definitions. We point out that Schechter's essential spectrum (1.68) has a particularly suggestive interpretation. In fact, the formula

$$\sigma_{\mathrm{es}}(L) = \bigcap_{K \in \mathfrak{K}\mathfrak{L}(X)} \sigma(L + K) \tag{1.70}$$

shows that $\sigma_{\mathrm{es}}(L)$ consists precisely of those points in $\sigma(L)$ which cannot be "removed" by compact perturbations of L.

There is yet another definition which is useful in applications to elliptic boundary value problems. If λ is an isolated point of $\sigma(L)$, by the *spectral projection* associated with λ one means the operator $P_\lambda \in \mathfrak{L}(X)$ defined by

$$P_\lambda := \frac{1}{2\pi i} \int_{\Gamma(\lambda)} (zI - L)^{-1}\, dz = \frac{1}{2\pi i} \int_{\Gamma(\lambda)} R(z; L)\, dz, \tag{1.71}$$

where $\Gamma(\lambda)$ is a simply closed contour around λ such that λ is the only point of $\sigma(L)$ inside $\Gamma(\lambda)$. If P_λ is finite dimensional (i.e. $\dim R(P_\lambda) < \infty$), then

$$n(\lambda; L) = \dim \bigcup_{k=1}^{\infty} N((\lambda I - L)^k) = \dim R(P_\lambda),$$

see (1.20). Such an eigenvalue is usually called an *isolated finite dimensional eigenvalue*.

We say that λ belongs to the *essential spectrum in Browder's sense* and write $\lambda \in \sigma_{\mathrm{eb}}(L)$ if either the range of $\lambda I - L$ is not closed, or λ is an accumulation point of $\sigma(L)$, or λ is an eigenvalue of L of infinite multiplicity (1.20). Thus, the only points $\lambda \in \sigma(L) \setminus \sigma_{\mathrm{eb}}(L)$ are isolated finite dimensional eigenvalues. As before, by $\rho_{\mathrm{eb}}(L)$ we mean the complement of $\sigma_{\mathrm{eb}}(L)$. It is not hard to prove the inclusion

$$\sigma_{\mathrm{es}}(L) \subseteq \sigma_{\mathrm{eb}}(L) \tag{1.72}$$

which complements (1.69).

All essential spectra introduced so far are closed subsets of $\sigma(L)$, hence compact. They are empty if the underlying Banach space is finite dimensional. Moreover, for a self-adjoint operator they all coincide. In the following proposition we recall some relations between these subsets.

Proposition 1.5. *The following inclusions and equalities are true*:

(a) $\sigma_{\mathrm{ek}}(L) \cup \sigma_{\mathrm{p}}(L) = \sigma_{\mathrm{q}}(L)$.

(b) $\partial\sigma_{\mathrm{eb}}(L) \subseteq \partial\sigma_{\mathrm{es}}(L) \subseteq \partial\sigma_{\mathrm{ew}}(L) \subseteq \partial\sigma_{\mathrm{ek}}(L)$.

(c) $\partial\sigma(L) \subseteq \rho_{\mathrm{eb}}(L) \cup \sigma_{\mathrm{ew}}(L)$.

Proof. The assertion (a) is obvious. To prove (b), suppose first that $\lambda \in \partial\sigma_{\text{eb}}(L)$. If λ is isolated, then $\lambda \in \sigma_{\text{es}}(L)$ by definition. On the other hand, assume that λ is not isolated and $\lambda \notin \sigma_{\text{es}}(L)$. Let C be the connected component of $\rho_{\text{es}}(L)$ containing λ. By (1.70), we may then find a compact operator K such that $\lambda \in \rho(L+K)$. Denoting by D the connected component of $\rho(L+K)$ containing λ we have $C \cap D \neq \emptyset$ and

$$R(\mu; L) = R(\mu; L+K)(I + KR(\mu; L+K))^{-1} \quad (\mu \in C \cap D).$$

This implies that λ can be at most an isolated singularity of $I + KR(\mu; L+K)$, and therefore also of $R(\mu; L)$. This contradicts our assumption, and so $\lambda \in \sigma_{\text{es}}(L)$. The assertion follows now from (a).

To prove the second inclusion in (b), suppose we can find $\lambda \in \partial\sigma_{\text{es}}(L) \setminus \partial\sigma_{\text{ew}}(L)$. Then $\lambda I - L$ is Fredholm, and so there exists $\delta > 0$ such that also $\mu I - L$ is Fredholm for $|\lambda - \mu| < \delta$, by the stability of the Fredholm property. Moreover, $\text{ind}(\mu I - L) = \text{ind}(\lambda I - L)$ for such μ. But $\lambda \in \sigma_{\text{es}}(L)$ implies that $\text{ind}(\lambda I - L) = 0$, by definition, and so $\text{ind}(\mu I - L) = 0$, a contradiction.

Now suppose that $\lambda \in \partial\sigma_{\text{ew}}(L) \setminus \partial\sigma_{\text{ew}}(L)$. Then $\lambda I - L$ is semi-Fredholm, and so there exists $\delta > 0$ such that also $\mu I - L$ is semi-Fredholm for $|\lambda - \mu| < \delta$, by the stability of the semi-Fredholm property. Moreover, $\text{ind}(\mu I - L) = \text{ind}(\lambda I - L)$ for such μ. We can take $\mu \in \rho_{\text{ew}}(L)$ so that $\text{ind}(\lambda I - L) = \text{ind}(\mu I - L) < \infty$ and $\lambda I - L$ is Fredholm. But this contradicts our assumption $\lambda \in \sigma_{\text{ew}}(L)$. The proof of the last inclusion in (b) is similar.

Let us now prove (c). Given $\lambda \in \partial\sigma(L)$, we distinguish three cases. First, if λ is an accumulation point of $\sigma(L)$, then $\lambda \in \partial\sigma_{\text{es}}(L) \subseteq \partial\sigma_{\text{ew}}(L) \subseteq \sigma_{\text{ew}}(L)$, by what we have proved above. Next, if λ is an isolated point of $\sigma(L)$ with $n(\lambda; L) < \infty$, then $\lambda \in \rho_{\text{eb}}(L)$, by definition. Finally, suppose that λ is isolated with $n(\lambda; L) = \infty$. Then the integral representation (1.71) of the spectral projection shows that $\lambda \in \sigma_{\text{ew}}(L)$. The proof is complete. □

Let us illustrate Proposition 1.5 by means of an example.

Example 1.8. In $X = l_2$ over $\mathbb{K} = \mathbb{C}$, consider the right shift operator (1.38). We already know that this operator has no eigenvalues, i.e. $\sigma_{\text{p}}(L) = \emptyset$, and that $\sigma(L) = \overline{\mathbb{D}}$. From the definitions of essential spectra it follows that

$$\sigma_{\text{ek}}(L) = \sigma_{\text{ew}}(L) = \mathbb{S}, \quad \sigma_{\text{es}}(L) = \sigma_{\text{eb}}(L) = \overline{\mathbb{D}}.$$

So from Proposition 1.5 (a) we may conclude that $\sigma_{\text{q}}(L) = \mathbb{S}$, and Proposition 1.5 (c) becomes

$$\mathbb{S} \subseteq (\overline{\mathbb{D}} \setminus \overline{\mathbb{D}}) \cup \mathbb{S}.$$

Moreover, all inclusions in Proposition 1.5 (b) are equalities in this example. ♡

To conclude, we mention the following result on spectra and essential spectra of compact operators which is an immediate consequence of the definition of essential spectra.

Theorem 1.4. *For $L \in \mathfrak{KL}(X)$ the equality*

$$\sigma_{\mathrm{ek}}(L) = \sigma_{\mathrm{ew}}(L) = \sigma_{\mathrm{es}}(L) = \sigma_{\mathrm{eb}}(L) = \{0\} \tag{1.73}$$

holds.

In analogy to the radius (1.8) of the whole spectrum, let us consider the *radii of the essential spectra*

$$r_\kappa(L) = \sup\{|\lambda| : \lambda \in \sigma_\kappa(L)\} \quad (\kappa \in \{\mathrm{ek}, \mathrm{es}, \mathrm{ew}, \mathrm{eb}\}). \tag{1.74}$$

The following theorem shows that, although the various essential spectra may be different, they all have the same "size".

Theorem 1.5. *Let X be a complex Banach space and $L \in \mathfrak{L}(X)$. Then the equality*

$$r_{\mathrm{ek}}(L) = r_{\mathrm{ew}}(L) = r_{\mathrm{es}}(L) = r_{\mathrm{eb}}(L) \tag{1.75}$$

holds.

Proof. By (1.69) and (1.72) it suffices to show that $r_{\mathrm{eb}}(L) \le r_{\mathrm{ek}}(L)$. Let $c_\infty[\sigma_{\mathrm{eb}}(L)]$ be the unbounded connected component of $\rho_{\mathrm{eb}}(L)$. Since $\mathbb{C} \setminus c_\infty[\sigma_{\mathrm{eb}}(L)]$ is compact we find $\lambda_0 \in \mathbb{C} \setminus c_\infty[\sigma_{\mathrm{eb}}(L)]$ such that

$$|\lambda_0| = \max\{|\lambda| : \lambda \in \mathbb{C} \setminus c_\infty[\sigma_{\mathrm{eb}}(L)]\}.$$

But $\lambda_0 \in \partial c_\infty[\sigma_{\mathrm{eb}}(L)]$ implies that $\lambda_0 \notin \rho_{\mathrm{ek}}(L)$, and therefore $|\lambda_0| \le r_{\mathrm{ek}}$. □

1.5 Notes, remarks and references

The basic notions and facts on spectral theory for bounded linear operators may be found in every book on functional analysis, see e.g. [36], [102], [249]. We point out that the Neumann series (1.12) which is fundamental in spectral theory converges only if the underlying normed space X is complete, as may be seen by the example

$$L(x_1, x_2, x_3, \dots) = \frac{1}{2}(x_2, x_3, x_4, \dots)$$

in the non-complete space $c_e \subset c_0$ of all *finite* sequences with the supremum norm.

The only property of the spectrum which is perhaps less known is its *upper semi-continuity* with respect to small perturbations, see Theorem 1.1 (i). The proof of this property, as well as Example 1.1, are taken from [158, Chapter IV, § 3]. The first paper which is concerned with continuity properties of spectra, viewed as multivalued maps from the algebra $\mathfrak{L}(X)$ into the complex plane, seems to be [205].

We will study an analogue of the resolvent operator (1.6) for *nonlinear* operators in detail in Section 5.6 below. A general discussion of resolvent operators and their connections with semigroups, which in part carry over to nonlinear operators, may be found in Chapter 4.3 of the book [35].

A notion which is quite useful in linear spectral theory, but we do not consider in this book, is that of *spectral resolutions*. This is a family $\{P_\lambda : -\infty < \lambda < +\infty\}$ of projections in a Hilbert space X with the property that every selfadjoint operator $L \in \mathfrak{L}(X)$ may be represented in the form

$$L = \int_{-\infty}^{+\infty} \lambda \, dP_\lambda.$$

Interestingly, Zarantonello [284], [285] has developed a parallel theory for certain classes of nonlinear operators, where now P_λ is a *nonlinear* projection (viz., the projection on a closed convex subset K of X which associates to each $x \in X$ the element $P(x) \in K$ of best approximation in K). A continuation of this idea may be found in the thesis [159], together with applications to homogeneous Nemytskij operators (see Section 4.3) and to nonlinear integral operators of Uryson type.

Theorem 1.2 shows that the spectral properties of compact linear operators are similar to those of matrices, apart from the exceptional role of the point 0. Theorem 1.3 is a natural extension of Theorem 1.2 due to Ambrosetti [4]. More material on measures of noncompactness and linear α-contractive operators may be found in the survey paper [231] or in the monograph [1]. The only nontrivial property of the measure of noncompactness (1.22) is the "nontrivial intersection property" given in Proposition 1.1 (h); this goes back to Kuratowski [168].

We remark that the measure of noncompactness (1.26) is closely related to the so-called *essential norm*

$$|||L||| := \inf\{\|L - K\| : K \in \mathfrak{K}\mathfrak{L}(X, Y)\} \tag{1.76}$$

of $L \in \mathfrak{L}(X, Y)$, i.e. the norm of L in the *Calkin algebra* $\mathfrak{L}(X, Y)/\mathfrak{K}\mathfrak{L}(X, Y)$ of all bounded modulo compact operators. In fact, it is not hard to see that

$$[L]_{\mathrm{A}} \le |||L||| \le \|L\|. \tag{1.77}$$

The first estimate in (1.77) follows from Proposition 1.2 (b), while the second estimate is a consequence of the trivial fact that the zero operator is compact. For instance, for the multiplication operator from Example 1.6 we have equality on both sides of (1.77). The second inequality in (1.77) is of course strict for any compact operator $L \neq \Theta$. We point out that the first inequality in (1.77) may also be strict. For example [134], let $X = l_2 \times c$ be equipped with the product norm

$$\|(x, y)\| := (\|x\|_{l_2}^2 + \|y\|_c^2)^{1/2} = \Big(\sum_{n=1}^{\infty} |x_n|^2 + \sup_n |y_n|^2\Big)^{1/2},$$

and define $L\colon X \to X$ by $L(x, y) = (\theta, x)$. Then

$$[L]_{\mathrm{A}} = \frac{1}{\sqrt{2}}, \quad |||L||| = \|L\| = 1.$$

This example is of course somewhat academic. However, there are also natural examples for noncompact operators L which satisfy $0 < [L]_A < \|L\|$ and are connected to essential spectra of elliptic differential operators over an unbounded domain, see [7].

The subdivision (1.32) of the spectrum is standard and may be found in most textbooks on functional analysis and operator theory, e.g. [249]. Examples 1.2–1.5 are also taken from [249].

In our definition of the approximate point spectrum (1.50), the defect spectrum (1.51) we followed [36]. We remark that the approximate point spectrum (1.50) is also called *limit spectrum* by some authors.

The proof of the relations between the subspectra of L and L^* stated in Proposition 1.3 may be found in many textbooks on functional analysis and operator theory, e.g., [249]. The equalities (1.58)–(1.62) may be proved in a straightforward way. For example, let us sketch the proof of the equality $\sigma_q(L) = R_e(a)$ in (1.61). The inclusion $\sigma(L) \subseteq R_e(a)$ is easy to prove. Conversely, for $\lambda \in R_e(a)$ we may find subsets $D_n \subseteq \{t : |a(t) - \lambda| \le 1/n\}$ of measure $\text{mes}\, D_n > 0$. Putting $x_n(t) := (\text{mes}\, D_n)^{-1/p}\chi_n(t)$, where χ_n denotes the characteristic function of D_n, we see that $\|x_n\| = 1$ and $\|\lambda x_n - Lx_n\| \le 1/n$, and so $\lambda \in \sigma_q(L)$, by (1.52). Of course, for proving the equality $\sigma_q(L) = R_e(a)$ in (1.62) for $p = \infty$, we just use χ_n directly without multiplying by $(\text{mes}\, D_n)^{-1/p}$.

The essential spectrum in Kato's sense has been introduced and studied in [158], the essential spectrum in Wolf's sense in [277], the essential spectrum in Schechter's sense in [235], and the essential spectrum in Browder's sense in [49]. For obvious reasons, Wolf's essential spectrum (1.67) is also called *Fredholm spectrum* by some authors. Observe that every operator $L \in \mathfrak{L}(X, Y)$ induces an element $\tilde{L}$ in the Calkin algebra $\mathfrak{L}(X, Y)/\mathfrak{K}\mathfrak{L}(X, Y)$ by means of the canonical definition $\hat{L} := L + \mathfrak{K}\mathfrak{L}(X, Y)$. In this terminology, the essential norm (1.76) of L is then nothing else but the norm of $\hat{L}$ in $\mathfrak{L}(X, Y)/\mathfrak{K}\mathfrak{L}(X, Y)$. An important criterion, called *Atkinson's theorem* [28], states that an operator $L \in \mathfrak{L}(X, Y)$ is Fredholm if and only if the induced operator $\hat{L}$ is *invertible* in $\mathfrak{L}(X, Y)/\mathfrak{K}\mathfrak{L}(X, Y)$. Consequently, we may consider Wolf's essential spectrum $\sigma_{ew}(L)$ as the full spectrum $\sigma(\hat{L})$ of $\hat{L}$ in the Calkin algebra $\mathfrak{L}(X)/\mathfrak{K}\mathfrak{L}(X)$. This implies, in particular, that the essential spectrum $\sigma_{ew}(L)$ (and hence also the essential spectra $\sigma_{es}(L)$ and $\sigma_{eb}(L)$, by (1.69) and (1.72) are always nonempty in case $\mathbb{K} = \mathbb{C}$.

Identifying the spectra $\sigma_{ew}(L)$ and $\sigma(\hat{L})$ allows us to prove the last argument in Proposition 1.5 (c) rather quickly. Indeed, if λ is an isolated point of $\sigma(L)$ with infinite multiplicity $n(\lambda; L)$, then the corresponding spectral projection (1.71) cannot be compact. Consequently,

$$\frac{1}{2\pi i}\int_{\Gamma(\lambda)} R(z; \hat{L})\, dz = \hat{P}_\lambda \ne \Theta,$$

where $\Gamma(\lambda)$ surrounds λ, and so $\lambda \in \sigma_{ew}(L)$.

Interestingly, the radii (1.75) of the various essential spectra do not only coincide, but also satisfy a Gel'fand-type formula (1.9) with the norm $\|L\|$ replaced by the α-

norm (1.26). In fact, if X is a complex Banach space and $L \in \mathfrak{L}(X)$, then the chain of equalities

$$r_{\mathrm{ek}}(L) = r_{\mathrm{ew}}(L) = r_{\mathrm{es}}(L) = r_{\mathrm{eb}}(L) = \inf_n \sqrt[n]{[L^n]_{\mathrm{A}}} = \lim_{n\to\infty} \sqrt[n]{[L^n]_{\mathrm{A}}} \qquad (1.78)$$

is true; for a short proof see [73]. If X is a real Banach space one has to pass to the so-called complexification $X_{\mathbb{C}}$ of X which consists, by definition, of all ordered pairs $(x, y) \in X \times X$, written usually $x + iy$. The set $X_{\mathbb{C}}$ is equipped with the algebraic vector space operations

$$(x + iy) + (u + iv) := (x + u) + i(y + v)$$

and

$$(\lambda + i\mu)(x + iy) := (\lambda x - \mu y) + i(\mu x + \lambda y).$$

A natural norm on $X_{\mathbb{C}}$ is the so-called projective tensor norm defined by

$$\|x + iy\| := \inf \sum_{k=1}^{n} |\lambda_k| \, \|z_k\|,$$

where the infimum is taken over all possible representations of the form $x + iy = \lambda_1 z_1 + \cdots + \lambda_n z_n$ with $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and $z_1, \ldots, z_n \in X$. Equivalently [162], one may define a norm on $X_{\mathbb{C}}$ by

$$\|x + iy\| := \max_{0\le t\le 2\pi} \|(\sin t)x + (\cos t)y\|.$$

Given a real Banach space X and an operator $L \in \mathfrak{L}(X)$, one may extend L to an operator $L_{\mathbb{C}} \in \mathfrak{L}(X_{\mathbb{C}})$ putting

$$L(x + iy) := Lx + iLy.$$

It is readily seen that then

$$\|L_{\mathbb{C}}\| = \|L\|, \quad [L_{\mathbb{C}}]_{\mathrm{A}} = [L]_{\mathrm{A}}, \quad [L_{\mathbb{C}}]_{\mathrm{a}} = [L]_{\mathrm{a}},$$

and so L and $L_{\mathbb{C}}$ have the same essential spectral radius, by (1.78).

It is well known that, although the spectral radius (1.9) of a bounded linear operator L may be strictly less than its norm, for any $\varepsilon > 0$ one may always find a norm $\|\cdot\|_\varepsilon$ which is equivalent to the original norm $\|\cdot\|$ on X and such that $r(L) \le \|L\|_\varepsilon \le r(L) + \varepsilon$, where $\|L\|_\varepsilon$ denotes the operator norm of L in $(X, \|\cdot\|_\varepsilon)$. There is an analogous result for the essential spectral radius (1.74) called *Leggett's theorem* (see [171, Theorem 1]) which states that for any $\varepsilon > 0$ one may always find a norm $\|\cdot\|_\varepsilon$ which is equivalent to the projective tensor norm $\|\cdot\|$ on $X_{\mathbb{C}}$ and such that $r_\kappa(L) \le [L_{\mathbb{C}}]_{A,\varepsilon} \le r_\kappa(L) + \varepsilon$, where $[L_{\mathbb{C}}]_{A,\varepsilon}$ denotes the α-norm (1.26) of $L_{\mathbb{C}}$ in $(X_{\mathbb{C}}, \|\cdot\|_\varepsilon)$, and $r_\kappa(L)$ is given by (1.74).

There is another characteristic for bounded linear operators which is of some interest in spectral theory. Given $L \in \mathfrak{L}(X)$, the number

$$[\![L]\!] := \begin{cases} \|L^{-1}\|^{-1} & \text{if } L \text{ is bijective,} \\ 0 & \text{otherwise} \end{cases} \tag{1.79}$$

is called *inner norm* of L (see [1]). This number is closely related to the *inner spectral radius*

$$r_i(L) := \inf\{|\lambda| : \lambda \in \sigma(L)\}, \tag{1.80}$$

inasmuch as the formula

$$r_i(L) = \lim_{n\to\infty} \sqrt[n]{[\![L^n]\!]} \tag{1.81}$$

holds true, which is of course the analogue to (1.9). In fact, for any bijection $L \in \mathfrak{L}(X)$ we have

$$\begin{aligned} r_i(L) &= \inf\{|\lambda| : \lambda \in \sigma(L)\} \\ &= (\sup\{|\lambda|^{-1} : \lambda \in \sigma(L)\})^{-1} \\ &= (\sup\{|\mu| : \mu \in \sigma(L^{-1})\})^{-1} \\ &= \frac{1}{r(L^{-1})}. \end{aligned}$$

and therefore

$$r_i(L) = r(L^{-1})^{-1} = \big(\lim_{n\to\infty} \sqrt[n]{\|L^{-n}\|}\big)^{-1} = \lim_{n\to\infty} \sqrt[n]{\|L^{-n}\|^{-1}} = \lim_{n\to\infty} \sqrt[n]{[\![L^n]\!]}.$$

If L is not bijective, then L^n is not bijective either for any n, and both sides in (1.81) are zero.

The proof of Proposition 1.5 (b) may be found in [192], that of Proposition 1.5 (c) in [149]. Theorem 1.4 may be found in [149]–[151], in the proof of Theorem 1.5 we followed [73]. We remark that one may prove some kind of polynomial spectral mapping theorem (see Theorem 1.1 (h)) also for the various subspectra we considered in this chapter. Indeed, in [150] it is shown that

$$\sigma_{\mathrm{p}}(p(L)) = p(\sigma_{\mathrm{p}}(L)), \quad \sigma_\pi(p(L)) = p(\sigma_\pi(L)), \quad \sigma_{\mathrm{eb}}(p(L)) = p(\sigma_{\mathrm{eb}}(L)),$$
$$\sigma_{\mathrm{ek}}(p(L)) = p(\sigma_{\mathrm{ek}}(L)), \quad \sigma_{\mathrm{es}}(p(L)) = p(\sigma_{\mathrm{es}}(L)), \quad \sigma_{\mathrm{ew}}(p(L)) = p(\sigma_{\mathrm{ew}}(L)),$$
$$\sigma_+(p(L)) = p(\sigma_+(L)), \quad \sigma_-(p(L)) = p(\sigma_-(L))$$

for any polynomial function $p\colon \mathbb{C} \to \mathbb{C}$.

The following example which shows how essential spectra naturally arise in the study of integral equations on unbounded intervals is taken from [73] (see also [68]). Denote by $L_1(\mathbb{R}, \mathbb{C})$ the Banach space of (classes of) measurable functions $x : \mathbb{R} \to \mathbb{C}$ with the usual norm

$$\|x\| = \int_{-\infty}^{+\infty} |x(t)|\, dt.$$

Given $k, y \in L_1(\mathbb{R}, \mathbb{C})$, consider the *Wiener–Hopf equation*

$$\lambda x(s) - \int_{-\infty}^{+\infty} k(s-t)x(t)\,dt = y(s) \quad (-\infty < s < \infty), \tag{1.82}$$

where $\lambda \in \mathbb{C}$ is fixed. We may rewrite (1.82) as operator equation $\lambda x - Lx = y$, where

$$Lx(s) := \int_{-\infty}^{+\infty} k(s-t)x(t)\,dt. \tag{1.83}$$

Applying the Fourier transform, i.e.

$$\hat{x}(\sigma) = \int_{-\infty}^{+\infty} x(s)e^{is\sigma}\,ds,$$

to both sides of (1.82) yields

$$\lambda\hat{x}(\sigma) - \hat{k}(\sigma)\hat{x}(\sigma) = \hat{y}(\sigma) \quad (-\infty < \sigma < \infty), \tag{1.84}$$

since the Fourier transform maps convolutions into pointwise products. Now, from the continuity of $\hat{k}$ and the fact that

$$\hat{k}(\pm\infty) := \lim_{\sigma\to\pm\infty} \hat{k}(\sigma) = 0$$

it follows that $\Gamma_\lambda := \{\lambda - \hat{k}(\sigma) : -\infty \le \sigma \le +\infty\}$ is a closed curve in the complex plane. Clearly, in case $\hat{k}(\sigma) \neq \lambda$ equation (1.84) may be solved to get $\hat{x}$ and, subsequently, x, provided we know the antitransform of $\hat{y}(\sigma)/(\lambda - \hat{k}(\sigma))$. In fact, it may be shown that $\lambda I - L$ is a Fredholm operator on $L_1(\mathbb{R}, \mathbb{C})$ if and only if $\hat{k}(\sigma) \neq \lambda$ for all $\sigma \in [-\infty, +\infty]$. For the index we get then the nice formula

$$\operatorname{ind}(\lambda I - L) = -w(\Gamma_\lambda, 0)$$

involving the *winding number* $w(\Gamma_\lambda, 0)$ of Γ_λ around the origin. Moreover, one may show that

$$\sigma_{\text{ew}}(L) = \{\hat{k}(\sigma) : -\infty \le \sigma \le +\infty\},$$
$$r_{\text{ew}}(L) = \max\{|\hat{k}(\sigma)| : -\infty \le \sigma \le +\infty\},$$

and

$$\sigma(L) = \sigma_{\text{es}}(L) = \sigma_{\text{ew}}(L) \cup \{\lambda : w(\Gamma_\lambda, 0) \neq 0\}.$$

For the special example $k(s) = e^{-|s|}$, say, we have $\hat{k}(\sigma) = 2/(\sigma^2 + 1)$ and thus

$$\sigma(L) = \sigma_{\text{ew}}(L) = \sigma_{\text{eb}}(L) = [0, 2],$$

since Γ_λ does not wind around the origin for $\lambda \notin [0, 2]$.

An interesting completely different approach to spectra and essential spectra of linear operators in Hilbert spaces was proposed recently by Stuart [244]. Let H be a real Hilbert space with scalar product $\langle \cdot, \cdot \rangle$, and let $L \in \mathfrak{L}(H)$ be a selfadjoint operator, so $\sigma(L) \subset \mathbb{R}$. Denote by H_1 the domain of the operator $|L|^{1/2}$, equipped with the natural scalar product

$$\langle x, y\rangle_1 = \langle x, y\rangle + \langle |L|^{1/2}x, |L|^{1/2}y\rangle.$$

Then there exist unique bounded linear operators A and B such that $\langle Ax, y\rangle_1 = \langle Lx, y\rangle$ for all $x \in H$ and $y \in H_1$, and $\langle Bx, y\rangle_1 = \langle x, y\rangle$ for all $x, y \in H_1$. The main result in [244] states that the resolvent set $\rho(L)$ of L consists precisely of all $\lambda \in \mathbb{R}$ such that the (Gâteaux differentiable) quadratic functional $J_\lambda\colon H_1 \to \mathbb{R}$ defined by

$$J_\lambda(x) := \langle (A - \lambda B)x, x\rangle_1 \quad (x \in H_1)$$

satisfies a *Palais–Smale condition*, which means that every sequence $(x_n)_n$ in H_1 such that $(J_\lambda(x_n))_n$ is bounded and $\|J_\lambda'(x_n)\| \to 0$ contains a convergent subsequence.

Chapter 2

Some Characteristics of Nonlinear Operators

In this chapter we introduce and study some numerical characteristics for continuous nonlinear operators F between two Banach spaces X and Y. Such characteristics describe and "quantify" certain mapping properties of F, such as compactness, Lipschitz continuity, or quasiboundedness. All these characteristics are necessary to define the various nonlinear spectra in subsequent chapters; we request the reader's indulgence until then.

2.1 Some metric characteristics of nonlinear operators

Throughout this chapter X and Y are two Banach spaces, and $F: X \to Y$ is a continuous (in general, nonlinear) operator. The set $\mathfrak{C}(X, Y)$ of all continuous operators from X into Y is of course a linear space; moreover, $\mathfrak{C}(X) := \mathfrak{C}(X, X)$ is an algebra with respect to composition. Given $F \in \mathfrak{C}(X, Y)$, we put

$$[F]_{\mathrm{Lip}} = \sup_{x \neq y} \frac{\|F(x) - F(y)\|}{\|x - y\|} \tag{2.1}$$

and

$$[F]_{\mathrm{lip}} = \inf_{x \neq y} \frac{\|F(x) - F(y)\|}{\|x - y\|}, \tag{2.2}$$

and write $F \in \mathfrak{Lip}(X, Y)$ if $[F]_{\mathrm{Lip}} < \infty$. In case $X = Y$ we simply write $\mathfrak{Lip}(X)$ instead of $\mathfrak{Lip}(X, X)$. Note that (2.1) is a seminorm on the space $\mathfrak{Lip}(X, Y)$, and $[F]_{\mathrm{Lip}} = 0$ means that F is constant. On the subspace $\mathfrak{Lip}_0(X, Y)$ of all $F \in \mathfrak{Lip}(X, Y)$ satisfying $F(\theta) = \theta$, where θ denotes the zero vector, (2.1) is even a norm.

In the following Proposition 2.1 we recall some useful properties of the characteristics (2.1) and (2.2). Recall that an operator F is called *closed* if it maps closed sets onto closed sets.

Proposition 2.1. *The characteristics* (2.1) *and* (2.2) *have the following properties* ($F, G \in \mathfrak{C}(X, Y)$):

(a) $[F]_{\mathrm{lip}} > 0$ *implies that F is injective and closed.*

(b) $[G]_{\mathrm{lip}}\,[F]_{\mathrm{lip}} \le [GF]_{\mathrm{Lip}} \le [G]_{\mathrm{lip}}\,[F]_{\mathrm{Lip}}$.

(c) $[F]_{\mathrm{lip}} - [G]_{\mathrm{Lip}} \leq [F+G]_{\mathrm{lip}} \leq [F]_{\mathrm{lip}} + [G]_{\mathrm{Lip}}$.

(d) $|[F]_{\mathrm{lip}} - [G]_{\mathrm{lip}}| \leq [F-G]_{\mathrm{Lip}}$*; in particular,* $[F-G]_{\mathrm{Lip}} = 0$ *implies* $[F]_{\mathrm{lip}} = [G]_{\mathrm{lip}}$.

(e) $[F^{-1}]_{\mathrm{Lip}} = [F]_{\mathrm{lip}}^{-1}$ *if* F *is a homeomorphism.*

Proof. The condition $[F]_{\mathrm{lip}} > 0$ means that we may find $k > 0$ such that $\|F(x) - F(y)\| \geq k\|x - y\|$ for all $x, y \in X$. So $F(x) = F(y)$ implies $x = y$, i.e. the injectivity of F. To see that F is closed, let $M \subseteq X$ be a closed set, and let $(y_n)_n$ be a sequence in $F(M)$ which converges to $y \in Y$. Choose a sequence $(x_n)_n$ in M such that $F(x_n) = y_n$. With $k > 0$ as before we get then $\|x_m - x_n\| \leq k\|y_m - y_n\|$ which implies that $(x_n)_n$ is a Cauchy sequence. Denoting by x the limit of this sequence, the continuity of F implies that $F(x) = y$, and so $y \in F(M)$. The proof of (a) is complete.

The properties (b)–(d) are straightforward consequences of calculation rules for suprema and infima. Finally, (e) follows from the chain of equalities

$$
\begin{aligned}
[F^{-1}]_{\mathrm{Lip}} &= \sup_{u \neq v} \frac{\|F^{-1}(u) - F^{-1}(v)\|}{\|u - v\|} = \sup_{x \neq y} \frac{\|x - y\|}{\|F(x) - F(y)\|} \\
&= \left(\inf_{x \neq y} \frac{\|F(x) - F(y)\|}{\|x - y\|} \right)^{-1} = \frac{1}{[F]_{\mathrm{lip}}},
\end{aligned}
$$

and so Proposition 2.1 is proved. □

Given $F \in \mathfrak{C}(X, Y)$, consider now the two characteristics

$$
[F]_{\mathrm{Q}} = \limsup_{\|x\| \to \infty} \frac{\|F(x)\|}{\|x\|} \tag{2.3}
$$

and

$$
[F]_{\mathrm{q}} = \liminf_{\|x\| \to \infty} \frac{\|F(x)\|}{\|x\|}. \tag{2.4}
$$

In case $[F]_{\mathrm{Q}} < \infty$ we write $F \in \mathfrak{Q}(X, Y)$ and call F *quasibounded*. Again, for $X = Y$ we simply write $\mathfrak{Q}(X)$ instead of $\mathfrak{Q}(X, X)$. Also the characteristic (2.3) is a seminorm on the space $\mathfrak{Q}(X, Y)$; here $[F]_{\mathrm{Q}} = 0$ means that F has *strictly sublinear growth* on large spheres, i.e. $\|F(x)\| = o(\|x\|)$ as $\|x\| \to \infty$.

Proposition 2.2. *The characteristics* (2.3) *and* (2.4) *have the following properties* $(F, G \in \mathfrak{C}(X, Y))$:

(a) $[F]_{\mathrm{q}} > 0$ *implies that* F *is coercive, i.e.*

$$
\lim_{\|x\| \to \infty} \|F(x)\| = \infty. \tag{2.5}
$$

(b) $[G]_{\mathrm{q}}\,[F]_{\mathrm{q}} \le [GF]_{\mathrm{Q}} \le [G]_{\mathrm{q}}\,[F]_{\mathrm{Q}}$.

(c) $[F]_{\mathrm{q}} - [G]_{\mathrm{Q}} \le [F+G]_{\mathrm{q}} \le [F]_{\mathrm{q}} + [G]_{\mathrm{Q}}$.

(d) $|[F]_{\mathrm{q}} - [G]_{\mathrm{q}}| \le [F-G]_{\mathrm{Q}}$*; in particular,* $[F-G]_{\mathrm{Q}} = 0$ *implies* $[F]_{\mathrm{q}} = [G]_{\mathrm{q}}$.

(e) $[F^{-1}]_{\mathrm{Q}} = [F]_{\mathrm{q}}^{-1}$ *if* F *is a homeomorphism.*

Proof. First of all, from $[F]_{\mathrm{q}} > 0$ it follows that there exists $k > 0$ such that $\|F(x)\| \ge k\|x\|$ for $\|x\|$ sufficiently large, and hence (2.5) holds. The properties (b)–(d) are straightforward consequences of calculation rules for limsup and liminf. Finally, (e) follows from the chain of equalities

$$[F^{-1}]_{\mathrm{Q}} = \limsup_{\|y\|\to\infty} \frac{\|F^{-1}(y)\|}{\|y\|} = \limsup_{\|x\|\to\infty} \frac{\|x\|}{\|F(x)\|} = \left(\liminf_{\|x\|\to\infty} \frac{\|F(x)\|}{\|x\|}\right)^{-1} = \frac{1}{[F]_{\mathrm{q}}},$$

and so we are done. □

Finally, let us introduce the characteristics

$$[F]_{\mathrm{B}} = \sup_{x\ne\theta} \frac{\|F(x)\|}{\|x\|} \tag{2.6}$$

and

$$[F]_{\mathrm{b}} = \inf_{x\ne\theta} \frac{\|F(x)\|}{\|x\|}. \tag{2.7}$$

In case $[F]_{\mathrm{B}} < \infty$ we write $F \in \mathfrak{B}(X,Y)$ and call F *linearly bounded*. As before, for $X = Y$ we simply write $\mathfrak{B}(X,X) =: \mathfrak{B}(X)$. The characteristic (2.6) is of course a norm on the space $\mathfrak{B}(X,Y)$, since $[F]_{\mathrm{B}} = 0$ implies $F = \Theta$, the zero operator.

Proposition 2.3. *The characteristics* (2.6) *and* (2.7) *have the following properties* $(F, G \in \mathfrak{C}(X,Y))$:

(a) $[F]_{\mathrm{b}} > 0$ implies that F is bounded away from zero.

(b) $[G]_{\mathrm{b}}\,[F]_{\mathrm{b}} \le [GF]_{\mathrm{B}} \le [G]_{\mathrm{b}}\,[F]_{\mathrm{B}}$.

(c) $[F]_{\mathrm{b}} - [G]_{\mathrm{B}} \le [F+G]_{\mathrm{b}} \le [F]_{\mathrm{b}} + [G]_{\mathrm{B}}$.

(d) $|[F]_{\mathrm{b}} - [G]_{\mathrm{b}}| \le [F-G]_{\mathrm{B}}$*; in particular,* $[F-G]_{\mathrm{B}} = 0$ *implies* $[F]_{\mathrm{b}} = [G]_{\mathrm{b}}$.

(e) $[F^{-1}]_{\mathrm{B}} = [F]_{\mathrm{b}}^{-1}$ *if* F *is a homeomorphism.*

Proof. The proof is very similar to that of the preceding Proposition 2.2. From $[F]_{\mathrm{b}} > 0$ it follows that there exists $k > 0$ such that $\|F(x)\| \ge k\|x\|$ for all $x \in X$, and this precisely means that F is bounded away from zero. The properties (b)–(d) are again straightforward consequences of calculation rules for suprema and infima.

Finally, (e) follows from the fact that $F(x) = \theta$ if and only if $x = \theta$, and from

$$[F^{-1}]_{\mathrm{B}} = \sup_{y \neq \theta} \frac{\|F^{-1}(y)\|}{\|y\|} = \sup_{x \neq \theta} \frac{\|x\|}{\|F(x)\|} = \left(\inf_{x \neq \theta} \frac{\|F(x)\|}{\|x\|} \right)^{-1} = \frac{1}{[F]_{\mathrm{b}}},$$

and so Proposition 2.3 is proved. □

For a linear operator $L \in \mathfrak{L}(X, Y)$ we have of course $[L]_{\mathrm{B}} = \|L\|$, by the definition of the norm. In particular, if L is a linear isomorphism, then Proposition 2.3 (e) implies that

$$[L]_{\mathrm{b}} = [L^{-1}]_{\mathrm{B}}^{-1} = \|L^{-1}\|^{-1} = \left\bracevert L \right\bracevert,$$

where $\left\bracevert L \right\bracevert$ denotes the inner norm (1.79) of L. It is also useful to mention some relations between the above characteristics; we omit the (trivial) proof of the following lemma.

Lemma 2.1. *The characteristics* (2.1)–(2.4) *and* (2.6)–(2.7) *are related by the inequalities*

$$[F]_{\mathrm{b}} \leq [F]_{\mathrm{q}} \leq [F]_{\mathrm{Q}} \leq [F]_{\mathrm{B}}. \tag{2.8}$$

Moreover, in case $F(\theta) = \theta$ *we have in addition*

$$[F]_{\mathrm{lip}} \leq [F]_{\mathrm{b}}, \quad [F]_{\mathrm{B}} \leq [F]_{\mathrm{Lip}}. \tag{2.9}$$

From Lemma 2.1 it follows, in particular, that every linearly bounded operator is quasibounded, and that every Lipschitz continuous operator is linearly bounded if it leaves zero fixed. These trivial connections may of course be verified directly.

2.2 A list of examples

Combining (2.8) and (2.9) in case $F(\theta) = \theta$, we get the chain of inequalities

$$[F]_{\mathrm{lip}} \leq [F]_{\mathrm{b}} \leq [F]_{\mathrm{q}} \leq [F]_{\mathrm{Q}} \leq [F]_{\mathrm{B}} \leq [F]_{\mathrm{Lip}}. \tag{2.10}$$

There are in principle 32 possibilities for equality and strict inequality in (2.10) which, surprisingly, may all indeed occur. We collect the possible combinations in the following table and then illustrate them by a series of 32 examples.

Table 2.1

$[F]_{\mathrm{lip}} = [F]_{\mathrm{b}} = [F]_{\mathrm{q}} = [F]_{\mathrm{Q}} = [F]_{\mathrm{B}} = [F]_{\mathrm{Lip}}$	Example 2.1
$[F]_{\mathrm{lip}} < [F]_{\mathrm{b}} = [F]_{\mathrm{q}} = [F]_{\mathrm{Q}} = [F]_{\mathrm{B}} = [F]_{\mathrm{Lip}}$	Example 2.2
$[F]_{\mathrm{lip}} = [F]_{\mathrm{b}} < [F]_{\mathrm{q}} = [F]_{\mathrm{Q}} = [F]_{\mathrm{B}} = [F]_{\mathrm{Lip}}$	Example 2.3
$[F]_{\mathrm{lip}} = [F]_{\mathrm{b}} = [F]_{\mathrm{q}} < [F]_{\mathrm{Q}} = [F]_{\mathrm{B}} = [F]_{\mathrm{Lip}}$	Example 2.4
$[F]_{\mathrm{lip}} = [F]_{\mathrm{b}} = [F]_{\mathrm{q}} = [F]_{\mathrm{Q}} < [F]_{\mathrm{B}} = [F]_{\mathrm{Lip}}$	Example 2.5
$[F]_{\mathrm{lip}} = [F]_{\mathrm{b}} = [F]_{\mathrm{q}} = [F]_{\mathrm{Q}} = [F]_{\mathrm{B}} < [F]_{\mathrm{Lip}}$	Example 2.6
$[F]_{\mathrm{lip}} < [F]_{\mathrm{b}} < [F]_{\mathrm{q}} = [F]_{\mathrm{Q}} = [F]_{\mathrm{B}} = [F]_{\mathrm{Lip}}$	Example 2.7
$[F]_{\mathrm{lip}} < [F]_{\mathrm{b}} = [F]_{\mathrm{q}} < [F]_{\mathrm{Q}} = [F]_{\mathrm{B}} = [F]_{\mathrm{Lip}}$	Example 2.8
$[F]_{\mathrm{lip}} < [F]_{\mathrm{b}} = [F]_{\mathrm{q}} = [F]_{\mathrm{Q}} < [F]_{\mathrm{B}} = [F]_{\mathrm{Lip}}$	Example 2.9
$[F]_{\mathrm{lip}} < [F]_{\mathrm{b}} = [F]_{\mathrm{q}} = [F]_{\mathrm{Q}} = [F]_{\mathrm{B}} < [F]_{\mathrm{Lip}}$	Example 2.10
$[F]_{\mathrm{lip}} = [F]_{\mathrm{b}} < [F]_{\mathrm{q}} < [F]_{\mathrm{Q}} = [F]_{\mathrm{B}} = [F]_{\mathrm{Lip}}$	Example 2.11
$[F]_{\mathrm{lip}} = [F]_{\mathrm{b}} < [F]_{\mathrm{q}} = [F]_{\mathrm{Q}} < [F]_{\mathrm{B}} = [F]_{\mathrm{Lip}}$	Example 2.12
$[F]_{\mathrm{lip}} = [F]_{\mathrm{b}} < [F]_{\mathrm{q}} = [F]_{\mathrm{Q}} = [F]_{\mathrm{B}} < [F]_{\mathrm{Lip}}$	Example 2.13
$[F]_{\mathrm{lip}} = [F]_{\mathrm{b}} = [F]_{\mathrm{q}} < [F]_{\mathrm{Q}} < [F]_{\mathrm{B}} = [F]_{\mathrm{Lip}}$	Example 2.14
$[F]_{\mathrm{lip}} = [F]_{\mathrm{b}} = [F]_{\mathrm{q}} < [F]_{\mathrm{Q}} = [F]_{\mathrm{B}} < [F]_{\mathrm{Lip}}$	Example 2.15
$[F]_{\mathrm{lip}} = [F]_{\mathrm{b}} = [F]_{\mathrm{q}} = [F]_{\mathrm{Q}} < [F]_{\mathrm{B}} < [F]_{\mathrm{Lip}}$	Example 2.16
$[F]_{\mathrm{lip}} = [F]_{\mathrm{b}} = [F]_{\mathrm{q}} < [F]_{\mathrm{Q}} < [F]_{\mathrm{B}} < [F]_{\mathrm{Lip}}$	Example 2.17
$[F]_{\mathrm{lip}} = [F]_{\mathrm{b}} < [F]_{\mathrm{q}} = [F]_{\mathrm{Q}} < [F]_{\mathrm{B}} < [F]_{\mathrm{Lip}}$	Example 2.18
$[F]_{\mathrm{lip}} = [F]_{\mathrm{b}} < [F]_{\mathrm{q}} < [F]_{\mathrm{Q}} = [F]_{\mathrm{B}} < [F]_{\mathrm{Lip}}$	Example 2.19
$[F]_{\mathrm{lip}} = [F]_{\mathrm{b}} < [F]_{\mathrm{q}} < [F]_{\mathrm{Q}} < [F]_{\mathrm{B}} = [F]_{\mathrm{Lip}}$	Example 2.20
$[F]_{\mathrm{lip}} < [F]_{\mathrm{b}} = [F]_{\mathrm{q}} = [F]_{\mathrm{Q}} < [F]_{\mathrm{B}} < [F]_{\mathrm{Lip}}$	Example 2.21
$[F]_{\mathrm{lip}} < [F]_{\mathrm{b}} = [F]_{\mathrm{q}} < [F]_{\mathrm{Q}} = [F]_{\mathrm{B}} < [F]_{\mathrm{Lip}}$	Example 2.22
$[F]_{\mathrm{lip}} < [F]_{\mathrm{b}} = [F]_{\mathrm{q}} < [F]_{\mathrm{Q}} < [F]_{\mathrm{B}} = [F]_{\mathrm{Lip}}$	Example 2.23
$[F]_{\mathrm{lip}} < [F]_{\mathrm{b}} < [F]_{\mathrm{q}} = [F]_{\mathrm{Q}} = [F]_{\mathrm{B}} < [F]_{\mathrm{Lip}}$	Example 2.24
$[F]_{\mathrm{lip}} < [F]_{\mathrm{b}} < [F]_{\mathrm{q}} = [F]_{\mathrm{Q}} < [F]_{\mathrm{B}} = [F]_{\mathrm{Lip}}$	Example 2.25
$[F]_{\mathrm{lip}} < [F]_{\mathrm{b}} < [F]_{\mathrm{q}} < [F]_{\mathrm{Q}} = [F]_{\mathrm{B}} = [F]_{\mathrm{Lip}}$	Example 2.26
$[F]_{\mathrm{lip}} < [F]_{\mathrm{b}} < [F]_{\mathrm{q}} < [F]_{\mathrm{Q}} < [F]_{\mathrm{B}} = [F]_{\mathrm{Lip}}$	Example 2.27
$[F]_{\mathrm{lip}} < [F]_{\mathrm{b}} < [F]_{\mathrm{q}} < [F]_{\mathrm{Q}} = [F]_{\mathrm{B}} < [F]_{\mathrm{Lip}}$	Example 2.28
$[F]_{\mathrm{lip}} < [F]_{\mathrm{b}} < [F]_{\mathrm{q}} = [F]_{\mathrm{Q}} < [F]_{\mathrm{B}} < [F]_{\mathrm{Lip}}$	Example 2.29
$[F]_{\mathrm{lip}} < [F]_{\mathrm{b}} = [F]_{\mathrm{q}} < [F]_{\mathrm{Q}} < [F]_{\mathrm{B}} < [F]_{\mathrm{Lip}}$	Example 2.30
$[F]_{\mathrm{lip}} = [F]_{\mathrm{b}} < [F]_{\mathrm{q}} < [F]_{\mathrm{Q}} < [F]_{\mathrm{B}} < [F]_{\mathrm{Lip}}$	Example 2.31
$[F]_{\mathrm{lip}} < [F]_{\mathrm{b}} < [F]_{\mathrm{q}} < [F]_{\mathrm{Q}} < [F]_{\mathrm{B}} < [F]_{\mathrm{Lip}}$	Example 2.32

Some of the following examples may be given in the simplest case $X = Y = \mathbb{R}$. Alternatively, we may often choose $X = Y$ as arbitrary Banach space and $F\colon X \to X$

in the form

$$F(x) = \varphi(\|x\|)e, \tag{2.11}$$

where $\varphi\colon [0,\infty) \to [0,\infty)$ is a scalar function (usually piecewise linear), and $e \in X$ is a fixed element with $\|e\| = 1$.

Example 2.1. Let F be an isometry between two Banach spaces X and Y satisfying $F(\theta) = \theta$. Then all characteristics are equal to 1. ♡

Example 2.2. Let $X = Y$ and F be as in (2.11) with $\varphi(t) = t$. Then

$$[F]_{\mathrm{lip}} = 0, \quad [F]_{\mathrm{b}} = [F]_{\mathrm{q}} = [F]_{\mathrm{Q}} = [F]_{\mathrm{B}} = [F]_{\mathrm{Lip}} = 1. \qquad \heartsuit$$

Example 2.3. Let $X = Y$ and F be as in (2.11) with

$$\varphi(t) = \begin{cases} 0 & \text{if } t \le 1, \\ t-1 & \text{if } t \ge 1. \end{cases}$$

Then

$$[F]_{\mathrm{lip}} = [F]_{\mathrm{b}} = 0, \quad [F]_{\mathrm{q}} = [F]_{\mathrm{Q}} = [F]_{\mathrm{B}} = [F]_{\mathrm{Lip}} = 1. \qquad \heartsuit$$

Example 2.4. Let $X = l_\infty$ and let F be the (linear) left shift operator $F(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots)$. Then

$$[F]_{\mathrm{lip}} = [F]_{\mathrm{b}} = [F]_{\mathrm{q}} = 0, \quad [F]_{\mathrm{Q}} = [F]_{\mathrm{B}} = [F]_{\mathrm{Lip}} = 1. \qquad \heartsuit$$

Example 2.5. Let $X = Y$ and F be as in (2.11) with

$$\varphi(t) = \begin{cases} t & \text{if } t \le 1, \\ 1 & \text{if } t \ge 1. \end{cases}$$

Then

$$[F]_{\mathrm{lip}} = [F]_{\mathrm{b}} = [F]_{\mathrm{q}} = [F]_{\mathrm{Q}} = 0, \quad [F]_{\mathrm{B}} = [F]_{\mathrm{Lip}} = 1. \qquad \heartsuit$$

Example 2.6. Let $X = \mathbb{R}$, $Y = \mathbb{C}$, and $F(t) = te^{\phi(t)}$, where $\phi(t) = \frac{\pi}{2} i \, \max\{|t|, 1\}$. Then

$$[F]_{\mathrm{lip}} = [F]_{\mathrm{b}} = [F]_{\mathrm{q}} = [F]_{\mathrm{Q}} = [F]_{\mathrm{B}} = 1, \quad [F]_{\mathrm{Lip}} = \sqrt{1 + \pi^2/4}. \qquad \heartsuit$$

Example 2.7. Let $X = Y$ and F be as in (2.11) with

$$\varphi(t) = \begin{cases} t & \text{if } t \le 1, \\ 1 & \text{if } 1 \le t \le 2, \\ t-1 & \text{if } t \ge 2. \end{cases}$$

Then

$$[F]_{\mathrm{lip}} = 0, \quad [F]_{\mathrm{b}} = \tfrac{1}{2}, \quad [F]_{\mathrm{q}} = [F]_{\mathrm{Q}} = [F]_{\mathrm{B}} = [F]_{\mathrm{Lip}} = 1. \qquad \heartsuit$$

Example 2.8. Let $X = Y = \mathbb{R}$ and

$$F(x) = \begin{cases} -\frac{1}{2}x & \text{if } x \le -2, \\ 1 & \text{if } -2 \le x \le -1, \\ -x & \text{if } -1 \le x \le 0, \\ x & \text{if } x \ge 0. \end{cases}$$

Then

$$[F]_{\mathrm{lip}} = 0, \quad [F]_{\mathrm{b}} = [F]_{\mathrm{q}} = \tfrac{1}{2}, \quad [F]_{\mathrm{Q}} = [F]_{\mathrm{B}} = [F]_{\mathrm{Lip}} = 1. \qquad \heartsuit$$

Example 2.9. Let $X = Y$ and F be as in (2.11) with

$$\varphi(t) = \begin{cases} t & \text{if } 0 \le t \le 1, \\ \frac{1}{2}t + \frac{1}{2} & \text{if } t \ge 1. \end{cases}$$

Then

$$[F]_{\mathrm{lip}} = 0, \quad [F]_{\mathrm{b}} = [F]_{\mathrm{q}} = [F]_{\mathrm{Q}} = \tfrac{1}{2}, \quad [F]_{\mathrm{B}} = [F]_{\mathrm{Lip}} = 1. \qquad \heartsuit$$

Example 2.10. Let $X = \mathbb{R}$, $Y = \mathbb{C}$, and let $F(t) = |t|e^{\phi(t)}$, where $\phi(t) = 2\pi i \max\{|t|, 1\}$. Then

$$[F]_{\mathrm{lip}} = 0, \quad [F]_{\mathrm{b}} = [F]_{\mathrm{q}} = [F]_{\mathrm{Q}} = [F]_{\mathrm{B}} = 1, \quad [F]_{\mathrm{Lip}} = \sqrt{1 + 4\pi^2}.$$

To see that $[F]_{\mathrm{Lip}} = 2\pi + 1$ it suffices to note that $|F'(t)| \to \sqrt{1 + 4\pi^2}$ as $t \uparrow 1$. $\heartsuit$

Example 2.11. Let $X = Y = \mathbb{R}$ and

$$F(x) = \begin{cases} -\frac{1}{2}x - \frac{1}{2} & \text{if } x \le -1, \\ 0 & \text{if } -1 \le x \le 0, \\ x & \text{if } x \ge 0. \end{cases}$$

Then

$$[F]_{\mathrm{lip}} = [F]_{\mathrm{b}} = 0, \quad [F]_{\mathrm{q}} = \tfrac{1}{2}, \quad [F]_{\mathrm{Q}} = [F]_{\mathrm{B}} = [F]_{\mathrm{Lip}} = 1. \qquad \heartsuit$$

Example 2.12. Let $X = Y = \mathbb{R}$ and

$$F(x) = \begin{cases} -\frac{1}{2}x + \frac{1}{2} & \text{if } x \le -1, \\ -x & \text{if } -1 \le x \le 0, \\ 0 & \text{if } 0 \le x \le 1, \\ x - 1 & \text{if } 1 \le x \le 2, \\ ]\frac{1}{2}x & \text{if } x \ge 2. \end{cases}$$

Then

$$[F]_{\mathrm{lip}} = [F]_{\mathrm{b}} = 0, \quad [F]_{\mathrm{q}} = [F]_{\mathrm{Q}} = \tfrac{1}{2}, \quad [F]_{\mathrm{B}} = [F]_{\mathrm{Lip}} = 1. \qquad \heartsuit$$

Example 2.13. Let $X = Y$ and F be as in (2.11) with

$$\varphi(t) = \begin{cases} 0 & \text{if } t \le 1, \\ t-1 & \text{if } 1 \le t \le 2, \\ \frac{1}{2}t & \text{if } t \ge 2. \end{cases}$$

Then

$$[F]_{\mathrm{lip}} = [F]_{\mathrm{b}} = 0, \quad [F]_{\mathrm{q}} = [F]_{\mathrm{Q}} = [F]_{\mathrm{B}} = \tfrac{1}{2}, \quad [F]_{\mathrm{Lip}} = 1. \qquad \heartsuit$$

Example 2.14. Let $X = Y = \mathbb{R}$ and

$$F(x) = \begin{cases} 0 & \text{if } x \le 0, \\ x & \text{if } 0 \le x \le 1, \\ \frac{1}{2}x + \frac{1}{2} & \text{if } x \ge 1. \end{cases}$$

Then

$$[F]_{\mathrm{lip}} = [F]_{\mathrm{b}} = [F]_{\mathrm{q}} = 0, \quad [F]_{\mathrm{Q}} = \tfrac{1}{2}, \quad [F]_{\mathrm{B}} = [F]_{\mathrm{Lip}} = 1. \qquad \heartsuit$$

Example 2.15. Let $X = Y = \mathbb{R}$ and

$$F(x) = \begin{cases} 0 & \text{if } x \le 1, \\ x-1 & \text{if } 1 \le x \le 2, \\ \frac{1}{2}x & \text{if } x \ge 2. \end{cases}$$

Then

$$[F]_{\mathrm{lip}} = [F]_{\mathrm{b}} = [F]_{\mathrm{q}} = 0, \quad [F]_{\mathrm{Q}} = [F]_{\mathrm{B}} = \tfrac{1}{2}, \quad [F]_{\mathrm{Lip}} = 1. \qquad \heartsuit$$

Example 2.16. Let $X = Y$ and F be as in (2.11) with

$$\varphi(t) = \begin{cases} 0 & \text{if } t \le 1, \\ t-1 & \text{if } 1 \le t \le 2, \\ 1 & \text{if } t \ge 2. \end{cases}$$

Then

$$[F]_{\mathrm{lip}} = [F]_{\mathrm{b}} = [F]_{\mathrm{q}} = [F]_{\mathrm{Q}} = 0, \quad [F]_{\mathrm{B}} = \tfrac{1}{2}, \quad [F]_{\mathrm{Lip}} = 1. \qquad \heartsuit$$

Example 2.17. Let $X = Y = \mathbb{R}$ and

$$F(x) = \begin{cases} 0 & \text{if } x \le 1, \\ x-1 & \text{if } 1 \le x \le 2, \\ \frac{1}{4}x + \frac{1}{2} & \text{if } x \ge 2. \end{cases}$$

Then

$$[F]_{\mathrm{lip}} = [F]_{\mathrm{b}} = [F]_{\mathrm{q}} = 0, \quad [F]_{\mathrm{Q}} = \tfrac{1}{4}, \quad [F]_{\mathrm{B}} = \tfrac{1}{2}, \quad [F]_{\mathrm{Lip}} = 1. \qquad \heartsuit$$

Example 2.18. Let $X = Y$ and F be as in (2.11) with

$$\varphi(t) = \begin{cases} 0 & \text{if } t \le 1, \\ t-1 & \text{if } 1 \le t \le 2, \\ \frac{1}{4}t + \frac{1}{2} & \text{if } t \ge 2. \end{cases}$$

Then

$$[F]_{\text{lip}} = [F]_{\text{b}} = 0, \quad [F]_{\text{q}} = [F]_{\text{Q}} = \tfrac{1}{4}, \quad [F]_{\text{B}} = \tfrac{1}{2}, \quad [F]_{\text{Lip}} = 1. \qquad \heartsuit$$

Example 2.19. Let $X = Y = \mathbb{R}$ and

$$F(x) = \begin{cases} -\frac{1}{4}x & \text{if } x \le 0, \\ 0 & \text{if } 0 \le x \le 1, \\ x-1 & \text{if } 1 \le x \le 2, \\ \frac{1}{2}x & \text{if } x \ge 2. \end{cases}$$

Then

$$[F]_{\text{lip}} = [F]_{\text{b}} = 0, \quad [F]_{\text{q}} = \tfrac{1}{4}, \quad [F]_{\text{Q}} = [F]_{\text{B}} = \tfrac{1}{2}, \quad [F]_{\text{Lip}} = 1. \qquad \heartsuit$$

Example 2.20. Let $X = Y = \mathbb{R}$ and

$$F(x) = \begin{cases} -\frac{1}{4}x - \frac{1}{4} & \text{if } x \le -1, \\ 0 & \text{if } -1 \le x \le 0, \\ x & \text{if} 0 \le x \le 1, \\ \frac{1}{2}x + \frac{1}{2} & \text{if } x \ge 1. \end{cases}$$

Then

$$[F]_{\text{lip}} = [F]_{\text{b}} = 0, \quad [F]_{\text{q}} = \tfrac{1}{4}, \quad [F]_{\text{Q}} = \tfrac{1}{2}, \quad [F]_{\text{B}} = [F]_{\text{Lip}} = 1. \qquad \heartsuit$$

Example 2.21. Let $X = Y$ and F be as in (2.11) with

$$\varphi(t) = \begin{cases} \frac{1}{2}t & \text{if } 0 \le t \le 1, \\ t - \frac{1}{2} & \text{if } 1 \le t \le 2, \\ \frac{1}{2}t + \frac{1}{2} & \text{if } t \ge 2. \end{cases}$$

Then

$$[F]_{\text{lip}} = 0, \quad [F]_{\text{b}} = [F]_{\text{q}} = [F]_{\text{Q}} = \tfrac{1}{2}, \quad [F]_{\text{B}} = \tfrac{3}{4}, \quad [F]_{\text{Lip}} = 1. \qquad \heartsuit$$

Example 2.22. Let $X = Y = \mathbb{R}$ and

$$F(x) = \begin{cases} -\frac{1}{2}x & \text{if } x \le 0, \\ \frac{1}{2}x & \text{if } 0 \le x \le 1, \\ \frac{3}{2}x - 1 & \text{if } 1 \le x \le 2, \\ x & \text{if } x \ge 2. \end{cases}$$

Then

$$[F]_{\text{lip}} = 0 \quad [F]_{\text{b}} = [F]_{\text{q}} = \tfrac{1}{2}, \quad [F]_{\text{Q}} = [F]_{\text{B}} = 1, \quad [F]_{\text{Lip}} = \tfrac{3}{2}. \qquad \heartsuit$$

Example 2.23. Let $X = Y = \mathbb{R}$ and

$$F(x) = \begin{cases} -\frac{1}{4}x & \text{if } x \le 0, \\ x & \text{if } 0 \le x \le 1, \\ \frac{1}{2}x + \frac{1}{2} & \text{if } x \ge 1. \end{cases}$$

Then

$$[F]_{\text{lip}} = 0, \quad [F]_{\text{b}} = [F]_{\text{q}} = \tfrac{1}{4}, \quad [F]_{\text{Q}} = \tfrac{1}{2}, \quad [F]_{\text{B}} = [F]_{\text{Lip}} = 1. \qquad \heartsuit$$

Example 2.24. Let $X = Y$ and F be as in (2.11) with

$$\varphi(t) = \begin{cases} t & \text{if } t \le 1, \\ 1 & \text{if } 1 \le t \le 2, \\ 2t - 3 & \text{if } 2 \le t \le 3, \\ t & \text{if } t \ge 3. \end{cases}$$

Then

$$[F]_{\text{lip}} = 0, \quad [F]_{\text{b}} = \tfrac{1}{2}, \quad [F]_{\text{q}} = [F]_{\text{Q}} = [F]_{\text{B}} = 1, \quad [F]_{\text{Lip}} = 2. \qquad \heartsuit$$

Example 2.25. Let $X = Y$ and F be as in (2.11) with

$$\varphi(t) = \begin{cases} t & \text{if } t \le 1, \\ 1 & \text{if } 1 \le t \le 2, \\ \frac{2}{3}t - \frac{1}{3} & \text{if } t \ge 2. \end{cases}$$

Then

$$[F]_{\text{lip}} = 0, \quad [F]_{\text{b}} = \tfrac{1}{2}, \quad [F]_{\text{q}} = [F]_{\text{Q}} = \tfrac{2}{3}, \quad [F]_{\text{B}} = [F]_{\text{Lip}} = 1. \qquad \heartsuit$$

Example 2.26. Let $X = Y = \mathbb{R}$ and

$$F(x) = \begin{cases} -\frac{1}{2}x & \text{if } x \le 0, \\ \frac{1}{4}x & \text{if } 0 \le x \le 1, \\ x - \frac{3}{4} & \text{if } x \ge 1. \end{cases}$$

Then

$$[F]_{\text{lip}} = 0, \quad [F]_{\text{b}} = \tfrac{1}{4}, \quad [F]_{\text{q}} = \tfrac{1}{2}, \quad [F]_{\text{Q}} = [F]_{\text{B}} = [F]_{\text{Lip}} = 1. \qquad \heartsuit$$

Example 2.27. Let $X = Y = \mathbb{R}$ and

$$F(x) = \begin{cases} -\frac{1}{2}x - \frac{3}{8} & \text{if } x \le -1, \\ -\frac{1}{8}x & \text{if } -1 \le x \le 0, \\ x & \text{if } 0 \le x \le 1, \\ \frac{1}{4}x + \frac{3}{4} & \text{if } x \ge 1. \end{cases}$$

Then

$$[F]_{\text{lip}} = 0, \quad [F]_{\text{b}} = \tfrac{1}{8}, \quad [F]_{\text{q}} = \tfrac{1}{4}, \quad [F]_{\text{Q}} = \tfrac{1}{2}, \quad [F]_{\text{B}} = [F]_{\text{Lip}} = 1. \qquad \heartsuit$$

Example 2.28. Let $X = Y = \mathbb{R}$ and

$$F(x) = \begin{cases} -\frac{3}{4}x & \text{if } x \le 0, \\ \frac{1}{2}x & \text{if } 0 \le x \le 1, \\ \frac{3}{2}x - 1 & \text{if } 1 \le x \le 2, \\ x & \text{if } x \ge 2. \end{cases}$$

Then

$$[F]_{\text{lip}} = 0, \quad [F]_{\text{b}} = \tfrac{1}{2}, \quad [F]_{\text{q}} = \tfrac{3}{4}, \quad [F]_{\text{Q}} = [F]_{\text{B}} = 1, \quad [F]_{\text{Lip}} = \tfrac{3}{2}. \qquad \heartsuit$$

Example 2.29. Let $X = Y$ and F be as in (2.11) with

$$\varphi(t) = \begin{cases} \frac{1}{2}t & \text{if } t \le 1, \\ t - \frac{1}{2} & \text{if } 1 \le t \le 2, \\ \frac{2}{3}t + \frac{1}{6} & \text{if } t \ge 2. \end{cases}$$

Then

$$[F]_{\text{lip}} = 0, \quad [F]_{\text{b}} = \tfrac{1}{2}, \quad [F]_{\text{q}} = [F]_{\text{Q}} = \tfrac{2}{3}, \quad [F]_{\text{B}} = \tfrac{3}{4}, \quad [F]_{\text{Lip}} = 1. \qquad \heartsuit$$

Example 2.30. Let $X = Y = \mathbb{R}$ and

$$F(x) = \begin{cases} -\frac{1}{2}x & \text{if } x \leq 0, \\ \frac{1}{2}x & \text{if } 0 \leq x \leq 1, \\ \frac{3}{2}x - 1 & \text{if } 1 \leq x \leq 2, \\ \frac{3}{4}x + \frac{1}{2} & \text{if } x \geq 2. \end{cases}$$

Then

$$[F]_{\text{lip}} = 0, \quad [F]_{\text{b}} = [F]_{\text{q}} = \tfrac{1}{2}, \quad [F]_{\text{Q}} = \tfrac{3}{4}, \quad [F]_{\text{B}} = 1, \quad [F]_{\text{Lip}} = \tfrac{3}{2}. \quad \heartsuit$$

Example 2.31. Let $X = Y = \mathbb{R}$ and

$$F(x) = \begin{cases} -\frac{1}{4}x & \text{if } x \leq 0, \\ 0 & \text{if } 0 \leq x \leq 1, \\ 2x - 2 & \text{if } 1 \leq x \leq 2, \\ \frac{1}{2}x + 1 & \text{if } x \geq 2. \end{cases}$$

Then

$$[F]_{\text{lip}} = [F]_{\text{b}} = 0, \quad [F]_{\text{q}} = \tfrac{1}{4}, \quad [F]_{\text{Q}} = \tfrac{1}{2}, \quad [F]_{\text{B}} = 1, \quad [F]_{\text{Lip}} = 2. \quad \heartsuit$$

Example 2.32. Let $X = Y = \mathbb{R}$ and

$$F(x) = \begin{cases} -\frac{3}{5}x & \text{if } x \leq 0, \\ \frac{1}{2}x & \text{if } 0 \leq x \leq 1, \\ \frac{3}{2}x - 1 & \text{if } 1 \leq x \leq 2, \\ \frac{3}{4}x + \frac{1}{2} & \text{if } x \geq 2. \end{cases}$$

Then

$$[F]_{\text{lip}} = 0, \quad [F]_{\text{b}} = \tfrac{1}{2}, \quad [F]_{\text{q}} = \tfrac{3}{5}, \quad [F]_{\text{Q}} = \tfrac{3}{4}, \quad [F]_{\text{B}} = 1, \quad [F]_{\text{Lip}} = \tfrac{3}{2}. \quad \heartsuit$$

2.3 Compact and α-contractive nonlinear operators

The characteristics introduced above all contain some kind of "metric" information on the operator F. In rather the same way as we have introduced the "topological" measure of noncompactness for bounded linear operators in Section 1.2, we can do this for nonlinear operators.

Let X and Y be two infinite dimensional Banach spaces. A continuous operator $F : X \to Y$ is called *α-Lipschitz* if there exists some $k > 0$ such that

$$\alpha(F(M)) \leq k\alpha(M) \tag{2.12}$$

for any bounded subset $M \subset X$, where $\alpha(M)$ denotes the measure of noncompactness (1.22). In this case we put

$$[F]_{\mathrm{A}} = \inf\{k : k > 0,\ (2.12) \text{ holds}\} \tag{2.13}$$

and call the characteristic $[F]_{\mathrm{A}}$ the *measure of noncompactness* (or α*-norm*) of F. We write $F \in \mathfrak{A}(X, Y)$ if $[F]_{\mathrm{A}} < \infty$, and $\mathfrak{A}(X, X) =: \mathfrak{A}(X)$ as before. As in the linear case, for $[F]_{\mathrm{A}} < 1$ the operator F is called α*-contractive* (or a *ball contraction*). Similarly, F is called α*-nonexpansive* if $[F]_{\mathrm{A}} \leq 1$. We point out that, in contrast to the linear case, a bounded nonlinear operator F need not satisfy at all an estimate of the form (2.12).

Parallel to (2.12) and (2.13), we will also be interested in the "reverse" condition

$$\alpha(F(M)) \geq k\alpha(M) \tag{2.14}$$

for any bounded subset $M \subset X$, and in the "lower" characteristic

$$[F]_{\mathrm{a}} = \sup\{k : k > 0,\ (2.14) \text{ holds}\}. \tag{2.15}$$

As in the linear case, the equivalent representations

$$[F]_{\mathrm{A}} = \sup_{\alpha(M)>0} \frac{\alpha(F(M))}{\alpha(M)} \tag{2.16}$$

and

$$[F]_{\mathrm{a}} = \inf_{\alpha(M)>0} \frac{\alpha(F(M))}{\alpha(M)} \tag{2.17}$$

are useful in infinite dimensional spaces. It is not hard to see that the characteristic (2.13) is a seminorm on $\mathfrak{A}(X, Y)$. As in the linear case, we have $[F]_{\mathrm{A}} = 0$ if and only F is *compact*, i.e. the image $F(M)$ of each bounded set $M \subset X$ is precompact in Y. By $\mathfrak{K}(X, Y)$ (and, in particular, by $\mathfrak{K}(X) := \mathfrak{K}(X, X)$) we denote the class of all compact operators between X and Y.

We collect again some basic properties of the characteristics (2.13) and (2.15) in the following Proposition 2.4. Recall that a nonlinear operator $F \colon X \to Y$ is called *proper* if the preimage $F^{-}(N) = \{x \in X : F(x) \in N\}$ of each compact set $N \subset Y$ is compact in X, and *proper on closed bounded sets* if this is true for the restriction of F to closed bounded sets. We will study the important class of proper operators in detail in the next chapter.

Proposition 2.4. *The characteristics* (2.13) *and* (2.15) *have the following properties* ($F, G \in \mathfrak{C}(X, Y)$, $\lambda \in \mathbb{K}$):

(a) $[F]_{\mathrm{a}} > 0$ *and* $[F]_{\mathrm{q}} > 0$ *imply that* F *is proper.*

(b) $[F]_{\mathrm{a}} > 0$ *implies that* F *is proper on closed bounded sets.*

(c) $[G]_{\mathrm{a}}\,[F]_{\mathrm{a}} \leq [GF]_{\mathrm{A}} \leq [G]_{\mathrm{a}}\,[F]_{\mathrm{A}}$.

(d) $[F]_a - [G]_A \le [F+G]_a \le [F]_a + [G]_A$.

(e) $|[F]_a - [G]_a| \le [F-G]_A$*; in particular,* $[F-G]_A = 0$ *implies* $[F]_a = [G]_a$.

(f) $[F^{-1}]_A = [F]_a^{-1}$ *if F is a homeomorphism.*

(g) $[F]_A \le [F]_{\mathrm{Lip}}$.

(h) $[F]_{\mathrm{lip}} \le [F]_a$ *if* $\dim X = \infty$.

(i) $[\lambda I - F]_A = [\lambda I - F]_a = |\lambda|$ *if* $\dim X = \infty$ *and F is compact.*

(j) $[F]_a \le [F]_Q$ *if* $\dim X = \infty$.

Proof. From $[F]_a > 0$ it follows that we may find $k > 0$ such that $\alpha(F(M)) \ge k\alpha(M)$ for each bounded $M \subset X$. By our additional assumption $[F]_q > 0$, we know that F is coercive (see Proposition 2.2 (a)). So, given a compact set $N \subset Y$ we conclude that $F^-(N)$ is bounded and

$$\alpha(F^-(N)) \le \frac{1}{k}\alpha(F(F^-(N)) \le \frac{1}{k}\alpha(N)$$

which shows that $F^-(N)$ is precompact. Moreover, the continuity of F implies that $F^-(N)$ is also closed, hence compact. So we have proved (a). The same reasoning shows that F is proper on bounded closed sets if only $[F]_a > 0$.

The properties (c), (d) and (e) follow immediately from the definition of the characteristics (2.13) and (2.15) and from the properties of the measure of noncompactness α established in Proposition 1.1.

Property (f) is trivial if X is finite dimensional (we use the convention $1/0 = \infty$ and $1/\infty = 0$ here). If X is infinite dimensional, the assertion follows from the fact that $\alpha(F(M)) = 0$ if and only if $\alpha(M) = 0$ and from the chain of equalities

$$\begin{aligned} [F^{-1}]_A = \sup_{\alpha(N)>0} \frac{\alpha(F^{-1}(N))}{\alpha(N)} &= \sup_{\alpha(M)>0} \frac{\alpha(M)}{\alpha(F(M))} \\ = \left(\inf_{\alpha(M)>0} \frac{\alpha(F(M))}{\alpha(M)} \right)^{-1} &= \frac{1}{[F]_a}. \end{aligned}$$

If $\{z_1, \dots, z_m\}$ is an ε-net for $M \subset X$ and $[F]_{\mathrm{Lip}} < \infty$, then obviously $\{F(z_1), \dots, F(z_m)\}$ is a $[F]_{\mathrm{Lip}}\varepsilon$-net for $F(M) \subset Y$, which proves (g). The estimate (h) is trivial if $[F]_{\mathrm{lip}} = 0$; so assume that $[F]_{\mathrm{lip}} > 0$. Choosing $k \in (0, [F]_{\mathrm{lip}})$ we have then $\|F(x) - F(y)\| \ge k\|x-y\|$ for all $x, y \in X$. Consequently, the same reasoning as in (g) shows that $\alpha(F(M)) \ge k\alpha(M)$ for any bounded $M \subset X$, and so $[F]_a \ge k$ which proves (h).

The assertion (i) follows from (d) and (e) and the fact that $[I]_A = [I]_a = 1$ in any infinite dimensional space.

Finally, to prove (j) suppose that F is quasibounded and choose $k > [F]_Q$. We find then $R > 0$ such that $\|F(x)\| \le k\|x\|$ for all $x \in X$ satisfying $\|x\| \ge R$. In particular,

$F(S_R(X)) \subseteq B_{kR}(Y)$, and hence $\alpha(F(S_R(X))) \leq k\alpha(B_R(X))) = k\alpha(S_R(X))$. Since X is infinite dimensional, we may apply formula (2.17) and get

$$[F]_{\mathrm{a}} \leq \frac{\alpha(F(S_R(X)))}{\alpha(S_R(X))} \leq k$$

from which the assertion follows, since $k > [F]_{\mathrm{Q}}$ is arbitrary. □

Observe that equality (i) in Proposition 2.4 is false if $\dim X < \infty$, since then $[I]_{\mathrm{A}} = 0$ and $[I]_{\mathrm{a}} = \infty$. The next example shows that the converse of Proposition 2.4 (b) is not true, i.e. the properness of F on closed bounded sets does not imply that $[F]_{\mathrm{a}} > 0$.

Example 2.33. Let X be an infinite dimensional Banach space and $F\colon X \to X$ be defined by

$$F(x) = \|x\|x. \tag{2.18}$$

Then F is a proper homeomorphism with continuous inverse

$$F^{-1}(y) = \begin{cases} \dfrac{y}{\sqrt{\|y\|}} & \text{if } y \neq \theta, \\ \theta & \text{if } y = \theta. \end{cases} \tag{2.19}$$

On the other hand, the sphere $S_{1/n}(X)$ is mapped by F onto the sphere $S_{1/n^2}(X)$ which implies that $[F]_{\mathrm{a}} = 0$. ♡

We also remark that the inequality $[F]_{\mathrm{a}} \leq [F]_{\mathrm{Q}}$ in Proposition 2.4 (j) is sharp in the sense that $[F]_{\mathrm{Q}}$ cannot be replaced by any of the lower characteristics contained in the estimates (2.8) and (2.9). In fact, for the linear left shift operator F from Example 2.4 we have

$$[F]_{\mathrm{lip}} = [F]_{\mathrm{b}} = [F]_{\mathrm{q}} = 0, \quad [F]_{\mathrm{Q}} = [F]_{\mathrm{a}} = 1.$$

Proposition 2.4 shows that, in contrast to the metric characteristics introduced in the previous section, the topological characteristics $[F]_{\mathrm{A}}$ and $[F]_{\mathrm{a}}$ have essentially different behaviour in finite and infinite dimensional spaces. This is not surprising, since the Heine–Borel compactness criterion characterizes precisely the finite dimensional normed spaces.

There are bounded nonlinear operators which frequently occur in applications and satisfy the strict inequality $0 < [F]_{\mathrm{A}} < [F]_{\mathrm{Lip}}$. As a typical example, let us consider *radial retractions*. Recall that a subset $M \subseteq X$ is called *retract* of a larger subset $N \supseteq M$ if there exists a continuous map $\rho\colon N \to M$ ("retraction") such that $\rho(x) = x$ on M. In other words, M is a retract of N if and only if the identity on M may be extended to a continuous map on N.

Example 2.34. Let X be an infinite dimensional Banach space, and let $\rho\colon X \to X$ denote the radial retraction

$$\rho(x) := \begin{cases} x & \text{if } \|x\| \le r, \\ r\dfrac{x}{\|x\|} & \text{if } \|x\| > r \end{cases} \tag{2.20}$$

of X onto the closed ball $B_r(X)$ which maps the exterior $X \setminus B_r(X)$ of $B_r(X)$ onto the sphere $S_r(X)$. Then

$$[\rho]_A = 1, \quad 1 \le [\rho]_{\text{Lip}} \le 2. \tag{2.21}$$

The first equality in (2.21) is easy to prove: on the one hand, for any bounded set $M \subset X$, we have $\rho(M) \subseteq \text{co}[M \cup \{\theta\}]$ and hence

$$\alpha(\rho(M)) \le \alpha(\text{co}[M \cup \{\theta\}]) = \alpha(M \cup \{\theta\}) = \alpha(M),$$

by Proposition 1.1 (e) and (f). On the other, from $\rho(B_r(X)) = B_r(X)$ it follows that $[\rho]_{\text{A}} = 1$.

To prove the second estimate in (2.21) we have to distinguish three cases. If $\|x\| \le r$ and $\|y\| \le r$ then of course $\|\rho(x) - \rho(y)\| = \|x - y\|$. If $\|x\| > r$ and $\|y\| \le r$ then

$$\begin{aligned} \|\rho(x) - \rho(y)\| &= \left\| \frac{rx}{\|x\|} - y \right\| \le \frac{r}{\|x\|}\|x - y\| + \left\| \frac{ry}{\|x\|} - y \right\| \\ &\le \|x - y\| + \frac{\|y\|}{\|x\|}(\|x\| - r) \le \|x - y\| + \|x\| - \|y\| \le 2\|x - y\|. \end{aligned}$$

Finally, if $\|x\| > r$ and $\|y\| > r$ then

$$\begin{aligned} \|\rho(x) - \rho(y)\| &= \left\| \frac{rx}{\|x\|} - \frac{ry}{\|y\|} \right\| \le \frac{r}{\|x\|}\|x - y\| + r\|y\| \left| \frac{1}{\|x\|} - \frac{1}{\|y\|} \right| \\ &\le \|x - y\| + \frac{r}{\|x\|} \,|\, \|y\| - \|x\| \,|\, \le 2\|x - y\|. \end{aligned}$$

We have proved that $[\rho]_{\text{Lip}} \le 2$; the lower estimate $[\rho]_{\text{Lip}} \ge 1$ is of course trivial. ♡

It is well known that retractions are an important tool in fixed point theory. As a matter of fact, the statement of Brouwer's fixed point theorem is equivalent to the *non-existence* of a continuous retraction of the closed unit ball $B(\mathbb{R}^n)$ in $\mathbb{R}^n$ onto its boundary $S(\mathbb{R}^n)$. Likewise, the fact that in every infinite dimensional Banach space X such retractions exist means that Brouwer's theorem fails in any such space. This is precisely the point where compactness comes in: in Schauder's fixed point theorem one has to impose a compactness condition either on the domain or on the range of the operator involved. As a consequence, retractions from the closed unit ball of any Banach space onto its boundary can never be compact. In fact, if $\rho\colon B(X) \to S(X)$

were a compact retraction, then $F = -\rho$ would be a compact selfmap of $B(X)$, and so would have a fixed point $\hat{x} \in S(X)$, by Schauder's fixed point theorem, which leads to the contradiction $-\hat{x} = \hat{x}$ and $\|\hat{x}\| = 1$. Theorem 2.1 below shows that it is also impossible to find an α-contractive retraction of this type, i.e. $\rho\colon B(X) \to S(X)$ satisfying $[\rho|_{B(X)}]_{\mathrm{A}} < 1$. Nevertheless, the following quite remarkable example shows that there do exist such retractions with $[\rho|_{B(X)}]_{\mathrm{A}}$ being arbitrarily close to 1!

Example 2.35. In the Banach space $X = C[0, 1]$ of all continuous real functions on $[0, 1]$, define $F\colon B(X) \to X$ by

$$F(x)(t) := \begin{cases} x\left(\frac{2t}{1+\|x\|}\right) & \text{if } 0 \le t < \frac{1+\|x\|}{2}, \\ x(1) & \text{if } \frac{1+\|x\|}{2} \le t \le 1. \end{cases}$$

It is not hard to see that F is continuous, $\|F(x)\| = \|x\|$ for $x \in B(X)$, and $F(x) = x$ for $x \in S(X)$. Moreover, for each $x \in B(X)$ the function $F(x)$ attains its norm in the interval $[0, \frac{1}{2}(1 + \|x\|)]$. Finally, the Arzelà–Ascoli theorem and a simple geometric reasoning show that $\alpha(F(M)) \le \alpha(M)$ for all $M \subseteq B(X)$, i.e., $[F|_{B(X)}]_{\mathrm{A}} \le 1$.

Now we define, for each $u \in (0, \infty)$, another mapping $P_u\colon B(X) \to X$ by

$$P_u(x)(t) = \max\left\{0, \frac{u}{2}\,(2t - \|x\| - 1)\right\}.$$

Observe that P_u is continuous and compact, and $P_u(x)(t) \equiv 0$ on $[0, \frac{1}{2}(1 + \|x\|)]$ for each $x \in B(X)$. In particular, $P_u(x) \equiv \theta$ for $x \in S(X)$.

For $u > 0$, a retraction $\rho_u\colon B(X) \to S(X)$ may now be defined by

$$\rho_u(x) = \frac{F(x) + P_u(x)}{\|F(x) + P_u(x)\|}. \tag{2.22}$$

Observe that the operator $F + P_u$ is α-nonexpansive, because $[F|_{B(X)}]_{\mathrm{A}} \le 1$ and $[P_u|_{B(X)}]_{\mathrm{A}} = 0$. Moreover, $F + P_u = I$ on $S(X)$, and so also $\rho_u = I$ on $S(X)$. On the other hand, for $x \in B(X)$ we have the lower estimates

$$\begin{aligned} \|F(x) + P_u(x)\| &\ge \max\{\|x\|, F(x)(1) + P_u(x)(1)\} \\ &= \max\left\{\|x\|, x(1) + \frac{u}{2}(1 - \|x\|)\right\} \\ &\ge \max\left\{\|x\|, \frac{u}{2}(1 - \|x\|) - \|x\|\right\} \ge \frac{u}{u+4}, \end{aligned}$$

because the last term attains its minimum $u/(u + 4)$ for functions x with $\|x\| = u/(u+4)$. So the operator (2.22) is well-defined and in fact retracts the unit ball $B(X)$ onto its boundary $S(X)$. Moreover, a straightforward calculation shows that

$$\alpha(\rho_u(M)) \le \frac{u+4}{u}\alpha(M) \quad (M \subseteq B(X)), \tag{2.23}$$

which means that $[\rho_u|_{B(X)}]_{\mathrm{A}} \le (u + 4)/u$. Passing in (2.23) to the limit $u \to \infty$ we see that the α of the retraction may be made arbitrarily close to 1. ♡

We will use Example 2.35 in subsequent chapters to construct several operators with surprising or even "pathological" properties.

We pass now to an important result for α-contractive operators which is known as *Darbo's fixed point theorem* and which we will use later several times.

Theorem 2.1. *Let X be a Banach space, $M \subset X$ nonempty, bounded, closed, and convex, and $F: M \to M$ an α-contractive operator. Then F has a fixed point in M, i.e. an $x \in M$ with $F(x) = x$.*

Proof. We define a sequence $(M_n)_n$ of subsets $M_n \subset X$ recursively by

$$M_0 := M, \ M_1 := \overline{\text{co}}\, F(M_0), \ldots, \ M_{n+1} := \overline{\text{co}}\, F(M_n).$$

One may easily see by induction that $(M_n)_n$ is monotonically decreasing (with respect to inclusion), and each M_n is a closed convex subset of M. Choosing $k \in ([F]_{\text{A}}, 1)$ we get

$$\alpha(M_1) = \alpha(F(M)) \leq k\alpha(M)$$

and, more generally,

$$\alpha(M_{n+1}) = \alpha(F(M_n)) \leq k\alpha(F(M_{n-1})) \leq \cdots \leq k^n\alpha(M),$$

hence $\alpha(M_n) \to 0$ as $n \to \infty$. By Proposition 1.1 (h), this implies that the set

$$M_\infty := \bigcap_{n=0}^{\infty} M_n$$

is nonempty, compact and convex. Moreover, this set is invariant under F since $F(M_n) \subseteq \overline{\text{co}}\, F(M_n) = M_{n+1}$ implies $F(M_\infty) \subseteq M_\infty$. From Schauder's fixed point theorem it follows that F has a fixed point in $M_\infty \subseteq M$. □

As we have seen above, both compact operators and contractions (i.e., $[F]_{\text{Lip}} < 1$) are particular examples of α-contractive operators. This implies that *Darbo's fixed point theorem generalizes both the classical Schauder fixed point principle and* (a special version of) *Banach's contraction mapping principle*. On the other hand, since we have proved Darbo's fixed point principle by means of Schauder's fixed point principle, these two results are actually equivalent.

The following fixed point theorem is an easy but quite useful consequence of Theorem 2.1; we will use this many times in what follows.

Theorem 2.2. *Let X be a Banach space and $F: X \to X$ an α-contractive operator with $[F]_{\text{Q}} < 1$. Then the operator $I - F$ is surjective; in particular, F has a fixed point in X.*

Proof. Fix $z \in X$ and put

$$F_z(x) := F(x) + z. \tag{2.24}$$

The condition $[F_z]_Q = [F]_Q < 1$ ensures that we may find $q \in ([F]_Q, 1)$ and $b > 0$ such that

$$\|F_z(x)\| \leq q\|x\| + b \quad (x \in X).$$

Consequently, for $R \geq b/(1-q)$ the operator F_z maps the ball $B_R(X)$ into itself, and the assertion follows from Theorem 2.1. □

For the sake of completeness we still mention a slight generalization of Theorem 2.1 which goes back to Sadovskij. Let us call a continuous operator $F: X \to Y$ *condensing* if

$$\alpha(F(M)) < \alpha(M) \tag{2.25}$$

for every bounded set $M \subset X$ with noncompact closure. Such operators also satisfy a fixed point theorem.

Theorem 2.3. *Let X be a Banach space, $M \subset X$ nonempty, bounded, closed, and convex, and $F: M \to M$ a condensing operator. Then F has a fixed point in M.*

Proof. First we show that there exists a nonempty set $K \subseteq M$ such that $K \subseteq F(K)$. To this end, fix $x \in M$ and consider the orbit $\Omega(x) := \{x, F(x), F^2(x), F^3(x), \dots\}$ of x. Since $\Omega(x) = F(\Omega(x)) \cup \{x\}$ and F is condensing, we have $\alpha(\Omega(x))) = 0$, i.e. $\Omega(x)$ is precompact. We claim that the set K of all accumulation points of $\Omega(x)$ satisfies $K \subseteq F(K)$. Indeed, K is nonempty since $\Omega(x)$ is precompact. Given $y \in K$, choose a sequence $(n_k)_k$ of natural numbers such that $F^{n_k}(x) \to y$ as $k \to \infty$. Again by the precompactness of $\Omega(x)$, the sequence $(F^{n_k-1}(x))_k$ has a convergent subsequence; without loss of generality let $F^{n_k-1}(x) \to z$ as $k \to \infty$. The continuity of F implies then that $y = F(z)$, and so $y \in F(K)$ as claimed.

Now denote by $\mathfrak{M}$ the family of all closed convex sets between K and M which are invariant under F, i.e.

$$\mathfrak{M} = \{N : K \subseteq N \subseteq M,\ \overline{\text{co}}\, N = N,\ F(N) \subseteq N\}.$$

Since $M \in \mathfrak{M}$, this family is nonempty. The intersection $\hat{M}$ of all sets from $\mathfrak{M}$ is then a minimal element of $\mathfrak{M}$.

We define a map $\Phi : \mathfrak{M} \to \mathfrak{M}$ by

$$\Phi(N) := \overline{\text{co}}\, F(N).$$

The inclusion $F(K) \supseteq K$ proved above shows that Φ maps indeed $\mathfrak{M}$ into itself. Since $F(N) \subseteq N$ for all $N \in \mathfrak{M}$ we obtain

$$\Phi(N) = \overline{\text{co}}\, F(N) \subseteq \overline{\text{co}}\, N = N \quad (N \in \mathfrak{M}).$$

In particular, for $N = \hat{M}$ we have $\Phi(\hat{M}) \subseteq \hat{M}$, and so even $\Phi(\hat{M}) = \hat{M}$, by the minimality of $\hat{M}$. Finally, from

$$\alpha(F(\hat{M})) = \alpha(\overline{\text{co}}\, F(\hat{M}) = \alpha(\Phi(\hat{M})) = \alpha(\hat{M})$$

it follows that $\hat{M}$ is precompact, since F is condensing. Schauder's fixed point theorem implies that F has a fixed point in $\hat{M} \subseteq M$. □

The following example shows that Theorem 2.3 is a nontrivial generalization of Theorem 2.1.

Example 2.36. Let X be an infinite dimensional Banach space and $F\colon X \to X$ be defined by

$$F(x) = \begin{cases} (1 - \|x\|)x & \text{if } \|x\| \le 1, \\ \theta & \text{if } \|x\| > 1; \end{cases} \tag{2.26}$$

then F is condensing but not α-contractive. In fact, since $F(M) \subseteq \overline{\text{co}}(M \cup \{\theta\})$ for any set $M \subseteq B(X)$, we trivially have $[F]_{\text{A}} \le 1$. Given $M \subseteq B(X)$ with noncompact closure, fix r with $0 < r < \alpha(M)$. For $M_1 := M \cap B_r(X)$ we have then, on the one hand,

$$\alpha(F(M_1)) \le \alpha(M_1) \le \alpha(B_r(X)) = r < \alpha(M),$$

while for $M_2 := M \setminus B_r(X)$ we have, on the other,

$$F(M_2) \subseteq \{tx : 0 \le t \le 1 - r,\ x \in M_2\} \subseteq \overline{\text{co}}\,[(1 - r)M \cup \{\theta\}],$$

hence

$$\alpha(F(M_2)) \le (1 - r)\alpha(M) < \alpha(M).$$

We conclude that $\alpha(F(M)) = \max\{\alpha(F(M_1)), \alpha(F(M_2))\} < \alpha(M)$, i.e. F is condensing. Nevertheless, F is not α-contractive. Indeed, if (2.12) were true with some $k < 1$, for the sphere $S_r(X)$ with $0 < r < 1 - k$ we would have

$$kr < (1 - r)r = \alpha(F(S_r(X))) \le k\alpha(S_r(X)) = kr,$$

a contradiction. ♡

If X is an infinite dimensional space, there are in principle 8 possibilities for equality and strict inequality occurring in the chain of estimates

$$[F]_{\text{lip}} \le [F]_{\text{a}} \le [F]_{\text{A}} \le [F]_{\text{Lip}} \tag{2.27}$$

which is a combination of Proposition 2.4 (g) and (h). The following 8 examples correspond to these possibilities. In all these example we may choose $X = l_\infty$ and, surprisingly, F linear.

Example 2.37. Let $L(x_1, x_2, x_3, x_4, x_5, \dots) = (x_1, x_2, x_3, x_4, x_5, \dots)$; then

$$[L]_{\text{lip}} = [L]_{\text{a}} = [L]_{\text{A}} = [L]_{\text{Lip}} = 1.$$

Of course, the identity on any infinite dimensional space will do the same job. ♡

Example 2.38. Let $L(x_1, x_2, x_3, x_4, x_5, \dots) = (x_2, x_3, x_4, x_5, x_6, \dots)$; then

$$[L]_{\text{lip}} = 0, \quad [L]_{\text{a}} = [L]_{\text{A}} = [L]_{\text{Lip}} = 1.$$ ♡

Example 2.39. Let $L(x_1, x_2, x_3, x_4, x_5, \dots) = (x_1, 0, x_3, 0, x_5, \dots)$; then

$$[L]_{\text{lip}} = [L]_{\text{a}} = 0, \quad [L]_{\text{A}} = [L]_{\text{Lip}} = 1.$$

To see that $[L]_{\text{a}} = 0$, observe that the set $M = \{x : x = (0, x_2, 0, x_4, 0, \dots), \|x\| = 1\}$ satisfies $\alpha(M) = 1$. ♡

Example 2.40. Let $L(x_1, x_2, x_3, x_4, x_5, \dots) = (x_1, 0, 0, 0, 0, \dots)$; then

$$[L]_{\text{lip}} = [L]_{\text{a}} = [L]_{\text{A}} = 0, \quad [L]_{\text{Lip}} = 1.$$ ♡

Example 2.41. Let $L(x_1, x_2, x_3, x_4, x_5, \dots) = (2x_2, x_3, 2x_4, x_5, 2x_6, \dots)$; then

$$[L]_{\text{lip}} = 0, \quad [L]_{\text{a}} = 1, \quad [L]_{\text{A}} = [L]_{\text{Lip}} = 2.$$ ♡

Example 2.42. Let $L(x_1, x_2, x_3, x_4, x_5, \dots) = (2x_2, x_3, x_4, x_5, x_6, \dots)$; then

$$[L]_{\text{lip}} = 0, \quad [L]_{\text{a}} = [L]_{\text{A}} = 1, \quad [L]_{\text{Lip}} = 2.$$ ♡

Example 2.43. Let $L(x_1, x_2, x_3, x_4, x_5, \dots) = (2x_1, 0, x_3, 0, x_5, \dots)$; then

$$[L]_{\text{lip}} = [L]_{\text{a}} = 0, \quad [L]_{\text{A}} = 1, \quad [L]_{\text{Lip}} = 2.$$ ♡

Example 2.44. Let $L(x_1, x_2, x_3, x_4, x_5, \dots) = (3x_2, x_3, 2x_4, x_5, 2x_6, \dots)$; then

$$[L]_{\text{lip}} = 0, \quad [L]_{\text{a}} = 1, \quad [L]_{\text{A}} = 2, \quad [L]_{\text{Lip}} = 3.$$ ♡

To obtain a general view of these examples, we collect them again in the following table.

Table 2.2

$[F]_{\mathrm{lip}} = [F]_{\mathrm{a}} = [F]_{\mathrm{A}} = [F]_{\mathrm{Lip}}$	Example 2.37
$[F]_{\mathrm{lip}} < [F]_{\mathrm{a}} = [F]_{\mathrm{A}} = [F]_{\mathrm{Lip}}$	Example 2.38
$[F]_{\mathrm{lip}} = [F]_{\mathrm{a}} < [F]_{\mathrm{A}} = [F]_{\mathrm{Lip}}$	Example 2.39
$[F]_{\mathrm{lip}} = [F]_{\mathrm{a}} = [F]_{\mathrm{A}} < [F]_{\mathrm{Lip}}$	Example 2.40
$[F]_{\mathrm{lip}} < [F]_{\mathrm{a}} < [F]_{\mathrm{A}} = [F]_{\mathrm{Lip}}$	Example 2.41
$[F]_{\mathrm{lip}} < [F]_{\mathrm{a}} = [F]_{\mathrm{A}} < [F]_{\mathrm{Lip}}$	Example 2.42
$[F]_{\mathrm{lip}} = [F]_{\mathrm{a}} < [F]_{\mathrm{A}} < [F]_{\mathrm{Lip}}$	Example 2.43
$[F]_{\mathrm{lip}} < [F]_{\mathrm{a}} < [F]_{\mathrm{A}} < [F]_{\mathrm{Lip}}$	Example 2.44

2.4 Special subsets of scalars

Let X be a Banach space over $\mathbb{K}$ and $F\colon X \to X$ a continuous operator. We define several subsets of $\mathbb{K}$ by means of the lower characteristics (2.2), (2.4), (2.7) and (2.15), namely

$$\sigma_{\mathrm{lip}}(F) = \{\lambda \in \mathbb{K} : [\lambda I - F]_{\mathrm{lip}} = 0\}, \tag{2.28}$$

$$\sigma_{\mathrm{q}}(F) = \{\lambda \in \mathbb{K} : [\lambda I - F]_{\mathrm{q}} = 0\}, \tag{2.29}$$

$$\sigma_{\mathrm{b}}(F) = \{\lambda \in \mathbb{K} : [\lambda I - F]_{\mathrm{b}} = 0\}, \tag{2.30}$$

and

$$\sigma_{\mathrm{a}}(F) = \{\lambda \in \mathbb{K} : [\lambda I - F]_{\mathrm{a}} = 0\}. \tag{2.31}$$

Observe that we have already defined a subspectrum $\sigma_{\mathrm{q}}(L)$ for linear operators L in (1.50); so we have to show that this notation is compatible. But this follows immediately from the fact that $[L]_{\mathrm{q}} = 0$ if and only if there exists some sequence $(e_n)_n$ in $S(X)$ such that $\|Le_n\| \to 0$ as $n \to \infty$, see (1.50).

The following Proposition 2.5 gives some information on the sets (2.28)–(2.31). In particular, the inclusions in (a) show that $\sigma_\kappa(F)$ is always contained in an "annular domain" for $\kappa \in \{\mathrm{lip, q, b, a}\}$.

Proposition 2.5. *The sets* (2.28)–(2.31) *have the following properties*:

(a) *The inclusions*

$$\sigma_{\mathrm{lip}}(F) \subseteq \{\lambda \in \mathbb{K} : [F]_{\mathrm{lip}} \le |\lambda| \le [F]_{\mathrm{Lip}}\}, \tag{2.32}$$

$$\sigma_{\mathrm{q}}(F) \subseteq \{\lambda \in \mathbb{K} : [F]_{\mathrm{q}} \le |\lambda| \le [F]_{\mathrm{Q}}\}, \tag{2.33}$$

$$\sigma_{\mathrm{b}}(F) \subseteq \{\lambda \in \mathbb{K} : [F]_{\mathrm{b}} \le |\lambda| \le [F]_{\mathrm{B}}\}, \tag{2.34}$$

and

$$\sigma_{\mathrm{a}}(F) \subseteq \{\lambda \in \mathbb{K} : [F]_{\mathrm{a}} \le |\lambda| \le [F]_{\mathrm{A}}\}, \tag{2.35}$$

are true.

(b) $\sigma_q(F) \subseteq \sigma_b(F)$.

(c) $\sigma_b(F) \subseteq \sigma_{lip}(F)$ *if* $F(\theta) = \theta$.

(d) $\sigma_a(F) \subseteq \sigma_{lip}(F)$.

(e) $\sigma_a(F) = \{0\}$ *if* F *is compact and* $\dim X = \infty$.

(f) $\partial\sigma(L) \subseteq \sigma_q(L) = \sigma_b(L) = \sigma_{lip}(L) \subseteq \sigma(L)$ *if* $L \in \mathfrak{L}(X)$.

(g) $\partial\sigma(L) = \sigma_q(L) = \sigma_b(L) = \sigma_{lip}(L) = \sigma(L)$ *if* $L \in \mathfrak{K}\mathfrak{L}(X)$.

(h) $\sigma_a(L) = \sigma_+(L)$ *and* $\sigma_a(L^*) = \sigma_-(L)$ *if* $L \in \mathfrak{L}(X)$.

(i) $\sigma_a(L) \subseteq \sigma_q(L)$ *if* $L \in \mathfrak{L}(X)$.

Proof. To prove (2.32), fix $\lambda \in \sigma_{lip}(F)$. We have then $[\lambda I - F]_{lip} = 0$, by definition, and Proposition 2.1 (c) implies that $[F]_{lip} - [\lambda I]_{Lip} \leq 0$ and $[\lambda I]_{lip} - [F]_{Lip} \leq 0$. Consequently,

$$[F]_{lip} \leq [\lambda I]_{Lip} = |\lambda| = [\lambda I]_{lip} \leq [F]_{Lip},$$

so (2.32) is true. The inclusions (2.33)–(2.35) are proved similarly.

The inclusions in (b), (c) and (d) follow straightforwardly from (2.8), (2.9) and (2.27), respectively.

To prove (e), let X be infinite dimensional and $F : X \to X$ be compact. Then $[F]_A = [F]_a = 0$, and so the assertion follows from (2.35).

Now let $L : X \to X$ be bounded and linear. We show that $\sigma_{lip}(L) \subseteq \sigma_q(L)$, and hence $\sigma_q(L) = \sigma_b(L) = \sigma_{lip}(L)$, by (b) and (c). Given $\lambda \in \sigma_{lip}(L)$, by linearity we may find a sequence $(x_n)_n$ in $X \setminus \{\theta\}$ such that

$$\lim_{n\to\infty} \frac{\|\lambda x_n - Lx_n\|}{\|x_n\|} = 0.$$

In case $\|x_n\| \to \infty$ we immediately have $\lambda \in \sigma_q(L)$ and so we are done. In case $\|x_n\| \to 0$ we define $y_n := x_n/\|x_n\|^2$ and get $\|y_n\| \to \infty$ and

$$\lim_{n\to\infty} \frac{\|\lambda y_n - Ly_n\|}{\|y_n\|} = \lim_{n\to\infty} \frac{\|x_n\|^{-2}\|\lambda x_n - Lx_n\|}{\|x_n\|^{-1}} = \lim_{n\to\infty} \frac{\|\lambda x_n - Lx_n\|}{\|x_n\|} = 0,$$

hence again $\lambda \in \sigma_q(L)$. Finally, if the sequence $(x_n)_n$ is both bounded and bounded away from zero we put $z_n := nx_n$ and get $\|z_n\| \to \infty$ and

$$\lim_{n\to\infty} \frac{\|\lambda z_n - Lz_n\|}{\|z_n\|} = \lim_{n\to\infty} \frac{n\|\lambda x_n - Lx_n\|}{n\|x_n\|} = \lim_{n\to\infty} \frac{\|\lambda x_n - Lx_n\|}{\|x_n\|} = 0,$$

hence $\lambda \in \sigma_q(L)$ as before.

We have proved the equalities in (f). The first inclusion in (f) has already been shown in Proposition 1.4 (a), and the last inclusion is trivial.

Now, if $L : X \to X$ is compact and linear, from Theorem 1.2 we conclude that $\sigma(L)$ has no interior points, and so $\partial\sigma(L) = \sigma(L)$ which proves (g).

Finally, (h) is just a reformulation of Proposition 1.2 (k) and (l), while (i) is a trivial consequence of (d) and (f). □

By means of the Examples 2.1–2.32 and 2.36–2.43 one may show that the inclusions in Proposition 2.5 (b) and (c) may be strict; by Proposition 2.5 (f), such examples cannot involve linear operators. The following two examples show that the spectral set (2.31) plays a somewhat exceptional role: it does not contain the boundary of the spectrum even in the linear case, and it is not comparable with the other spectral sets in the nonlinear case.

Example 2.45. Let $X = l_\infty$ and let $L \in \mathfrak{KL}(X)$ be the projection onto the first coordinate as in Example 2.40. Then

$$\partial\sigma(L) = \sigma(L) = \{0, 1\}, \quad \sigma_{\mathrm{a}}(L) = \{0\}.$$

Here the last equality follows from the fact that the "reduced" unit ball $M = \{x \in B(X) : x_1 = 0\}$ is mapped by $\lambda I - L$ into a set whose measure of noncompactness is precisely $|\lambda|$. ♡

We remark that Proposition 2.5 (e) also implies that the subspectrum $\sigma_{\mathrm{a}}(L)$ in general does not contain the boundary of the spectrum $\sigma(L)$.

Example 2.46. Let X be an infinite dimensional Banach space, $e \in X$ fixed with $\|e\| = 1$, and

$$F(x) = \|x\|e.$$

Then $\sigma_{\mathrm{a}}(F) = \{0\}$, by Proposition 2.5 (e), but $0 \notin \sigma_{\mathrm{b}}(F) \cap \sigma_{\mathrm{q}}(F)$, since $[F]_{\mathrm{b}} = [F]_{\mathrm{q}} = 1$. ♡

We will use the spectral sets (2.29)–(2.31) over and over again in Chapters 6–9. For the time being we just describe some of their properties.

Theorem 2.4. *The sets* (2.28)–(2.31) *are closed for any* $F \in \mathfrak{C}(X)$. *Moreover, the set* (2.28) [(2.29), (2.30), (2.31), *respectively*] *is even compact for* $F \in \mathfrak{Lip}(X)$ [$F \in \mathfrak{Q}(X)$, $F \in \mathfrak{B}(X)$, $F \in \mathfrak{A}(X)$, *respectively*].

Proof. We prove the closedness of the set (2.28), the proof for the other sets is similar. From Proposition 2.1 (d) it follows that

$$|[\lambda I - F]_{\mathrm{lip}} - [\mu I - F]_{\mathrm{lip}}| \le |\lambda - \mu| \quad (\lambda, \mu \in \mathbb{K})$$

which immediately implies that $\sigma_{\mathrm{lip}}(F)$ is closed. For $F \in \mathfrak{Lip}(X)$ the inclusion (2.32) shows that $\sigma_{\mathrm{lip}}(F)$ is also bounded, hence compact. □

One cannot expect that the sets (2.28)–(2.31) are also bounded for general $F \in \mathfrak{C}(X)$. To see this, let us consider some very simple examples.

Example 2.47. Let $X = C[0,1]$ and $F\colon X \to X$ be defined by

$$F(x)(t) = \begin{cases} 0 & \text{if } x(t) \le 1, \\ \sin \pi x^2(t) & \text{if } x(t) \ge 1. \end{cases}$$

Then $F \in \mathfrak{B}(X) \cap \mathfrak{A}(X)$, but $F \notin \mathfrak{Lip}(X)$, and $\sigma_{\mathrm{lip}}(F) = \mathbb{R}$. ♡

Example 2.48. Let $X = C[0,1]$ and $F\colon X \to X$ be defined by $F(x)(t) = x(1)x(t)$. Then $F \in \mathfrak{C}(X)$, but $F \notin \mathfrak{Q}(X) \cup \mathfrak{A}(X)$, and

$$\sigma_{\mathrm{q}}(F) = \sigma_{\mathrm{b}}(F) = \sigma_{\mathrm{lip}}(F) = \sigma_{\mathrm{a}}(F) = \mathbb{R}.$$

To see that $\sigma_{\mathrm{a}}(F) = \mathbb{R}$, consider the set $M_\lambda = \{x \in X : x(1) = \lambda,\ \|x\| = |\lambda| + 1\}$. ♡

Example 2.49. Let $F\colon l_\infty \to l_\infty$ be defined by

$$F(x_1, x_2, x_3, \ldots) = (\sqrt{|x_1|}, 0, 0, \ldots).$$

Then $F \in \mathfrak{Q}(X) \cap \mathfrak{A}(X)$, but $F \notin \mathfrak{B}(X)$, and $\sigma_{\mathrm{b}}(F) = \sigma_{\mathrm{lip}}(F) = [0, \infty)$. ♡

Example 2.50. Let $X = l_\infty$ as in the preceding example, but

$$F(x_1, x_2, x_3, \ldots) = (\sqrt{|x_1|}, \sqrt{|x_2|}\sqrt{|x_3|}, \ldots).$$

Then $F \in \mathfrak{Q}(X)$, but $F \notin \mathfrak{B}(X) \cup \mathfrak{A}(X)$, and $\sigma_{\mathrm{a}}(F) = [0, \infty)$. ♡

Example 2.51. Let $F\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$F(x) = \begin{cases} 0 & \text{if } x = 2n, n \in \mathbb{N}, \\ x^2 & \text{if } x = 2n-1, n \in \mathbb{N}, \\ \text{linear} & \text{otherwise.} \end{cases}$$

Then $F \in \mathfrak{A}(X)$, but $F \notin \mathfrak{Q}(X)$, and

$$\sigma_{\mathrm{lip}}(F) = \sigma_{\mathrm{b}}(F) = \sigma_{\mathrm{q}}(F) = [0, \infty).$$ ♡

2.5 Notes, remarks and references

The characteristic (2.1) is widely used in nonlinear analysis, since Lipschitz constants are natural substitutes for operator norms if one passes from linear to nonlinear problems. The first systematic examination of (2.2) may be found in [181] and [82]. The characteristic (2.3) was introduced by Granas [136] under the name *quasinorm* of F, the corresponding lower characteristic (2.4) was studied for the first time, to the best of our knowledge, together with the spectral set (2.29), in [124]. The characteristic

(2.4) plays a prominent role in bifurcation theory for nonlinear operators which are not necessarily differentiable, see [125]; we will return to such problems in Chapter 6.

The spectral set (2.29) was studied subsequently in [129], [130]. In particular, it is shown by degree-theoretical methods in [129] that $\lambda I - F$ is always surjective for $F \in \mathfrak{A}(X) \cap \mathfrak{Q}(X)$ satisfying some natural additional conditions. For linear operators, these conditions are well known. The characteristics (2.6) and (2.7) may also be considered as natural "nonlinear analogues" to norms and have been introduced in [82], see also [8], [15].

All the examples in Section 2.2 are of course straightforward, maybe only the Examples 2.6 and 2.10 are not completely trivial.

There is a huge amount of literature on α-contractive and condensing operators, we mention the survey [231] and the monograph [1], where one may also find the proof of some properties stated in Proposition 2.4. We point out that the inequality in Proposition 2.4 (g) may be strict for many important operators arising in applications, and therefore there are reasonable examples of α-contractive operators which are not contractive; see [7] for a series of such examples.

Radial retractions occur in many places in nonlinear analysis and topology, we took the calculations in Example 2.34 from [184]. In view of the relations (2.21) which illustrate the difference between the Lipschitz- and the α-norm of a retraction, the following result is interesting [74]: *Let ρ be the radial retraction* (2.20) *in a Banach space X with* $\dim X \geq 3$*; then* $[\rho]_{\mathrm{Lip}} = 1$ *if and only if X is Hilbert.*

The remarkable Example 2.35 is due to Wośko, see [279]. We point out, however, that the first example of this type has been given already in 1974 by Furi and Martelli [119], [120], also in the space $X = C[0, 1]$ of continuous functions. Furi and Martelli first consider, for $u \in (1, \infty)$, the operator $F_u \colon B(X) \to B(X)$ defined by

$$F_u(x)(t) = \min\{1, u(1-t)|x(t)| + ut\};$$

this operator satisfies a Lipschitz condition with constant u, and so $[F_u|_{B(X)}]_{\mathrm{A}} \leq u$. Afterwards they define a corresponding retraction $\rho_u \colon B(X) \to S(X)$ by means of a geometrical reasoning. This retraction satisfies the estimates

$$\alpha(\rho_u(M)) \leq \frac{2u}{u-1}\alpha(M) + \frac{u+1}{u-1}\alpha(F_u(M)) \leq \frac{u^2+3u}{u-1}\alpha(M) \quad (M \subseteq B(X)),$$

and so $[\rho_u|_{B(X)}]_{\mathrm{A}} \leq (u^2 + 3u)/(u-1)$. Note that this expression attains its minimal value 9 for $u = 3$. A detailed discussion of the existence (and explicit construction) of Lipschitz continuous retractions of $B(X)$ onto $S(X)$ may be found in the excellent book [132].

We have chosen Wośko's example because it is somewhat simpler, and it gives a sharper estimate. We may rephrase Example 2.35 in terms of the following general problem from Banach space geometry. Given a Banach space X, we call the number

$$\begin{aligned} W(X) := \inf\{k : k > 0,\ &\text{there exists a retraction} \\ &\rho\colon B(X) \to S(X) \text{ with } [\rho]_{\mathrm{A}} \leq k\} \end{aligned} \tag{2.36}$$

the *Mośko constant* of the space X. From Darbo's fixed point principle (Theorem 2.1) it follows that $W(X) \geq 1$ for every space X; so it is interesting to give upper estimates (or better lower estimates) for $W(X)$. Example 2.35 shows that $W(C[0,1]) = 1$, but it is an open problem whether or not $W(X) = 1$ in every Banach space X. We shall come back to this in Section 7.6.

Theorem 2.1 (which generalizes Schauder's celebrated fixed point principle [234]) may be found in [71], Theorem 2.2 in [268], [269], and Theorem 2.3 in [230]. Example 2.36 which illustrates the difference between Darbo's and Sadovskij's fixed point theorem is taken from [208]. A systematic account of all these fixed point theorems and their application to differential and integral equations may be found in the books [71] and [286]. There is another remarkable fixed point theorem due to Nussbaum [209] for α-contractive operators on *spheres* which reads as follows.

Theorem 2.5. *Let X be an infinite dimensional Banach space and*

$$F: S_r(X) \to S_r(X)$$

an α-contractive operator. Then F has a fixed point in $S_r(X)$.

The proof of Theorem 2.5 is completely different from that of Theorem 2.1 and requires rather subtle topological arguments, see [209]. We will see in Chapter 10 that Theorem 2.5 may also be obtained as a corollary of the so-called *Birkhoff–Kellogg theorem* on eigenvalues of nonlinear operators. We remark that Theorem 2.5 was extended from α-contractive to condensing operators in [186]. The example of a rotation in $X = \mathbb{R}^2$ shows that Theorem 2.5 is false in finite dimensions.

The spectral sets (2.29) and (2.31) have been studied in detail in [122], the set (2.28) has been introduced in [181], and the set (2.30) has been considered first in [82]. Of course, all scalars λ which belong to one of these sets characterize some "lack of regularity" of the operator $\lambda I - F$; this is made more precise in Section 2.5. In the proofs of most results from Section 2.4 we followed the theses [82] and [155].

Chapter 3

Invertibility of Nonlinear Operators

In this chapter we give several conditions on a nonlinear operator $F: X \to Y$ under which the local invertibility of F implies its global invertibility. The simplest such condition is, by the classical Banach–Mazur lemma, the properness of F which we already introduced in Section 2.3. Closely related properties are ray-properness, coercivity, and ray-coercivity. We establish all possible relations between these and similar notions, and we show by means of some counterexamples that other relations cannot occur.

3.1 Proper and ray-proper operators

As in the preceding chapter, X and Y are Banach spaces, and $F: X \to Y$ is a continuous operator. Recall that F is called *proper* if the pre-image $F^{-}(C) = \{x \in X : F(x) \in C\} \subset X$ of any compact set $C \subset Y$ is compact. Further, we call F *ray-proper* if the pre-image $F^{-}([\theta, y]) \subset X$ of the "ray" $[\theta, y] = \{ty : 0 \leq t \leq 1\}$ is compact for any $y \in Y$. Finally, F is called *closed* if the image $F(M)$ of every closed subset M of X is a closed subset of Y. Some simple but important interconnections between these properties are collected in the following

Theorem 3.1. *With X, Y, and F as above, the following statements are true*:

(a) *Every proper operator F is ray-proper, but not vice versa.*

(b) *F is proper if and only if F is closed and ray-proper.*

(c) *If L is linear, properness of L is equivalent to the existence and boundedness of L^{-1} on $R(L)$.*

Proof. The statement (a) is a simple consequence of the fact that any ray $[\theta, y]$, being a bounded and closed subset of a one-dimensional subspace, is compact. Of course, (a) is also a straightforward consequence of (b).

To prove (b), suppose first that F is proper and $C \subseteq X$ is closed. Let $(y_n)_n = (F(x_n))_n$ be a sequence in $F(C)$ such that $y_n \to y \in Y$ as $n \to \infty$. Then the set $\{y, y_1, y_2, y_3, \dots\}$ is compact, and so is the set $F^{-}(\{y, y_1, y_2, y_3, \dots\})$, by the properness of F. In particular, we may find a subsequence $(x_{n_k})_k$ of $(x_n)_n$ with $x_{n_k} \to x \in C$ as $k \to \infty$. By continuity, we have $F(x) = y$, and so $y \in F(C)$.

Conversely, suppose that F is ray-proper and closed, and let $C \subset Y$ be compact. Let $(A_n)_n$ be a decreasing sequence of nonempty closed subsets of $F^{-}(C)$; we have to show that the sets A_n have a nonempty intersection. Since F is a closed operator, the sets $F(A_n)$ are closed in the compact set C, and so there exists some $y \in C$ which belongs to $F(A_n)$ for all $n \in \mathbb{N}$.

By assumption, the set $F^{-}([\theta, y])$ is compact, and so the sets $B_n := A_n \cap F^{-}([\theta, y])$ form a decreasing sequence of nonempty compact subsets of X. It follows that the sets B_n, hence also the sets A_n, have a nonempty intersection, and so we are done.

It remains to prove (c). Clearly, every linear isomorphism $L: X \to R(L)$ is proper. Conversely, if L is not injective, then L maps a nontrivial subspace into $\{\theta\}$, and so L cannot be proper. Similarly, if L^{-1} exists, but is unbounded, there exists a sequence $(y_n)_n$ in Y such that $y_n \to \theta$, but $\|L^{-1}y_n\| \geq \varepsilon > 0$ for all $n \in \mathbb{N}$. Then the set $C := \{\theta, y_1, y_2, y_3, \dots\} \subset Y$ is compact, but its pre-image $L^{-}(C) \subset X$ is not, and so L is not proper. □

The above properties are of particular interest in invertibility results for nonlinear operators. As a matter of fact, each of these properties is "missing" if an operator is only locally invertible, but not globally invertible (see Theorem 3.2 below for a precise formulation). Results of this type are not only of theoretical interest, but also important in view of applications: thus, to prove the global invertibility of a proper operator simply reduces to proving its local invertibility which may often be achieved by other well-known means (e.g., by the inverse function theorem and its various generalizations).

Before stating a corresponding theorem, we have to recall another notion. Let us call an operator $F: X \to Y$ *ray-invertible* if the following holds: for each $y \in Y$ there exists a continuous path $\gamma: [0, 1] \to X$ such that $F(\gamma(\tau)) = \tau y$ for $0 \leq \tau \leq 1$. Observe that a ray-invertible operator is always onto, since $y = F(\gamma(1)) \in F(X)$ for a suitably chosen path γ. We also point out that, if there exists a path with the required properties, it is always unique, provided that F is a local homeomorphism. To see this, suppose that $\gamma_1, \gamma_2: [0, 1] \to X$ be continuous with $\gamma_1(0) = \gamma_2(0) = 0$ and $F(\gamma_1(\tau)) = F(\gamma(\tau)) = \tau y$ for $0 \leq \tau \leq 1$. Let τ_0 be the largest value such that $\gamma_1(\tau) = \gamma_2(\tau)$ for $\tau < \tau_0$, and assume that $\tau_0 < 1$. Then also $\gamma_1(\tau_0) = \gamma_2(\tau_0)$. Since $F(\gamma_1(\tau)) = F(\gamma_2(\tau))$ and F is a homeomorphism near $\gamma_1(\tau_0) = \gamma_2(\tau_0)$, by assumption, we obtain that $\gamma_1(\tau) = \gamma_2(\tau)$ also for $\tau > \tau_0$ sufficiently close to τ_0. This contradiction shows that $\tau_0 = 1$, i.e., $\gamma_1 = \gamma_2$.

Theorem 3.2. *Let $F: X \to Y$ be an operator satisfying $F(\theta) = \theta$. Then F is a global homeomorphism if and only if any one of the following four conditions is satisfied*:

(a) *F is a local homeomorphism and proper.*

(b) *F is a local homeomorphism and ray-proper.*

(c) *F is a local homeomorphism and closed.*

(d) *F is a local homeomorphism and ray-invertible.*

Proof. We first prove (d) and then show how the other three assertions follow from (d). It is easy to see that every global homeomorphism $F\colon X \to Y$ is ray-invertible. Suppose that $F\colon X \to Y$ is a ray-invertible local homeomorphism. The last property implies that F is an open mapping, and the ray-invertibility implies that F is onto. So we only have to show that F is injective.

Given $\hat{y} \in Y$, we find a (unique) continuous path $\gamma\colon [0,1] \to X$ such that $F(\gamma(\tau)) = \tau\hat{y}$ for $0 \le \tau \le 1$. Fix $x := \gamma(1)$ and choose $\hat{x} \in X$ such that $F(x) = F(\hat{x}) = \hat{y}$. Define $z\colon [0,1] \to X$ by $z(t) := t(\hat{x} - x) + x$, so $z(0) = x$, $z(1) = \hat{x}$, and $F(z(0)) = F(z(1)) = \hat{y}$. Since F is ray-invertible, for each $t \in [0,1]$ we find a (unique) continuous path $\gamma_t\colon [0,1] \to X$ such that $\gamma_t(0) = \theta$ and $F(\gamma_t(\tau)) = \tau F(z(t))$ for $0 \le \tau \le 1$.

Now, for t being sufficiently close to 1, we must have $\gamma_t(1) = z(t)$, because F is a local homeomorphism, and thus $\gamma_1(1) = z(1) = \hat{x}$. On the other hand, the relation $F(\gamma_1(\tau)) = \tau\hat{y}$ and the uniqueness of γ_1 imply that $\gamma_1(\tau) = \gamma(\tau)$ for all $\tau \in [0,1]$. In particular, $\hat{x} = \gamma_1(1) = \gamma(1) = x$, and so we are done.

Let us now show that (b) is a consequence of (d). To this end, it suffices to prove that every ray-proper local homeomorphism is ray-invertible. Fix $y \in Y$. Without loss of generality we may assume that $F(\theta) = \theta$. Since F is a local homeomorphism, we find $\delta > 0$ and a continuous path $\gamma\colon [0,\delta) \to X$ such that $\gamma(0) = \theta$ and $F(\gamma(\tau)) = \tau y$ for $0 \le \tau < \delta$. Denote by $\hat{\delta}$ the largest number such that γ may be extended to $[0,\hat{\delta})$ with $F(\gamma(\tau)) = \tau y$ for $0 \le \tau < \hat{\delta}$; we claim that $\hat{\delta} = 1$.

Suppose that $\hat{\delta} < 1$. Let $(\tau_n)_n$ be some sequence in $[0,\hat{\delta})$ with $\tau_n \to \hat{\delta}$ as $n \to \infty$. Since F is ray-proper, by assumption, the set $F^-([\theta, y])$ is compact, and so the sequence $(\gamma(\tau_n))_n$ admits a convergent subsequence, say $\gamma(\tau_{n_k}) \to \hat{x} \in F^-([\theta, y])$ as $k \to \infty$. The continuity of F implies that $F(\hat{x}) = F(\gamma(\hat{\delta})) = \hat{\delta}y$. Since F is locally invertible, we may extend γ to $[0,\delta)$ for some $\delta > \hat{\delta}$, contradicting the definition of $\hat{\delta}$.

The assertion (a) is an immediate consequence of (b) and the fact that properness implies ray-properness, by Theorem 3.1 (a).

It remains to prove (c). By what we have already proved, it suffices to show that every closed local homeomorphism is ray-invertible. Fix $y \in Y$. If $y = \theta$ it is easy to see that $\gamma(\tau) \equiv \theta$ is the only path satisfying $F(\gamma(\tau)) = \theta$ for $0 \le \tau \le 1$. So let $y \ne \theta$, and define $\gamma\colon [0,\hat{\delta}) \to X$ and $(\tau_n)_n$ as in the proof of (b). We show that the sequence $(\gamma(\tau_n))_n$ again contains a convergent subsequence, and then the remaining part of the proof goes as before.

Suppose that $(\gamma(\tau_n))_n$ contains no convergent subsequence. Then we find $\delta > 0$ such that $\|\gamma(\tau_m) - \gamma(\tau_n)\| \ge \delta$ for $m \ne n$. This implies that the set $C := \{\gamma(\tau_1), \gamma(\tau_2), \gamma(\tau_3), \dots\}$ is closed in X, and so $F(C) = \{\tau_1 y, \tau_2 y, \tau_3 y, \dots\}$ is closed in Y, by assumption. Consequently, $\hat{\delta}y \in F(C)$, and $F(\gamma(\tau_n)) = \tau_n y = \hat{\delta}y$ for some $n \in \mathbb{N}$. But $\hat{\delta} > \tau_n$ for all n, and so $y = \theta$, a contradiction. The proof is complete. □

The standard second-year calculus example of a local homeomorphism which is not globally invertible is $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$F(x_1, x_2) = (e^{x_1} \cos x_2, e^{x_1} \sin x_2). \tag{3.1}$$

Of course, the operator (3.1) is not ray-proper since $F^-([\theta, y]) = (-\infty, 0) \times \{2k\pi : k \in \mathbb{Z}\}$ for $y = (1, 0)$, and thus not proper either. Similarly, F is not closed since $F(\mathbb{R} \times \{0\}) = (0, \infty) \times \{0\}$. Finally, F is not ray-invertible since it is not onto.

Let us now make a comparison of the various conditions on F arising in Theorems 3.1 and 3.2. A careful analysis of the implications of these theorems shows that there are only 10 combinations which do not lead to a contradiction; we collect them in the following table.

Table 3.1

global$\cong$	local$\cong$	proper	ray-proper	ray-invertible	closed	
yes	yes	yes	yes	yes	yes	Example 3.1
no	no	yes	yes	yes	yes	Example 3.2
no	yes	no	no	no	no	Example 3.3
no	no	no	yes	yes	no	Example 3.4
no	no	no	no	yes	yes	Example 3.5
no	no	no	no	yes	no	Example 3.6
no	no	no	yes	no	no	Example 3.7
no	no	no	no	no	yes	Example 3.8
no	no	yes	yes	no	yes	Example 3.9
no	no	no	no	no	no	Example 3.10

To show that these possibilities actually occur, we give a series of 10 examples. The only example which is not straightforward is Example 3.4.

Example 3.1. Let $X = Y = \mathbb{R}$ and $F(x) = x$. Obviously, F has all 6 properties. ♡

Example 3.2. Let $X = \mathbb{R}^3$, $Y = \mathbb{R}^2$, and

$$F(x_1, x_2, x_3) = (r \cos t, r \sin t) \quad (r^2 = x_1^2 + x_2^2 + x_3^2,\ t = \pi x_1/r).$$

Clearly, F is not locally invertible. On the other hand, F is proper, since $\|F(x)\| = \|x\|$ for all $x \in \mathbb{R}^3$. It is also clear that F is closed. ♡

Observe that Example 3.2 is not possible for $Y = \mathbb{R}$, since any continuous surjective operator $F\colon X \to \mathbb{R}$, with $\dim X \geq 2$, has the property that $F^-(\{y\})$ is unbounded for any $y \in \mathbb{R}$.

Example 3.3. Let $X = Y = \mathbb{R}$ and $F(x) = \arctan x$. Then F is locally invertible, but has none of the remaining properties. ♡

Example 3.4. Let $X = Y$ be a real separable Hilbert space with orthonormal basis $\{e_1, e_2, e_3, \dots\}$. For $0 < \delta < \frac{1}{\sqrt{2}}$, let

$$B_n := \{x \in X : \|x - e_n\| \leq \delta\}, \quad B := \bigcup_{n=1}^{\infty} B_n.$$

Obviously, $B_m \cap B_n = \emptyset$ for $m \neq n$. We define F by

$$F(x) := \begin{cases} x & \text{if } x \in X \setminus B, \\ \frac{1}{n}\big(1 - \frac{\|x-e_n\|}{\delta}\big)e_n + \frac{\|x-e_n\|}{\delta}x & \text{if } x \in B_n. \end{cases}$$

Then F is a continuous operator with $F(B_n) = T_n$, where T_n is the "ice cream cone"

$$T_n = \text{co}(\{\tfrac{1}{n}e_n\} \cup B_n) \quad (n = 1, 2, 3, \dots).$$

It is not hard to see that F is neither globally nor locally invertible. Furthermore, F is not proper, since the set $F^-(\{e_1, \frac{1}{2}e_2, \frac{1}{3}e_3, \dots\} \cup \{0\})$ contains the noncompact set $\{e_1, e_2, e_3, \dots\}$. Similarly, F is not closed, since the set $F(\{e_1, e_2, e_3, \dots\}) = \{e_1, \frac{1}{2}e_2, \frac{1}{3}e_3, \dots\}$ is not closed. We claim that F is both ray-proper and ray-invertible.

Fix $y \in Y$. If the ray $[\theta, y]$ does not meet any of the sets T_n, we have $F^-([\theta, y]) = [\theta, y]$, and there is nothing to prove. In the opposite case, we have $[\theta, y] \cap T_n \neq \emptyset$ for just one n. Since T_n is closed and convex we get then $[\theta, y] \cap T_n = \{ty : t_- \leq t \leq t_+\}$ for some scalars $t_-, t_+ \geq 0$.

If $y = \eta e_n$ for some $\eta > 0$, then all points $x \in F^-([\theta, y])$ belong to the ray $\{te_n : 0 \leq t < \infty\}$. By definition, F maps the generic point $x = te_n$ of this ray to the point $F(te_n) = \gamma_n(t)e_n$, where γ_n is the piecewise linear function

$$\gamma_n(t) = \begin{cases} t & \text{if } 0 \leq t \leq 1 - \delta, \\ \frac{1}{n} + \frac{1-n+n\delta}{n\delta}(t-1) & \text{if } 1 - \delta \leq t \leq 1, \\ \frac{1}{n} + \frac{n+n\delta-1}{n\delta}(t-1) & \text{if } 1 \leq t \leq 1 + \delta, \\ t & \text{if } 1 + \delta \leq t < \infty. \end{cases}$$

This shows that, for every $\eta > 0$, the pre-image $F^-([\theta, \eta e_n])$ is some interval containing θ. On the other hand, if y is not of the form $y = \eta e_n$ for some $\eta > 0$, then $F^-([\theta, y])$ is contained in the plane $\Pi = \text{span}\,\{e_n, y\}$. The restriction of F to this plane is the identity outside $B_n \cap \Pi$; in particular, the boundary Σ of $B_n \cap \Pi$ is kept fixed. To understand the action of F on the interior of the ball $B_n \cap \Pi$, observe that the two segments joining e_n with the two points, where the tangent rays starting from $\frac{1}{n}e_n$ hit the boundary Σ, are mapped by F to the segments joining $\frac{1}{n}e_n$ with these boundary points. (Geometrically, F maps the "night-cap" with peak e_n continuously to the "night-cap" with peak $\frac{1}{n}e_n$, keeping the boundary Σ fixed.) This geometric reasoning shows that the pre-image of the portion of $[\theta, y]$ inside $B_n \cap \Pi$ is a continuous curve in $B_n \cap \Pi$ which is mapped 1-1 onto $[\theta, y] \cap B_n \cap \Pi$. Altogether, we conclude that F is both ray-proper and ray-invertible. ♡

Example 3.5. Let $X = \mathbb{R}^2$, $Y = \mathbb{R}$, and $F(x_1, x_2) = x_1$. Then F is closed and ray-invertible, but has none of the remaining properties. ♡

Example 3.6. Let $X = \mathbb{R}^2$, $Y = \mathbb{R}$, and $F(x_1, x_2) = x_1 \cos x_2$. Then F is ray-invertible, but has none of the remaining properties. ♡

Example 3.7. Let $X = Y = \mathbb{R}^2$ and $F = H \circ G$, where

$$G(x_1, x_2) = (|x_1|, \arctan x_2), \qquad H(y_1, y_2) = (e^{y_1} \cos y_2, e^{y_1} \sin y_2).$$

It is not hard to see that F is ray-proper. On the other hand, F is not proper, since the pre-image

$$F^-(S(Y)) = F^-(\{(y_1, y_2) : y_1^2 + y_2^2 = 1\}) = \{0\} \times \mathbb{R}$$

is not compact. Similarly, F is not closed since the set

$$F(\{0\} \times \mathbb{R}) = \{(y_1, y_2) : y_1^2 + y_2^2 = 1,\ y_1 > 0\}$$

is not closed in $\mathbb{R}^2$. It is clear that F is neither ray-invertible nor globally invertible. By Theorem 3.2, F cannot be locally invertible either. ♡

Example 3.8. Let $X = Y = \mathbb{R}$ and $F(x) \equiv 0$. Obviously, F is closed, but has none of the remaining properties. ♡

Example 3.9. Let $X = Y = \mathbb{R}$ and $F(x) = |x|$. Obviously, F is proper and closed, but not ray-invertible. ♡

Example 3.10. Let $X = Y = \mathbb{R}$ and $F(x) = \sin x$. Obviously, F is neither ray-proper nor ray-invertible. To see that F is not closed, consider the discrete set $\{2\pi n + \frac{1}{n} : n \in \mathbb{N}\}$. ♡

It is interesting to compare Example 3.7 with the following (linear!) operator between infinite dimensional spaces:

Example 3.7′. Let $X = Y = C[0, 1]$, equipped with the usual maximum norm, and let F be the operator defined by

$$F(x)(s) = \int_0^s x(t)\, dt. \tag{3.2}$$

The range of F is the subspace $C_0^1[0,1] = \{y \in C^1[0,1] : y(0) = 0\}$ of X, and F is invertible on this subspace with inverse $F^{-1}(y) = y'$. However, F^{-1} is *not* bounded, since we consider Y equipped with the C-norm, and thus F is not proper, by Theorem 3.1 (c). It is easy to check that

$$F^{-}([\theta, y]) = \begin{cases} [\theta, y'] & \text{if } y \in C_0^1[0,1], \\ \{\theta\} & \text{otherwise,} \end{cases}$$

and hence F is ray-proper. On the other hand, F is not closed since $C_0^1[0,1]$ is not closed in Y. Finally, F is not ray-invertible either, since F is not onto. Thus, in this case Theorem 3.2 (a), (c) and (d) do not apply, but Theorem 3.2 (b) does. In fact, F cannot be a local homeomorphism, by Theorem 3.2 (b); this may of course also be verified directly.

Now take $X = C[0,1]$ and $Y = C_0^1[0,1]$ with norm $\|y\|_Y = \|y'\|_X$, and let F be defined again by (3.2). In this case the operator F is not any more compact, but a global homeomorphism with continuous inverse $F^{-1}(y) = y'$. Consequently, F is not only ray-proper now, but even proper, closed, and ray-invertible. ♡

3.2 Coercive and ray-coercive operators

Recall that an operator $F\colon X \to Y$ is called *coercive* if the pre-image $F^{-}(B) \subseteq X$ of any bounded set $B \subset Y$ is bounded in X, and *ray-coercive* if the pre-image $F^{-}([\theta, y])$ of $[\theta, y]$ is bounded in X for any $y \in Y$. It is easy to see that the coercivity of F is equivalent to the condition

$$\lim_{\|x\| \to \infty} \|F(x)\| = \infty, \tag{3.3}$$

which we already used in (2.5).

Theorem 3.3. *With X, Y, and F as above, the following statements are true*:

(a) *Every coercive operator F is ray-coercive, but not vice versa.*

(b) *Every ray-proper operator F is ray-coercive, the converse is true in case $X = \mathbb{R}^m$.*

(c) *Every coercive operator $F\colon \mathbb{R}^m \to Y$ is proper, the converse is true in case $Y = \mathbb{R}^n$.*

(d) *If L is linear, coercivity of L is equivalent to the existence and boundedness of L^{-1} on $R(L)$.*

Proof. The statement (a) is a simple consequence of the fact that any ray $[\theta, y]$ is a bounded subset of Y, while (b) follows from the fact that precompactness implies boundedness, and coincides with boundedness in finite dimensional spaces.

To prove (c), suppose first that $F\colon \mathbb{R}^m \to Y$ is coercive, and let $C \subset Y$ be compact. Then C is bounded and closed, and so is $F^{-}(C)$, by the coerciveness and continuity of F. But then $F^{-}(C)$ is already compact, being contained in a finite dimensional space. The converse also follows from the fact that compact sets and closed bounded sets coincide in finite dimensional Banach spaces.

It remains to prove (d). Clearly, every linear isomorphism $L\colon X \to R(L)$ is coercive. Conversely, if L is not injective, then L maps a nontrivial subspace into $\{\theta\}$, and so L cannot be coercive. Similarly, if L^{-1} exists, but is unbounded, there exists a bounded sequence $(y_n)_n$ in Y such that $\|L^{-1}y_n\| \to \infty$, and so L is not coercive. □

Again, the implications of Theorem 3.3 show that there are precisely 7 combinations which do not lead to a contradiction; we collect them in the following tablet.

Table 3.2

proper	ray-proper	coercive	ray-coercive	
yes	yes	yes	yes	Example 3.1
yes	yes	no	yes	Example 3.11
no	yes	yes	yes	Example 3.12
no	yes	no	yes	Example 3.7
no	no	yes	yes	Example 3.13
no	no	no	yes	Example 3.14
no	no	no	no	Example 3.8

We make some remarks on Table 3.2. The operator F from Example 3.1 is trivially coercive. The operator F from Example 3.7 is ray-coercive, by Theorem 3.3 (b), but not coercive, by Theorem 3.3 (c). The fact that F from Example 3.8 is not ray-coercive is again trivial. The following 4 nontrivial examples fit in the remaining rows in Table 3.2.

Example 3.11. Let $X = Y$ be as in Example 3.4, and put now, for $0 < \delta < \frac{1}{\sqrt{2}}$,

$$B_n := \{x \in X : \|x - ne_n\| \le n\delta\}, \quad B := \bigcup_{n=1}^{\infty} B_n,$$

and

$$K_n := \mathrm{co}(\{e_n\} \cup B_n), \quad K := \bigcup_{n=1}^{\infty} K_n.$$

Obviously, the sets K_n are all closed and satisfy $D(K_m, K_n) \ge \frac{1}{\sqrt{2}} - \delta > 0$ for $m \ne n$, where $D(A, B)$ denotes the distance of the sets A and B. Now we define F by

$$F(x) = \begin{cases} x & \text{if } x \in X \setminus B, \\ \left(1 - \frac{\|x-ne_n\|}{n\delta}\right)e_n + \frac{\|x-ne_n\|}{n\delta}x & \text{if } x \in B_n. \end{cases}$$

Then F is a continuous operator with $F(B_n) = K_n$. Since $F(ne_n) = e_n$, F cannot be coercive. However, F is proper. In fact, let $C \subset Y$ be compact. Then C meets only a finite number of the sets K_n; therefore it suffices to prove that the sets $F^-(C \setminus K)$ and $F^-(C \cap K_n)$ are compact.

By definition of F, we have $F^-(C \setminus K) = C \setminus K$. Moreover, for fixed n we have $F^-(C \cap K_n) \subseteq C \cup B_n$, since $F(B_n) = K_n$. Thus, we only have to show that the set $[F^-(C \cap K_n)] \cap B_n$ is compact.

For fixed n, consider the sequence $(y_k)_k$ defined by

$$y_k := \left(1 - \frac{\|x_k - ne_n\|}{n\delta}\right) e_n + \frac{\|x_k - ne_n\|}{n\delta} x_k$$

in B_n. Suppose that this sequence converges to some $y_* \in Y$. Without loss of generality we may assume that the real sequence $(\xi_k)_k$ with $\xi_k = \|x_k - ne_n\|$ converges to some $\xi \in [0, n\delta]$ as $k \to \infty$. If $\xi = 0$, then $(x_k)_k$ converges to ne_n as $k \to \infty$. If $\xi > 0$, then $\xi_k > 0$ for large k, hence

$$x_k = \frac{n\delta}{\xi_k}\left(y_k - \left[1 - \frac{\xi_k}{n\delta}\right] e_n\right) \to x_* = \frac{n\delta}{\xi}\left(y_* - \left[1 - \frac{\xi}{n\delta}\right] e_n\right)$$

as $k \to \infty$. In any case, $(x_k)_k$ contains a convergent subsequence, and thus the set $[F^-(C \cap K_n)] \cap B_n$ is compact as claimed. The ray-properness and ray-coercivity of F follows from Theorem 3.1 (b) and Theorem 3.3 (b). ♡

Example 3.12. Let $X = Y$ be as in Example 3.4, and let, again for $0 < \delta < \frac{1}{\sqrt{2}}$,

$$K_n := \{x \in X : \|x - \|x\|e_n\| \le \delta\|x\|\}, \quad K := \bigcup_{n=1}^{\infty} K_n.$$

Again, $K_m \cap K_n = \emptyset$ for $m \neq n$. Define $\varphi_n \colon X \setminus \{\theta\} \to \mathbb{R}$ by

$$\varphi_n(x) := \frac{1}{n} + \frac{n-1}{n} \frac{\|x - \|x\|e_n\|}{\delta\|x\|} \quad (n = 1, 2, 3, \dots),$$

and $F \colon X \to X$ by

$$F(x) := \begin{cases} x & \text{if } x \in X \setminus K, \\ \varphi_n(x)x & \text{if } x \in K_n \text{ and } \|x\| \le 1, \\ [\|x\| - 1 + (2 - \|x\|)\varphi_n(x)]\, x & \text{if } x \in K_n \text{ and } 1 \le \|x\| \le 2, \\ x & \text{if } x \in K_n \text{ and } \|x\| \ge 2. \end{cases}$$

Then F is continuous on X. Since $\|F(x)\| = \|x\|$ for $\|x\| \ge 2$, F is certainly coercive. On the other hand, F cannot be proper, because $F(e_n) = \frac{1}{n}e_n$. However, F is ray-proper. In fact, for any $y \in Y$ the pre-image $F^-([\theta, y])$ is bounded, by the coercivity of F, and contained in the ray $\{ty : 0 \le t < \infty\}$. ♡

Example 3.13. Let X and F as in Example 2.46. Being compact, F cannot be proper. On the other hand, F is coercive, since $\|F(x)\| = \|x\|$ for all $x \in X$. A trivial calculation shows that

$$F^{-}([\theta, y]) = \begin{cases} B_r(X) & \text{if } y = re \text{ for some } r > 0, \\ \{\theta\} & \text{otherwise.} \end{cases}$$

Consequently, F is ray-coercive but not ray-proper since X is infinite dimensional. ♡

Example 3.14. Let $X = Y$ be as in Example 3.4, and let $F(x) = \varphi(\|x\|)$, where $\varphi\colon [0, \infty) \to S(X)$ is the continuous vector function given by

$$\varphi(t) = \frac{(n-t)e_n + (t-n+1)e_{n+1}}{\|(n-t)e_n + (t-n+1)e_{n+1}\|} \quad (n-1 \le t \le n;\ n = 1, 2, 3, \ldots).$$

Since $\|F(x)\| = 1$ for all $x \in X$, F cannot be coercive. It is not difficult to check that the function φ is injective. Consequently, for any $y \in Y$ the pre-image $F^{-}(\{y\})$ is either empty or coincides with the sphere $\{x \in X : \|x\| = \varphi^{-1}(y)\}$. This shows that F cannot be ray-proper, let alone proper.

Finally, for any $y \in Y$ the pre-image $F^{-}([\theta, y])$ is again either empty or some sphere, and hence F is ray-coercive. ♡

3.3 Further properties of nonlinear operators

Combining the invertibility results in Theorem 3.2 with some of the properties of the numerical characteristics studied in Chapter 2, one may give further sufficient conditions under which a continuous operator $F\colon X \to Y$ is a global homeomorphism. We state some of theses conditions which will be particularly useful in subsequent chapters.

We begin with a particular case of Theorem 3.2.

Proposition 3.1. *Suppose that X and Y are infinite dimensional Banach spaces, and $F\colon X \to Y$ is a local homeomorphism satisfying $[F]_{\mathrm{a}} > 0$ and $[F]_{\mathrm{q}} > 0$. Then F is a global homeomorphism.*

Proof. The assumptions $[F]_{\mathrm{a}} > 0$ and $[F]_{\mathrm{q}} > 0$ imply that F is proper on the whole space X, by Proposition 2.4 (a). So the statement follows from Theorem 3.2 (a). □

Example 2.33 shows that the estimate $[F]_{\mathrm{a}} > 0$ is not necessary, and so Theorem 3.4 cannot be inverted. Moreover, as mentioned in Theorem 3.3 (c), the coercivity of $F\colon X \to Y$ implies its properness in case $\dim X < \infty$, and Example 3.13 shows that this may fail in case $\dim X = \infty$. The following result describes a class of operators for which coercivity always implies properness on closed bounded sets, no matter what the dimension of the underlying spaces is.

Proposition 3.2. *Let $F: X \to Y$ be a coercive operator which is representable as a sum $F = G + H$ with $[H]_A < [G]_a$. Then F is proper on closed bounded sets.*

Proof. Proposition 2.4 (d) implies that

$$0 < [G]_a - [H]_A \le [G + H]_a = [F]_a,$$

and hence F is proper on closed bounded sets, by Proposition 2.4 (b). □

Proposition 3.2 applies, for example, to operators of the form $F = G + H$, where G is Lipschitz continuous and injective with Lipschitz continuous inverse, and H is compact. In fact, for such operators we have

$$[H]_A = 0 < [G]_{lip} \le [G]_a.$$

In particular, taking $G(x) = \lambda x$ with $\lambda \in \mathbb{K}$ we see that $[G]_q = |\lambda|$ and so $\lambda I - H$ is always proper on the whole space X for H compact and $\lambda \neq 0$. This fact will be used several time in subsequent chapters. Another example of this type is the following.

Example 3.15. Let $X = l_2$ and $F: X \to X$ be defined by

$$F(x_1, x_2, x_3, \dots) := (\|x\|, x_1, x_2, \dots). \tag{3.4}$$

Then F is the sum $F = L + H$ of the linear right-shift operator

$$L(x_1, x_2, x_3, \dots) = (0, x_1, x_2, \dots) \tag{3.5}$$

(see Example 1.4) and the simple nonlinear operator

$$H(x) = \|x\|e, \quad (e := (1, 0, 0, \dots)) \tag{3.6}$$

(see Example 2.46). Since

$$\|F(x) - F(y)\|^2 = (\|x\| - \|y\|)^2 + \|x - y\|^2 \le 2\|x - y\|^2$$

and $\|F(e)\| = \|e + Le\| = \sqrt{2}$, we obviously have $[F]_{Lip} = \sqrt{2}$. It is easy to see that $[F]_Q = [F]_B = [F]_q = [F]_b = \sqrt{2}$ as well. On the other hand, the compactness of the operator (3.6) implies that

$$[H]_A = 0 < 1 = [L]_{lip} = [L]_a = [L]_q = \|L\|,$$

and so $F = L + H$ is proper on the whole space X. ♡

Recall that an operator $F \in \mathfrak{C}(X, Y)$ with $F(\theta) = \theta$ is called 1-*homogeneous* if

$$F(tx) = tF(x) \quad (x \in X,\ t > 0). \tag{3.7}$$

We will study 1-homogeneous and similar operators in detail in Section 9.6. For the time being, we only state a result which shows how to generate from a given operator another 1-homogeneous operator with the same measure of noncompactness.

Proposition 3.3. *Given $F : X \to Y$ with $F(\theta) = \theta$, define for $r > 0$ an operator $F_r : X \to Y$ by*

$$F_r(x) := \begin{cases} \frac{\|x\|}{r} F\left(\frac{rx}{\|x\|}\right) & \text{if } x \neq \theta, \\ \theta & \text{if } x = \theta. \end{cases} \tag{3.8}$$

Then F_r is 1-homogeneous, F_r and F coincide on $S_r(X)$, and

$$[F_r]_A = [F]_A. \tag{3.9}$$

Proof. Since $F_r(x) = \frac{1}{r} F_1(rx)$, we may restrict ourselves to the case $r = 1$, i.e., to the operator

$$F_1(x) := \begin{cases} \|x\| F\left(\frac{x}{\|x\|}\right) & \text{if } x \neq \theta, \\ \theta & \text{if } x = \theta. \end{cases} \tag{3.10}$$

It is clear that F_1 is 1-homogeneous and $F_1(x) \equiv F(x)$ on $S(X)$. Fix $q \in (0, 1)$ and put

$$R_n := \{x \in B(X) : q^{n+1} \leq \|x\| \leq q^n\} \quad (n = 0, 1, 2, \dots).$$

Moreover, we use the abbreviation $\nu(x) := x/\|x\|$ for $x \neq \theta$. Now, for any $M \subseteq R_n$ we have

$$\nu(M) \subseteq \overline{\text{co}}\,(\{\theta\} \cup q^{-(n+1)} M),$$

and so $\alpha(\nu(M)) \leq q^{-(n+1)} \alpha(M)$. Furthermore, from

$$F_1(M) \subseteq \bigcup \left\{ t F(\nu(M)) : q^{n+1} \leq t \leq q^n \right\} \subseteq \overline{\text{co}}\,(\{\theta\} \cup q^n F[\nu(M)])$$

it follows that $\alpha(F_1(M)) \leq q^n \alpha(F[\nu(M)])$. Combining these estimates yields

$$\alpha(F_1(M)) \leq q^n \alpha(F[\nu(M)]) \leq \frac{q^n}{q^{n+1}} [F]_A \alpha(M),$$

and so $[F_1]_A \leq [F]_A$, since $q \in (0, 1)$ was arbitrary. The converse inequality follows from the fact that $F_1(S(X)) = F(S(X))$. □

3.4 The mapping spectrum

This chapter is concerned with several sufficient (and in part also necessary) conditions for the invertibility of a nonlinear operator. In view of the definition (1.5) of the linear spectrum, it could be a tempting idea to define the spectrum of a nonlinear operator $F \in \mathfrak{C}(X)$ simply by

$$\Sigma(F) := \{\lambda \in \mathbb{K} : \lambda I - F \text{ is not a bijection}\}; \tag{3.11}$$

we will call (3.11) the *mapping spectrum* of F in what follows. More precisely, we could study the *injectivity spectrum*

$$\Sigma_i(F) := \{\lambda \in \mathbb{K} : \lambda I - F \text{ is not injective}\} \tag{3.12}$$

and the *surjectivity spectrum*

$$\Sigma_s(F) := \{\lambda \in \mathbb{K} : \lambda I - F \text{ is not surjective}\} \tag{3.13}$$

separately to get some information on $\Sigma(F) = \Sigma_i(F) \cup \Sigma_s(F)$. Of course, for linear operators L we have already studied the spectra

$$\Sigma_i(L) = \sigma_{\mathrm{p}}(L), \quad \Sigma_s(L) = \sigma_\delta(L)$$

in Section 1.3. In the nonlinear case it turns out, however, that this approach is too "naive" to be of any use. In fact, the simple definitions (3.12) and (3.13) make sense only in the linear case, since we have then the very rigid structure of linearity (which guarantees the linearity of the inverse of a linear operator), as well as such powerful tools like the closed graph theorem (which guarantees the boundedness of the inverse of a bounded operator).

We show now by a series of very simple examples that the mapping spectrum (3.11) has *none* of the familiar properties of a spectrum.

Example 3.16. Let $X = \mathbb{R}$ and $F: X \to X$ be defined by

$$F(x) = \sqrt{|x|}. \tag{3.14}$$

Then

$$\Sigma_i(F) = \Sigma(F) = \mathbb{R}, \quad \Sigma_s(F) = \{0\}.$$

So $\Sigma(F)$ may be unbounded. ♡

Example 3.17. Let $X = \mathbb{R}$ and $F: X \to X$ be defined by

$$F(x) = \begin{cases} x & \text{if } |x| > 1, \\ x^2 & \text{if } 0 \le x \le 1, \\ -x^2 & \text{if } -1 \le x \le 0. \end{cases} \tag{3.15}$$

A straightforward calculation shows that

$$\Sigma_i(F) = \Sigma(F) = (0, 1], \quad \Sigma_s(F) = \{1\}.$$

So $\Sigma(F)$ need not be closed. ♡

The surjectivity spectrum of a real function may be calculated in general by means of the following simple observation. Suppose that the limits

$$\lim_{x \to -\infty} \frac{F(x)}{x} =: a_-, \quad \lim_{x \to \infty} \frac{F(x)}{x} =: a_+$$

exist, and put $m := \min\{a_-, a_+\}$ and $M := \max\{a_-, a_+\}$. Then the intermediate value theorem implies that $\Sigma_s(F) = [m, M]$. In particular, $\Sigma_s(F)$ is bounded if and only if F is quasibounded. For instance, we have $m = M = 0$ in Example 3.16, and $m = M = 1$ in Example 3.17.

One of the most important properties of the linear spectrum (1.5) is that it is always nonempty in case $\mathbb{K} = \mathbb{C}$. It is not hard to see that also the mapping spectrum (3.11) is nonempty in case $X = \mathbb{K}$ (i.e., $\dim X = 1$) and $F(\theta) = \theta$. Indeed, we have then the trivial inclusion

$$\left\{ \frac{F(x)}{x} : x \in \mathbb{K},\ x \neq 0 \right\} \subseteq \Sigma_i(F) \subseteq \Sigma(F).$$

However, already in the case $X = \mathbb{C}^2$ this is no more true!

Example 3.18. Let $X = \mathbb{C}^2$ and $F\colon X \to X$ be defined by

$$F(z, w) = (\overline{w}, i\overline{z}). \tag{3.16}$$

Then for any $\lambda \in \mathbb{C}$, the map $(\lambda I - F)(z, w) = (\lambda z - \overline{w}, \lambda w - i\overline{z})$ is a bijection on $\mathbb{C}^2$ with inverse

$$(\lambda I - F)^{-1}(\zeta, \omega) = \left(\frac{\overline{\lambda}\zeta + \overline{\omega}}{i + |\lambda|^2}, -\frac{\overline{\lambda}\omega + i\overline{\zeta}}{i - |\lambda|^2} \right). \tag{3.17}$$

Consequently,

$$\Sigma_i(F) = \Sigma_s(F) = \Sigma(F) = \emptyset$$

in this example. So $\Sigma(F)$ may be empty. ♡

Observe that the square of the operator F in Example 3.18 has the form

$$F^2(z, w) = (-iz, iw),$$

i.e., is *linear*. So we have

$$\Sigma_i(F^2) = \Sigma_s(F^2) = \Sigma(F^2) = \{\pm i\}.$$

Of course, the injectivity spectrum (3.12) is closely related to the *point spectrum*

$$\sigma_{\mathrm{p}}(F) := \{\lambda \in \mathbb{K} : F(x) = \lambda x \text{ for some } x \neq \theta\}, \tag{3.18}$$

which is of course analogous to (1.21). As in the linear case, we will call the elements $\lambda \in \sigma_{\mathrm{p}}(F)$ the *eigenvalues* of F in what follows. Clearly, in case $F(\theta) = \theta$ we have the trivial inclusion

$$\sigma_{\mathrm{p}}(F) \subseteq \Sigma_i(F) \tag{3.19}$$

which, however, may be strict. For instance, we have $\sigma_{\mathrm{p}}(F) = \mathbb{R} \setminus \{0\} \subset \mathbb{R} = \Sigma_i(F)$ in Example 3.16, since every straight line through the origin with slope $\lambda \neq 0$ meets the graph of the function $F(x) = \sqrt{|x|}$ in some nonzero point. Of course, for linear operators L we always have $\sigma_{\mathrm{p}}(L) = \Sigma_i(L)$, by definition.

On the other hand, if F is a nonlinear operator such that $F(\theta) \neq \theta$, the point spectrum (3.18) is in general very large. Indeed, suppose that F maps $\mathbb{R}$ into $\mathbb{R}$ with $F(0) > 0$, say. Given $\varepsilon \in (0, F(0))$, by continuity we may find $\delta > 0$ such that $|F(x) - F(0)| \leq \varepsilon$ for $|x| \leq \delta$. But this means that the ratio $F(x)/x$ attains all values $\lambda \leq -(F(0) + \varepsilon)/\delta$, as well as all values $\lambda \geq (F(0) - \varepsilon)/\delta$, and thus $\sigma_{\mathrm{p}}(F)$ contains two unbounded intervals. Moreover, if X is an arbitrary Banach space, and $F(x) = Lx + z$ is *affine* (i.e., $L \in \mathfrak{L}(X)$ and $z \neq \theta$), then every $\lambda \in \mathbb{K}$ with $|\lambda| > r(L)$ (see (1.8)) belongs to $\sigma_{\mathrm{p}}(F)$.

Let us now take a closer look at the surjectivity spectrum (3.13). Recall that for $z \in X$ we have defined the translation F_z of F in (2.24) by

$$F_z(x) = F(x) + z \tag{3.20}$$

The next result provides a connection between the surjectivity spectrum $\Sigma_s(F)$ and the point spectra $\sigma_{\mathrm{p}}(F_z)$ of all translations (3.20).

Proposition 3.4. *The equality*

$$\mathbb{K} \setminus \Sigma_s(F) = \bigcap_{z \in X \setminus \{-F(\theta)\}} \sigma_{\mathrm{p}}(F_z) \tag{3.21}$$

holds for $F \in \mathfrak{C}(X)$.

Proof. Suppose first that $\lambda \in \sigma_{\mathrm{p}}(F_z)$ for all $z \neq -F(\theta)$. For every such z we may find $x_z \neq \theta$ such that $\lambda x_z = F_z(x_z) = F(x_z) + z$, hence $(\lambda I - F)(x_z) = z$. This shows that $R(\lambda I - F) \supseteq X \setminus \{-F(\theta)\}$. Since trivially also $-F(\theta) \in R(\lambda I - F)$, we see that $\lambda I - F$ is surjective.

Conversely, suppose that $\lambda I - F$ is surjective, and so for every $z \in X$ we find $x_z \in X$ such that $\lambda x_z - F(x_z) = z$. Moreover, if z is different from $-F(\theta)$, then x_z is different from θ. But this means precisely that $x = x_z$ is a nontrivial solution of the eigenvalue equation $F_z(x) = \lambda x$, and thus $\lambda \in \sigma_{\mathrm{p}}(F_z)$. □

The equality (3.21) may be easily illustrated by means of Example 3.16. In fact, the real function $F_z(x) = \sqrt{|x|} + z$ has point spectrum $\sigma_{\mathrm{p}}(F_z) = \mathbb{R} \setminus \{0\}$ for every $z \in \mathbb{R}$, and so $\Sigma_s(F) = \{0\}$.

If X is a Banach space and $L\colon X \to X$ is bounded and linear, the surjectivity spectrum $\Sigma_s(L)$ coincides with the approximate defect spectrum $\sigma_\delta(L)$ introduced in (1.51). In this case formula (3.21) becomes

$$\mathbb{K} \setminus \sigma_\delta(L) = \bigcap_{z \neq \theta} \sigma_{\mathrm{p}}(L_z) \tag{3.22}$$

and gives an interesting link between the approximate defect spectrum of the linear operator L, on the one hand, and the point spectra of the affine operators L_z, on the other.

To illustrate (3.22) consider, for example, the multiplication operator (1.16) in the space $X = l_p$ ($1 \le p \le \infty$). We already know that $\sigma_\delta(L) = \overline{A}$ for $1 \le p < \infty$, and $\sigma_\delta(L) = A$ for $p = \infty$, where $A = \{a_1, a_2, a_3, \dots\}$ (see (1.58) and (1.59)). Now, for $\lambda \in \mathbb{R}$ and $z = (z_1, z_2, z_3, \dots) \neq (0, 0, 0, \dots)$ the equation $L_z x = \lambda x$ has the formal solution $x = (x_1, x_2, x_3, \dots)$ with

$$x_n = \frac{z_n}{\lambda - a_n} \quad (n = 1, 2, 3, \dots),$$

which is nontrivial since $z \neq \theta$. The condition $\lambda \notin A$ is necessary and sufficient for $x \in l_p$ in case $1 \le p < \infty$, while the condition $\lambda \notin \overline{A}$ is necessary and sufficient for $x \in l_\infty$. So we get $\sigma_p(L_z) = \mathbb{R} \setminus A$ and $\sigma_p(L_z) = \mathbb{R} \setminus \overline{A}$, respectively, in accordance with (3.22).

Proposition 3.4 may be used to obtain the following boundedness result for the surjectivity spectrum: the spectrum $\Sigma_s(F)$ is bounded if and only if the sets $\mathbb{K} \setminus \sigma_p(F_z)$ with $z \in X$ are uniformly bounded. Of course, this criterion is not very practicable, since it requires the knowledge of the point spectra $\sigma_p(F_z)$. Instead, we mention another result which is useful for "localizing" $\Sigma_s(F)$. First we need the following proposition which seems to be of independent interest.

Proposition 3.5. *Let X be a real Banach space, and let $F: B_r(X) \to X$ be α-contractive. Suppose that $\lambda \in \mathbb{R}$ satisfies*

$$|\lambda - 1| < 1 - [F]_A. \tag{3.23}$$

Assume that F has no eigenvalues $\mu > \lambda$ with eigenvector on $S_r(X)$, i.e., $F(x) = \mu x$ with $\|x\| = r$ implies that $\mu \le \lambda$. Then there exists an $\hat{x} \in S_r(X)$ with $F(\hat{x}) = \lambda \hat{x}$.

Proof. Consider the operator $G: B_r(X) \to X$ defined by

$$G(x) := F(x) + (1 - \lambda)x.$$

Since $[G]_A \le [F]_A + |1 - \lambda| < 1$, the operator G is α-contractive. Denote by ρ the radial retraction (2.20) of X onto $B_r(X)$. Since ρG is again α-contractive, by (2.21), and maps the ball $B_r(X)$ into itself, Theorem 2.1 implies that there exists $\hat{x} \in B_r(X)$ with $\hat{x} = \rho(G(\hat{x}))$.

We claim that $\hat{x} \in S_r(X)$ and $\hat{x} = G(\hat{x})$, hence $F(\hat{x}) = \lambda \hat{x}$. Indeed, $\|\hat{x}\| < r$ would imply

$$\|\hat{x}\| = \|\rho(G(\hat{x}))\| = r \frac{\|G(\hat{x})\|}{\|G(\hat{x})\|} = r,$$

a contradiction. Suppose now that $\|\hat{x}\| = r$, but $\hat{x} \neq G(\hat{x})$. Then $\|G(\hat{x})\| > r$, hence

$$F(\hat{x}) = \frac{\|G(\hat{x})\|}{r} \hat{x} - (1 - \lambda)\hat{x} =: \mu \hat{x}$$

with $\mu = \frac{1}{r}\|G(\hat{x})\| - 1 + \lambda > \lambda$, contradicting our assumption. □

Observe that condition (3.23) implies $0 < \lambda < 2$. This restriction is due to the fact that the class of α-contractive operators is not closed under linear combinations, but only under convex combinations. If F is compact, (3.23) is of course fulfilled for *each* $\lambda \in (0, 2)$.

We also point out that Proposition 3.5 implies (and so is actually equivalent to) Darbo's fixed point theorem (Theorem 2.1) for balls. In fact, taking $\lambda = 1$ in (3.23) (i.e., $[F]_A < 1$) we see that the relations $F(x) = \mu x$, $\|x\| = r$, and $F(B_r(X)) \subseteq B_r(X)$ imply that $\mu \leq 1$, and so F has a fixed point on $S_r(X)$.

Theorem 3.4. *Let X be a real Banach space, and let $F\colon X \to X$ be α-contractive and quasibounded, i.e., $[F]_A < 1$ and $[F]_Q < \infty$. Then $\lambda I - F$ is surjective for all $\lambda \in \mathbb{R}$ satisfying*

$$\lambda > [F]_Q, \quad |1 - \lambda| < 1 - [F]_A. \tag{3.24}$$

Consequently, $\Sigma_s(F)$ is contained in the union of the two intervals $(-\infty, \max\{[F]_Q, [F]_A\}]$ *and* $[2 - [F]_A, \infty)$.

Proof. Fix $\lambda \in \mathbb{R}$ satisfying (3.24) and $z \in X$; we have to find $\hat{x} \in X$ with $\lambda\hat{x} - F(\hat{x}) = z$. Choose $\rho > 0$ so large that

$$\|z\| \leq (\lambda - [F]_Q)\rho, \tag{3.25}$$

and $r > \rho$ so large that $\|F(x)\| \leq [F]_Q\|x\|$ for $\|x\| \geq r$.

We apply Proposition 3.5 to the translation F_z defined by (3.20) on $S_r(X)$. Obviously, $[F_z]_A = [F]_A < 1$, so F_z is α-contractive, and (3.23) holds for F_z. Moreover, $[F_z]_Q = [F]_Q < \infty$. Suppose that there exist $\mu > \lambda$ and $x \in S_r(X)$ such that $F_z(x) = \mu x$. Then

$$\begin{aligned} 0 = \|\mu x - F_z(x)\| &\geq \|\mu x - F(x)\| - \|z\| \\ &\geq (\lambda - [F]_Q)\|x\| - \|z\| \geq (\lambda - [F]_Q)(r - \rho) > 0, \end{aligned}$$

a contradiction. So from Proposition 3.5 we conclude that there exists $\hat{x} \in S_r(X)$ with $\lambda\hat{x} = F_z(\hat{x})$, hence $\lambda\hat{x} - F(\hat{x}) = z$. □

Observe that in case $[F]_Q < 1$ we may choose $\lambda = 1$ in Theorem 3.4 and obtain Theorem 2.2 as a special case.

Let us now briefly discuss other properties of the mapping spectrum (3.11). The next example shows that the spectral mapping theorem (Theorem 1.1 (h)) fails even for very simple polynomials.

Example 3.19. Let $X = \mathbb{R}$ and $F\colon X \to X$ be defined by

$$F(x) := \begin{cases} 0 & \text{if } x \leq 1, \\ x - 1 & \text{if } 1 \leq x \leq 2, \\ 1 & \text{if } x \geq 2. \end{cases} \tag{3.26}$$

A simple calculation shows that $\Sigma(F) = [0, 1]$, and thus $p(\Sigma(F)) = [0, 1]$ for $p(z) = z^2$, say. On the other hand, the fact that $F^2(x) \equiv 0$ implies that $\Sigma(p(F)) = \{0\}$. ♡

Finally, let us see if the mapping spectrum (3.11) has any semicontinuity properties. Of course, since (3.11) coincides with (1.5) on the algebra $\mathfrak{L}(X)$ of continuous *linear* operators, it is trivially upper semicontinuous there, by Theorem 1.1 (i). We could ask whether or not the mapping spectrum is also upper semicontinuous on the operator classes $\mathfrak{Lip}(X)$, $\mathfrak{Q}(X)$, or $\mathfrak{B}(X)$, equipped with the seminorms (2.1), (2.3), and (2.6), respectively. The following example shows that this is not true.

Example 3.20. Let $X = \mathbb{R}$, and denote, for $0 \leq \delta < \frac{1}{2}$, by p_δ the polynomial function $p_\delta(x) = (x - \delta)^2 + 2\delta - \delta^2$. Let F be defined as in Example 3.17 and put

$$F_\delta(x) := \begin{cases} x & \text{if } x \leq -1, \\ -p_\delta(x) & \text{if } -1 < x < -\delta, \\ 0 & \text{if } -\delta \leq x \leq \delta, \\ p_\delta(x) & \text{if } \delta < x < 1, \\ x & \text{if } x \geq 1. \end{cases}$$

A simple calculation shows that $F_0 = F$ and $[F - F_\delta]_{\text{Lip}} < 2\delta$. Moreover, the open set $G := \mathbb{R} \setminus \{0\}$ contains $\Sigma(F)$, by what we have observed in Example 3.17. On the other hand, $\Sigma(F_\delta)$ is not contained in G for $0 < \delta < \frac{1}{2}$, since F_δ is not a bijection for such δ. This means that the map Σ which associates to each $F \in \mathfrak{Lip}(X)$ its mapping spectrum is not upper semicontinuous on $\mathfrak{Lip}_0(X)$ with norm (2.1) and hence, a fortiori, not upper semicontinuous on $\mathfrak{Q}(X)$ with seminorm (2.3) or $\mathfrak{B}(X)$ with norm (2.6) either. ♡

Let us now see what happens with *compact* nonlinear operators. As we have seen in Example 3.18, the mapping spectrum (3.11) may be empty even for compact operators. However, in infinite dimensional spaces we have the following result which is completely analogous to the linear case (see Theorem 1.2 (d)).

Theorem 3.5. *Let X be an infinite dimensional Banach space and $F \in \mathfrak{K}(X)$. Then $0 \in \Sigma_s(F) \subseteq \Sigma(F)$, hence $\Sigma(F) \neq \emptyset$.*

Proof. The proof is similar to that in the linear case, but we cannot suppose that the inverse of a continuous nonlinear operator is also continuous. Assume that $0 \notin \Sigma_s(F)$. Then F is surjective, and so for every $y \in X$ we have $y \in F(B_n(X))$ for some $n \in \mathbb{N}$. In other words, we have the representation

$$X = \bigcup_{n=1}^{\infty} F(B_n(X)) = \bigcup_{n=1}^{\infty} \overline{F(B_n(X))}.$$

By Baire's category theorem, at least one of the compact sets $\overline{F(B_n(X))}$ contains an interior point, and so X must be finite dimensional. □

Example 3.16 shows that, in contrast to Theorem 1.2 (a), the mapping spectrum of a compact nonlinear operator may be uncountable.

3.5 Excursion: topological degree theory

The fixed point theorems we discussed in Section 2.3 are of fundamental importance in nonlinear analysis. We have proved them by elementary means. Another method for obtaining these and many other existence results builds on a topological method known as *degree theory*, *index theory*, or *rotation of vector fields*. In this section we will briefly discuss some properties of the topological degree for compact or α-contractive perturbations of the identity in Banach spaces.

Let X be a Banach space, $\Omega \subset X$ an open bounded subset, and $F\colon \overline{\Omega} \to X$ an α-contractive (e.g., compact) operator. For every $y \in X \setminus (I-F)(\partial\Omega)$ one may then define an integer $\deg(I-F,\Omega,y)$, the *topological degree* of $I-F$ on $\overline{\Omega}$ with respect to y. This degree has the following characteristic properties.

Property 3.1 (Existence). *If* $\deg(I-F,\Omega,y) \neq 0$*, then the equation* $x - F(x) = y$ *has a solution in* Ω*; in particular,* $\deg(I-F,\Omega,\theta) \neq 0$ *implies that* F *has a fixed point in* Ω*.*

Property 3.2 (Normalization). *One has*

$$\deg(I,\Omega,y) = \begin{cases} 1 & \textit{if } y \in \Omega, \\ 0 & \textit{if } y \notin \overline{\Omega}. \end{cases}$$

Property 3.3 (Excision). *If* $\Omega' \subset \Omega$ *is open such that* $y \notin (I-F)(\overline{\Omega} \setminus \Omega')$*, then*

$$\deg(I-F,\Omega',y) = \deg(I-F,\Omega,y).$$

Property 3.4 (Additivity). *If* $\Omega = \Omega_1 \cup \Omega_2$, $\Omega_1 \cap \Omega_2 = \emptyset$*, and* $y \notin (I-F)(\partial\Omega_1 \cup \partial\Omega_2)$*, then*

$$\deg(I-F,\Omega,y) = \deg(I-F,\Omega_1,y) + \deg(I-F,\Omega_2,y).$$

Property 3.5 (Homotopy). *Suppose that* $H\colon \overline{\Omega} \times [0,1] \to X$ *is* α*-contractive and* $y \notin (I - H(\cdot\,,t))(\partial\Omega)$ *for* $0 \le t \le 1$*. Then* $\deg(I - H(\cdot\,,t),\Omega,y)$ *does not depend on* t*; in particular,*

$$\deg(I - H(\cdot\,,0),\Omega,y) = \deg(I - H(\cdot\,,1),\Omega,y).$$

Property 3.6 (Continuous dependence). *If* $\|y - z\| < \operatorname{dist}(y,(I-F)(\partial\Omega))$*, then*

$$\deg(I-F,\Omega,y) = \deg(I-F,\Omega,z).$$

Property 3.7 (Boundary dependence). *If* $F, G: \overline{\Omega} \to X$ *are* α*-contractive with* $F|_{\partial\Omega} = G|_{\partial\Omega}$ *and* $y \notin (I - F)(\partial\Omega)$*, then*

$$\deg(I - F, \Omega, y) = \deg(I - G, \Omega, y).$$

We point out that these 7 properties are not independent of each other; for example, the additivity property implies the excision property, and the homotopy property implies the boundary dependence property. In general, the homotopy property is by far the most important one, since it makes it possible to calculate the degree of a "complicated" operator through the degree of a "simpler" operator in the same homotopy class. A good example for this is Darbo's (in particular, Schauder's) fixed point principle on balls. In fact, suppose that $F: B_r(X) \to B_r(X)$ is an α-contractive operator. If F has a fixed point $\hat{x} \in S_r(X)$, then we are done. On the other hand, if $F(x) \neq x$ for all $x \in S_r(X)$, then the homotopy $H(x, t) := tF(x)$ is α-contractive on $\overline{\Omega} \times [0, 1]$ and satisfies $x - H(x, t) \neq \theta$ for all $(x, t) \in S_r(X) \times [0, 1]$, and so

$$\begin{aligned}\deg(I - F, B_r^o(X), \theta) &= \deg(I - H(\,\cdot\,, 1), B_r^o(X), \theta)\\ &= \deg(I - H(\,\cdot\,, 0), B_r^o(X), \theta)\\ &= \deg(I, B_r^o(X), \theta)\\ &= 1,\end{aligned}$$

by Properties 3.2 and 3.5. Consequently, Property 3.1 implies that the equation $x - F(x) = \theta$ has a solution $\hat{x} \in B_r^o(X)$. In any case, F has a fixed point in $B_r(X)$.

Apart from fixed point theorems, there are also more sophisticated existence results which may be obtained by degree methods. Among them we mention the famous *Borsuk theorem* which may be stated as follows. Let $\Omega \subset X$ be open, bounded, and symmetric (i.e., $-\Omega \subseteq \Omega$) with $\theta \in \Omega$. Suppose that $F: \overline{\Omega} \to X$ is compact (or α-contractive) and odd on $\partial\Omega$ (i.e., $F(-x) = -F(x)$ for $x \in \partial\Omega$) with $\theta \notin (I - F)(\partial\Omega)$. Then $\deg(I - F, \Omega, \theta)$ is odd (in particular, non-zero).

Although this is not a book on topological methods, let us briefly describe how a degree with the above 7 properties may be constructed explicitly step by step.

Step 1. Usually one starts with a C^1 operator $F: \overline{\Omega} \to \mathbb{R}^n$, where $\Omega \subset \mathbb{R}^n$ is open and bounded. Let $y \in \mathbb{R}^n \setminus F(\partial\Omega)$, and suppose in addition that the Fréchet derivative (Jacobian) $F'(x)$ of F is nonsingular at every point $x \in F^-(y)$. By the inverse function theorem, the set $F^-(y)$ is then discrete, hence finite. So the definition

$$\deg_B(F, \Omega, y) := \sum_{x \in F^-(y)} \operatorname{sign}\det F'(x) \tag{3.27}$$

makes sense and gives an integer, called the *Brouwer degree* of F on $\overline{\Omega}$ with respect to y.

Step 2. Now one drops the additional assumption on the nonsingularity of $F'(x)$ for each $x \in F^{-}(y)$. This is possible by Sard's theorem which asserts that the set of all $y \in \mathbb{R}^n$ for which $F'(x)$ is singular for some $x \in F^{-}(y)$ is a nullset in $\mathbb{R}^n$, and thus has no interior points. In other words, if $F'(x)$ is singular for some $x \in F^{-}(y)$, for each $\delta > 0$ we may find a $z \in \mathbb{R}^n$ such that $\|y - z\| \leq \delta$ and $F'(x)$ is nonsingular for all $x \in F^{-}(z)$. Moreover, one can show that all such points z have the same degree (3.27), provided that $\delta < \operatorname{dist}(y, F(\partial\Omega))$. So the definition

$$\deg_{\mathrm{B}}(F, \Omega, y) := \deg_{\mathrm{B}}(F, \Omega, z) \tag{3.28}$$

makes sense and is again called the Brouwer degree of F on $\overline{\Omega}$ with respect to y.

Step 3. To extend the definition (3.28) to continuous operators F, one uses the fact that, by the Stone–Weierstrass approximation theorem, for each $\delta > 0$ and $F \in \mathfrak{C}(\overline{\Omega}, \mathbb{R}^n)$ one may find a $G \in \mathfrak{C}^1(\overline{\Omega}, \mathbb{R}^n)$ such that $\max\{\|F(x) - G(x)\| : x \in \overline{\Omega}\} \leq \delta$. Moreover, one can show that all such operators G have the same degree (3.28), provided that $\delta < \operatorname{dist}(y, F(\partial\Omega))$. So the definition

$$\deg_{\mathrm{B}}(F, \Omega, y) := \deg_{\mathrm{B}}(G, \Omega, y) \tag{3.29}$$

makes sense and is again called the Brouwer degree of F on $\overline{\Omega}$ with respect to y.

Step 4. Now one considers infinite dimensional Banach spaces X, open and bounded subsets $\Omega \subset X$, compact operators $F\colon \overline{\Omega} \to X$, and points $y \in X \setminus (I - F)(\partial\Omega)$. Using the usual Schauder projections, one may show that for each $\delta > 0$ there exist a finite dimensional subspace $\hat{X}$ of X and a continuous operator $\hat{F}\colon \overline{\Omega} \to \hat{X}$ such that $\sup\{\|F(x) - \hat{F}(x)\| : x \in \overline{\Omega}\} \leq \delta$. Choosing $\hat{X}$ so large that $y \in \hat{X}$ and putting $\hat{\Omega} := \Omega \cap \hat{X}$, one may show then that $y \notin (I - \hat{F})(\partial\hat{\Omega})$, provided that $\delta < \operatorname{dist}(y, (I - F)(\partial\Omega))$. Moreover, $\deg_{\mathrm{B}}(I - \hat{F}, \hat{\Omega}, y)$ is the same for all choices of spaces $\hat{X}$ and operators $\hat{F}$. So the definition

$$\deg_{\mathrm{LS}}(I - F, \Omega, y) := \deg_{\mathrm{B}}(I - \hat{F}, \hat{\Omega}, y) \tag{3.30}$$

makes sense and is called the *Leray–Schauder degree* of $I - F$ on $\overline{\Omega}$ with respect to y.

Step 5. Finally, one may extend the Leray–Schauder degree to α-contractive perturbations of the identity. To this end, one uses the fact that, for every $F\colon \overline{\Omega} \to X$ with $[F]_{\mathrm{A}} < 1$, one may find a compact operator $G\colon \overline{\Omega} \to X$ which is homotopic to F on $\partial\Omega$. So the definition

$$\deg_{\mathrm{NS}}(I - F, \Omega, y) := \deg_{\mathrm{LS}}(I - G, \Omega, y) \tag{3.31}$$

makes sense and is called the *Nussbaum–Sadovskij degree* of $I - F$ on $\overline{\Omega}$ with respect to y.

All the degrees constructed in this way have the seven properties mentioned above. Moreover, one may show that *these properties determine the degree uniquely*, i.e., there is only one degree with these properties. This is a useful result, because there are several ways to construct the topological degree, say, for continuous operators in $\mathbb{R}^n$, but the result will be always the same.

As an example how degree methods apply to existence and uniqueness results in nonlinear analysis, we prove a theorem which will be used in Chapter 5.

Theorem 3.6. *Suppose that* $F: X \to X$ *satisfies* $[F]_A < 1$ *and* $[I - F]_{\text{lip}} > 0$. *Then* $I - F$ *is a global homeomorphism.*

Proof. As we have shown in Proposition 2.1 (a), the hypothesis $[I - F]_{\text{lip}} > 0$ implies that $I - F$ is injective and closed; in particular, the range $R(I - F)$ of $I - F$ is closed in X. We show that $R(I - F)$ is also open in X, and so $I - F$ is surjective.

So we have to show that every $y_0 \in R(I - F)$ is an interior point of $R(I - F)$. Let $x_0 \in X$ be any element in X such that $x_0 - F(x_0) = y_0$. Without loss of generality we may assume that $x_0 = \theta$ and $F(\theta) = \theta$, otherwise we pass from F to the operator $\tilde{F}(x) := F(x + x_0) - F(x_0)$ which also satisfies $[\tilde{F}]_A < 1$ and $[I - \tilde{F}]_{\text{lip}} > 0$. We claim that $(I - F)(B(X))$ contains an open neighbourhood of y_0; this will imply that y_0 is an interior point of $R(I - F)$, and so we are done.

To this end, we are going to prove that

$$\deg(I - F, B^o(X), \theta) \neq 0. \tag{3.32}$$

Consider the homotopy $H: B(X) \times [0, 1] \to X$ defined by

$$H(x, t) := F\Big(\Big(1 - \frac{t}{2}\Big)x\Big) - F\Big(-\frac{t}{2}x\Big) \quad (\|x\| \leq 1,\ 0 \leq t \leq 1).$$

Then $H(\cdot, 0) = F$, $H(\cdot, 1)$ is odd on $B(X)$, and

$$\begin{aligned} \|x - H(x, t)\| &= \Big\|\Big(1 - \frac{t}{2}\Big)x - F\Big(\Big(1 - \frac{t}{2}\Big)x\Big) + \frac{t}{2}x + F\Big(-\frac{t}{2}x\Big)\Big\| \\ &\geq [I - F]_{\text{lip}}\Big\|\Big(1 - \frac{t}{2}\Big)x + \frac{t}{2}x\Big\| = [I - F]_{\text{lip}}\|x\| = [I - F]_{\text{lip}} > 0 \end{aligned}$$

for $x \in S(X)$ and $0 \leq t \leq 1$. Furthermore, it is not hard to see that $\alpha(H(M \times [0, 1])) \leq \alpha(F(M)) \leq [F]_A \alpha(M)$ for $M \subseteq B(X)$. From Property 3.5 of the Nussbaum–Sadovskij degree and Borsuk's theorem for α-contractive operators we deduce that

$$\begin{aligned} \deg(I - F, B^o(X), \theta) &= \deg(I - H(\cdot, 0), B^o(X), \theta) \\ &= \deg(I - H(\cdot, 1), B^o(X), \theta) \equiv 1 \pmod 2, \end{aligned}$$

and so (3.32) follows. Now, since $I - F$ is injective and $(I - F)(\theta) = \theta$, we know that $(I - F)(x) \neq \theta$ for $x \in S(X)$. Moreover, $[I - F]_{\text{lip}} > 0$ implies that $I - F: B(X) \to X$ is a closed operator, and so there exists a $\delta > 0$ such that

$$\inf_{\|x\|=1} \|x - F(x)\| \geq \delta. \tag{3.33}$$

Fix any element $\hat{y}$ in X such that $\|\hat{y}\| < \delta/2$, and consider the homotopy $\hat{H}: B(X) \times [0, 1] \to X$ defined by $\hat{H}(x, t) := F(x) + t\hat{y}$. Then $\hat{H}(\cdot, 0) = F$, $\hat{H}(\cdot, 1) = F + \hat{y}$, and

$$\|x - \hat{H}(x, t)\| \geq \|x - F(x)\| - t\|\hat{y}\| \geq \delta - \|\hat{y}\| > \frac{\delta}{2}$$

for $x \in S(X)$ and $0 \leq t \leq 1$. We conclude that

$$\begin{aligned}\deg(I - F, B^o(X), \hat{y}) &= \deg(I - F - \hat{y}, B^o(X), \theta) \\ &= \deg(I - \hat{H}(\cdot, 1), B^o(X), \theta) \\ &= \deg(I - \hat{H}(\cdot, 0), B^o(X), \theta) \\ &= \deg(I - F, B^o(X), \theta) \\ &\neq 0,\end{aligned}$$

and so we find $\hat{x} \in B(X)$ such that $\hat{x} - F(\hat{x}) = \hat{y}$, by Property 3.1 of the degree. So we have shown that $(I - F)(B(X)) \supseteq B^o_{\delta/2}(X)$, and the proof is complete. □

Let us still discuss another application of the Nussbaum–Sadovskij degree which we will use in Chapter 12. As we have seen in Section 2.4, the spectral sets (2.28)–(2.31) have a natural meaning in the linear case. Thus, the three sets $\sigma_{\mathrm{lip}}(L)$, $\sigma_{\mathrm{b}}(L)$ and $\sigma_{\mathrm{q}}(L)$ all coincide with the approximate point spectrum (1.50), while the set $\sigma_{\mathrm{a}}(L)$ coincides with the "left Fredholm spectrum" (1.65). So, for linear L these spectra give precise information on the solvability of the linear equation

$$\lambda x - Lx = y \quad (y \in X). \tag{3.34}$$

So one could ask to what extent the spectral sets (2.28) - (2.31) also provide information on the solvability of the nonlinear equation

$$\lambda x - F(x) = y \quad (y \in X). \tag{3.35}$$

We already know that $\lambda \notin \sigma_{\mathrm{lip}}(F)$ implies that the operator $\lambda I - F$ is injective (Proposition 2.1 (a) and Proposition 2.5 (a)), and hence that the equation (3.35) has at most one solution for fixed y. On the other hand, the relation $\lambda \notin \sigma_\kappa(F)$ ($\kappa \in \{\mathrm{lip}, \mathrm{q}, \mathrm{b}\}$) does not imply the surjectivity of the operator $\lambda I - F$ even in the linear case, as is shown by the right-shift operator (1.38). However, we may give a positive result for all scalars λ which are "far enough" from the spectral set $\sigma_\kappa(F)$; this result is proved by degree-theoretic methods.

Given any nonempty closed set $\Sigma \subset \mathbb{K}$, we denote by $c_\infty[\Sigma]$ the unbounded connected component of $\mathbb{K} \setminus \Sigma$. Moreover, for $F \in \mathfrak{A}(X)$ we use the shortcut $D_A(F) := \{\lambda \in \mathbb{K} : |\lambda| \leq [F]_{\mathrm{A}}\}$.

Theorem 3.7. *Suppose that $F \in \mathfrak{A}(X) \cap \mathfrak{Q}(X)$. Then the operator $\lambda I - F$ is surjective provided that*

$$\lambda \in c_\infty[\sigma_{\mathrm{q}}(F) \cup D_A(F)], \tag{3.36}$$

and so equation (3.35) *has a solution for all $y \in X$ for λ belonging to the set* (3.36).

Proof. If $\lambda \in \mathbb{K}$ satisfies (3.36), then $|\lambda| > [F]_A$, and so F/λ is α-contractive. Let $y \in X$ be fixed. We show that there exist $R > 0$ such that

$$\deg(I - F/\lambda, B_R^o(X), y) = 1, \tag{3.37}$$

and so $\lambda x - F(x) = \lambda y$ for some $x \in B_R(X)$. Since y was arbitrary, we have proved that $\lambda I - F$ is surjective.

Since $\lambda \notin \sigma_q(F)$, we find some $R_1 > 0$ such that

$$\frac{\|\lambda x - F(x)\|}{\|x\|} \geq [\lambda I - F]_q > 0 \quad (\|x\| \geq R_1).$$

Choosing $R_2 > \lambda\|y\|/[\lambda I - F]_q$ we get for all $t \in [0, 1]$ and $x \in X$ with $\|x\| \geq \max\{R_1, R_2\}$

$$t\|y\| \leq \|y\| < \frac{\|x\|\,[\lambda I - F]_q}{\lambda} \leq \left\| \frac{F(x)}{\lambda} - x \right\|. \tag{3.38}$$

For $R_3 > \max\{R_1, R_2\}$, consider the homotopy $H_1 \colon B_{R_3}(X) \times [0, 1] \to X$ defined by

$$H_1(x, t) := \frac{1}{\lambda} F(x) + ty.$$

Then $H_1(\,\cdot\,, 0) = F/\lambda$, $H_1(\,\cdot\,, 1) = F/\lambda + y$, and $H_1(x, t) \neq x$ on $S_{R_3}(X) \times [0, 1]$, since

$$\|x - H_1(x, t)\| \geq \left\| x - \frac{F(x)}{\lambda} \right\| - t\|y\| > 0 \quad (\|x\| = R_3,\ 0 \leq t \leq 1),$$

by (3.38). Moreover, it is easy to see that $[H_1]_A = [F]_A/|\lambda| < 1$. So from Property 3.5 of the degree we conclude that

$$\begin{aligned} \deg(I - F/\lambda, B_{R_3}^o(X), y) &= \deg(I - F/\lambda - y, B_{R_3}^o(X), \theta) \\ &= \deg(I - H_1(\,\cdot\,, 1), B_{R_3}^o(X), \theta) \\ &= \deg(I - H_1(\,\cdot\,, 0), B_{R_3}^o(X), \theta) \\ &= \deg(I - F/\lambda, B_{R_3}^o(X), \theta). \end{aligned} \tag{3.39}$$

Now choose any point μ in the component (3.36) which satisfies $|\mu| > [F]_Q + 1$. Then we find a continuous path $\gamma : [0, 1] \to c_\infty[\sigma_q(F) \cup D_A(F)]$ such that $\gamma(0) = \lambda$ and $\gamma(1) = \mu$. Observing that $\gamma(t) \neq 0$ for $0 \leq t \leq 1$, we may consider the homotopy $H_2 \colon B_{R_3}(X) \times [0, 1] \to X$ defined by

$$H_2(x, t) := \frac{F(x)}{\gamma(t)}.$$

Clearly, $H_2(\,\cdot\,, 0) = F/\lambda$, $H_2(\,\cdot\,, 1) = F/\mu$, and $\gamma(t) \notin \sigma_q(F)$ for any $t \in [0, 1]$. Choosing $0 < \delta < [\gamma(t)I - F]_q$, we find for each t a radius $R_t > 0$ such that

$$\frac{\|\gamma(t)x - F(x)\|}{\|x\|} > [\gamma(t)I - F]_q - \delta \quad (\|x\| \geq R_t).$$

This implies that

$$\|x - H_2(x,t)\| = \left\| \frac{\gamma(t)x - F(x)}{\gamma(t)} \right\| > \frac{([\gamma(t)I - F]_{\mathrm{q}} - \delta)\|x\|}{|\gamma(t)|} > 0 \quad (\|x\| = R_{\mathrm{t}}).$$

In order to use the homotopy invariance of the degree, however, we have to find a radius $R_4 > 0$ which is *independent* of t and such that $x \neq H_2(x,t)$ on $S_{R_4}(X) \times [0, 1]$. We claim that actually

$$R_4 := \sup_{0 \le t \le 1} R_{\mathrm{t}} < \infty. \tag{3.40}$$

Indeed, suppose that (3.40) is false. Then there exists a sequence $(t_n)_n$ in $[0, 1]$ and an unbounded sequence $(x_n)_n$ in X such that $\|x_n - H_2(x_n, t_n)\| \to 0$ as $n \to \infty$. In other words, we have

$$\|\gamma(t_n)x_n - F(x_n)\| \to 0 \quad (n \to \infty). \tag{3.41}$$

Without loss of generality we may assume that $t_n \to t_*$ for some $t_* \in [0, 1]$. The estimate

$$\begin{aligned} \liminf_{\|x\|\to\infty} \frac{\|\gamma(t_*)x - F(x)\|}{\|x\|} &\le \lim_{n\to\infty} \frac{\|\gamma(t_*)x_n - F(x_n)\|}{\|x_n\|} \\ &\le \lim_{n\to\infty} |\gamma(t_*) - \gamma(t_n)| + \lim_{n\to\infty} \frac{\|\gamma(t_n)x_n - F(x_n)\|}{\|x_n\|} \\ &= 0 \end{aligned}$$

shows that $\gamma(t_*) \in \sigma_{\mathrm{q}}(F)$. But this contradicts the closedness of the spectral set $\sigma_{\mathrm{q}}(F)$ (see Theorem 2.4). So we conclude that

$$\|x - H_2(x,t)\| > 0 \quad (\|x\| = R_4).$$

Moreover, a straightforward but somewhat tedious computation shows that $[H_2]_{\mathrm{A}} < 1$. So from Property 3.5 of the degree we conclude again that

$$\begin{aligned} \deg(I - F/\lambda, B^o_{R_4}(X), \theta) &= \deg(I - H_2(\,\cdot\,, 0), B^o_{R_4}(X), \theta) \\ &= \deg(I - H_2(\,\cdot\,, 1), B^o_{R_4}(X), \theta) \\ &= \deg(I - F/\mu, B^o_{R_4}(X), \theta). \end{aligned} \tag{3.42}$$

Now we still define another homotopy $H_3 \colon B_{R_4} \times [0, 1] \to X$ by $H_3(x,t) := tF(x)/\mu$. Choose $R_5 > 0$ such that $\|F(x)\| \le [F]_{\mathrm{Q}}\,\|x\|$ for $\|x\| \ge R_5$. Then for $\|x\| = R_5$ and $0 \le t \le 1$ we have

$$\|x - H_3(x,t)\| \ge \|x\| - \frac{t}{\mu}\|F(x)\| \ge \|x\| - \frac{\|F(x)\|}{[F]_{\mathrm{Q}} + 1} \ge \|x\| - \frac{[F]_{\mathrm{Q}}}{[F]_{\mathrm{Q}} + 1}\|x\| > 0.$$

Moreover, $[H_3]_{\mathrm{A}} < 1$. So we deduce that

$$\begin{aligned}\deg(I - F/\mu, B^o_{R_5}(X), \theta) &= \deg(I - H_3(\,\cdot\,, 1), B^o_{R_5}(X), \theta) \\ &= \deg(I - H_3(\,\cdot\,, 0), B^o_{R_5}(X), \theta) \\ &= \deg(I, B^o_{R_5}(X), \theta) = 1.\end{aligned} \tag{3.43}$$

Now it suffices to take $R := \max\{R_1, R_2, R_3, R_4, R_5\}$ and to combine (3.39), (3.42), and (3.43). The proof is complete. □

The assertion of Theorem 3.7 may be stated more concisely as inclusion

$$c_\infty[\sigma_{\mathrm{q}}(F) \cup D_A(F)] \subseteq \mathbb{K} \setminus \Sigma_s(F),$$

where $\Sigma_s(F)$ denotes the surjectivity spectrum (3.13).

3.6 Notes, remarks and references

It is evident that invertibility results for nonlinear operators are of interest whenever one has to prove *existence and uniqueness of solutions* to nonlinear equations. Apart from the numerous general results discussed in this chapter, there are more specific constructions related to special classes of operators; for example, we mention the paper [254] for connections with so-called monotone operators which we will consider in more detail in Section 9.4.

Properness is a notion which has been used right from the beginning of nonlinear analysis. The concept of *ray-proper maps* has been introduced, apparently, in [145] (see also [198]), that of *ray-invertible maps* in [83].

The results contained in Theorem 3.2 (a) is sometimes called *Banach–Mazur lemma*, since it was published by Banach and Mazur in the paper [30] in 1934. However, this result has been proved earlier in the finite dimensional case by Hadamard [144] (under the name *monodromy lemma*), and in the infinite dimensional case by Caccioppoli [57]. The other three conditions in Theorem 3.2 are more recent: Theorem 3.2 (b) is due to Nashed and Hernandez [198], Theorem 3.2 (c) to Browder [48], and Theorem 3.2 (d) to Dörfner [83]. Most of the examples presented in Section 3.1 and 3.2 are straightforward, except for Example 3.4 [21], Example 3.7 [145], Example 3.7′ [83], and Examples 3.11–3.14 [21].

Ray-coercive operators have also been introduced in the thesis [145]. In [198] two results on the global invertibility of operators of the form $F = I - K$ with K being compact are given. Our Proposition 3.1 shows that such operators are always globally invertible if they are locally invertible. In particular, the hypothesis of [198, Cor. 3.5 and Cor. 3.7] that F be ray-coercive may be dropped.

As we have shown in Section 3.4, the mapping spectrum (3.11) is natural but useless. All examples which illustrate this are taken from the thesis [155], except for

Example 3.18 which was considered in [130] in a different context. We will return to this disappointing example several times in the following chapters.

All results on the surjectivity spectrum are due to Dörfner [82], except for Proposition 3.4 and Theorem 3.4 which may be found, in a slightly more general form, in [268]. Proposition 3.3 is taken from the survey article [122] and will be used several times in subsequent chapters.

There is a large amount of books on topological methods in nonlinear analysis which contain a part on the construction, properties, and applications of topological degree, e.g., [72], [73], [163]. In the Russian literature, e.g., in the monograph [163], the degree $\deg(F, \Omega, \theta)$ is often called *rotation of the vector field F on* Ω and denoted by $\gamma(F, \Omega)$. Theorem 3.6, which is a typical result based on degree-theoretic arguments, was proved for compact F by Granas [137], [138] and for α-contractive F by Petryshyn [216]. Theorem 3.7 which we have taken from the thesis [155] will be applied to existence results in Chapter 12.

Chapter 4

The Rhodius and Neuberger Spectra

In this chapter we start our investigation of spectra for nonlinear operators. More precisely, we study a spectrum for continuous operators introduced by Rhodius in 1984, and a spectrum for continuously differentiable operators introduced by Neuberger in 1969. In particular, we will be interested in the question what properties of the usual spectrum for bounded linear operators carry over to these spectra.

4.1 The Rhodius spectrum

In view of the importance of spectral theory for linear operators, it is not surprising that several attempts have been made to build a spectral theory also for nonlinear operators. This has often been done by means of the following simple-minded philosophy: keeping in mind that the resolvent set (1.4) of $L \in \mathfrak{L}(X)$ consists of all $\lambda \in \mathbb{K}$ such that $\lambda I - L$ is a bijection and $R(\lambda; L) = (\lambda I - L)^{-1} \in \mathfrak{L}(X)$, just replace $\mathfrak{L}(X)$ by some other class $\mathfrak{M}(X)$ of continuous nonlinear operators which contains the identity I on X. In this way, any such operator class $\mathfrak{M}(X)$ gives rise to a resolvent set

$$\rho(F) := \{\lambda \in \mathbb{K} : \lambda I - F \text{ is bijective and } R(\lambda; F) \in \mathfrak{M}(X)\}, \tag{4.1}$$

where

$$R(\lambda; F) = (\lambda I - F)^{-1} \tag{4.2}$$

is the *nonlinear resolvent operator*, and a spectrum

$$\sigma(F) := \mathbb{K} \setminus \rho(F) \tag{4.3}$$

for $F \in \mathfrak{M}(X)$. In case $F(\theta) = \theta$ we have then

$$\sigma_{\mathrm{p}}(F) \subseteq \Sigma(F) \subseteq \sigma(F),$$

see (3.11), which shows that the spectrum (4.3) always contains the eigenvalues of an operator which keeps zero fixed. As we shall see later, however, the properties of the spectra obtained in this "naive" way heavily depend on the choice of the operator class $\mathfrak{M}(X)$, and actually may be very bad. Later we will discuss other spectra which do not necessarily contain the eigenvalues, but have rather "nice" properties.

In this and the following chapter we will employ the definition (4.3) for $\mathfrak{M}(X)$ being the class of continuous, continuously differentiable, Lipschitz continuous, and linearly bounded operators. We begin with a spectrum whose definition goes back, as far as we know, to Rhodius. Let $\mathfrak{M}(X) = \mathfrak{C}(X)$, i.e., we consider the class of *all* continuous operators F on X. This very natural choice leads to the *Rhodius resolvent set*

$$\rho_{\mathrm{R}}(F) = \{\lambda \in \mathbb{K} : \lambda I - F \text{ is bijective and } R(\lambda; F) \in \mathfrak{C}(X)\} \tag{4.4}$$

and the *Rhodius spectrum*

$$\sigma_{\mathrm{R}}(F) = \mathbb{K} \setminus \rho_{\mathrm{R}}(F). \tag{4.5}$$

Thus, a point $\lambda \in \mathbb{K}$ belongs to $\rho_{\mathrm{R}}(F)$ if and only if $\lambda I - F$ is a (global) *homeomorphism* on X.

It is clear that in case of a bounded linear operator this gives the definition (1.5) of the usual spectrum. So one could expect that at least some of the properties of the linear spectrum carry over to the Rhodius spectrum. Unfortunately, this is not true even for the most elementary properties; we illustrate this with a series of simple examples which we will further use in the sequel.

Example 4.1. Let $X = \mathbb{R}$ and $F(x) = x^n$ with $n \in \mathbb{N}$, $n \geq 2$. Then $\sigma_{\mathrm{R}}(F) = \mathbb{R}$ if n is even, and $\sigma_{\mathrm{R}}(F) = (0, \infty)$ if n is odd. On the other hand, let $X = \mathbb{C}$ and $F(z) = z^n$ with $n \in \mathbb{N}$, $n \geq 2$. Then $\sigma_{\mathrm{R}}(F) = \mathbb{C}$ no matter what n is. Thus, the Rhodius spectrum is in general neither closed nor bounded. ♡

Example 4.2. Let $X = \mathbb{R}$ and $F(x) = \arctan x$. Then $\sigma_{\mathrm{R}}(F) = [0, 1)$. In fact, for any $\lambda \in (0, 1)$ the equation $\lambda x = \arctan x$ has a positive solution, hence $(0, 1) \subseteq \sigma_{\mathrm{R}}(F)$, and F itself is not onto, hence $0 \in \sigma_{\mathrm{R}}(F)$. On the other hand, the function $F_\lambda(x) = \lambda x - \arctan x$ is strictly decreasing for $\lambda < 0$ with $F_\lambda(x) \to \mp\infty$ as $x \to \pm\infty$, but strictly increasing for $\lambda \geq 1$ with $F_\lambda(x) \to \pm\infty$ as $x \to \pm\infty$. Consequently, $(-\infty, 0) \cup [1, \infty) \subseteq \rho_{\mathrm{R}}(F)$, and so $\sigma_{\mathrm{R}}(F) = [0, 1)$. ♡

One might ask if the Rhodius spectrum is always nonempty, as the linear spectrum, in case $\mathbb{K} = \mathbb{C}$. In case $\dim X = 1$, i.e., $X = \mathbb{C}$, and $F(0) = 0$ this is trivially true, since

$$\sigma_{\mathrm{R}}(F) \supseteq \sigma_{\mathrm{p}}(F) = \left\{ \frac{F(z)}{z} : z \in \mathbb{C},\ z \neq 0 \right\}. \tag{4.6}$$

As we have seen, already in case $\dim X = 2$ the mapping spectrum (3.11) may be empty. This is also true for the Rhodius spectrum.

Example 4.3. Let $X = \mathbb{C}^2$ and F be defined as in Example 3.18. Then $\sigma_{\mathrm{R}}(F) = \emptyset$, since the resolvent operator given in (3.17) is continuous on X for any $\lambda \in \mathbb{C}$. ♡

In view of the "bad" properties of the Rhodius spectrum, the question arises whether or not one may recover more properties of the linear spectrum by passing from $\mathfrak{C}(X)$ to a smaller operator class. One possible choice is the class $\mathfrak{C}^1(X)$ of all continuously Fréchet differentiable operators on X which we will study in the following two sections.

4.2 Fréchet differentiable operators

Recall that an operator $F\colon X \to Y$ is called (Fréchet) *differentiable* at $x_0 \in X$ if there is an operator $L \in \mathfrak{L}(X, Y)$ such that

$$\lim_{\|h\|\to 0} \frac{1}{\|h\|} \|F(x_0 + h) - F(x_0) - Lh\| = 0 \quad (h \in X). \tag{4.7}$$

In this case the (unique) linear operator L is called the (Fréchet) *derivative* of F at x_0 and denoted by $F'(x_0)$. If F is differentiable at each point $x \in X$ and the map $x \mapsto F'(x)$ is continuous from X into $\mathfrak{L}(X, Y)$, we write $F \in \mathfrak{C}^1(X, Y)$ and call F *continuously differentiable* as usual. If F is a bijection and both $F \in \mathfrak{C}^1(X, Y)$ and $F^{-1} \in \mathfrak{C}^1(Y, X)$, then F is called a *diffeomorphism*. Again we write $\mathfrak{C}^1(X, X) =: \mathfrak{C}^1(X)$ for short.

For the sake of completeness, let us recall some well-known examples of differentiable operators. First, every second year calculus student knows that an operator $F\colon \mathbb{R}^n \to \mathbb{R}^m$ is continuously differentiable if and only if all partial derivatives $\partial F_j/\partial x_i$ $(i = 1, \ldots, n;\ j = 1, \ldots, m)$ exist and are continuous; in this case,

$$F'(x_0) = \begin{pmatrix} \frac{\partial F_1}{\partial x_1}(x_0) & \frac{\partial F_1}{\partial x_2}(x_0) & \ldots\ldots & \frac{\partial F_1}{\partial x_n}(x_0) \\ \frac{\partial F_2}{\partial x_1}(x_0) & \frac{\partial F_2}{\partial x_2}(x_0) & \ldots\ldots & \frac{\partial F_2}{\partial x_n}(x_0) \\ \ldots & \ldots & \ldots\ldots & \ldots \\ \frac{\partial F_m}{\partial x_1}(x_0) & \frac{\partial F_m}{\partial x_2}(x_0) & \ldots\ldots & \frac{\partial F_m}{\partial x_n}(x_0) \end{pmatrix} \tag{4.8}$$

is a real $m \times n$-matrix. In particular, for $m = 1$ this matrix reduces to the *gradient*

$$\nabla F(x_0) = \left(\frac{\partial F}{\partial x_1}(x_0), \frac{\partial F}{\partial x_2}(x_0), \ldots, \frac{\partial F}{\partial x_n}(x_0) \right).$$

For operators $F\colon \mathbb{C}^n \to \mathbb{C}^m$, however, the requirement to be differentiable is much more restrictive. Consider, for example, the operator $F(z, w) = (\overline{w}, i\overline{z})$ in $X = \mathbb{C}^2$ from Example 3.18. As an operator $F\colon \mathbb{R}^4 \to \mathbb{R}^4$, F has the form

$$F(x, y, u, v) = (u, -v, y, x),$$

and hence, being linear, is differentiable everywhere with

$$F'(x_0, y_0, u_0, v_0) \equiv \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

On the other hand, F is not (complex) differentiable at any (!) point $(z, w) \in \mathbb{C}^2$.

In infinite dimensional spaces, the problem of differentiability becomes still more delicate. We illustrate this by means of a simple operator between Lebesgue spaces.

Example 4.4. Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function, i.e., $f(t,\cdot)$ is continuous on $\mathbb{R}$ for almost all $t \in [0,1]$, and $f(\cdot, u)$ is measurable on $[0,1]$ for all $u \in \mathbb{R}$. Given $p, q \in [1, \infty)$, suppose that f satisfies the growth estimate

$$|f(t,u)| \leq a(t) + b|u|^{p/q}$$

with some function $a \in L_q[0,1]$ and some constant $b \geq 0$. It is well known that this growth estimate is both necessary and sufficient for the continuity and boundedness of the *Nemytskij operator*

$$F(x)(t) = f(t, x(t)) \tag{4.9}$$

generated by f between $X = L_p[0,1]$ and $Y = L_q[0,1]$.

Now suppose, in addition, that the partial derivative $g(t,u) := \partial f(t,u)/\partial u$ of f with respect to the second argument exists and is also a Carathéodory function on $[0,1] \times \mathbb{R}$. We have to distinguish three cases.

First, let $p > q$. Then $F \in \mathfrak{C}^1(X,Y)$ if and only if the Nemytskij operator

$$G(x)(t) = g(t, x(t)) \tag{4.10}$$

generated by g maps $L_p[0,1]$ into $L_{pq/(p-q)}[0,1]$; in this case, the derivative of F at $x_0 \in L_p[0,1]$ is the linear multiplication operator given by

$$F'(x_0)h(t) = G(x_0)h(t) = g(t, x_0(t))h(t) \quad (h \in L_p[0,1]).$$

We have studied the spectral properties of such multiplication operators in Example 1.6. Second, let $p = q$. Then $F \in \mathfrak{C}^1(X,Y)$ if and only if the operator (4.10) maps $L_p[0,1]$ into $L_\infty[0,1]$; in this case, the derivative of F is constant on $L_p[0,1]$, and so $f(t,u)$ must be *linear* in u.

Third, let $p < q$. Then $F \in \mathfrak{C}^1(X,Y)$ if and only if the operator (4.10) is identically zero; in this case, the derivative of F is the zero operator $F'(x) \equiv \Theta$ on $L_p[0,1]$, and so $f(t,u)$ must be *independent of* u. ♡

Suppose that X and Y are two Banach spaces and $F \in \mathfrak{C}^1(X,Y)$. We are interested in the problem which properties of F carry over to its derivative $F'(x)$. The most important of these properties is compactness, as we shall show now. As a matter of fact, a more general statement is true.

Lemma 4.1. *Suppose that* $F : X \to Y$ *is differentiable at* $x_0 \in X$ *and satisfies* $[F]_{\mathrm{A}} < \infty$, *with* $[F]_{\mathrm{A}}$ *being the* α*-norm* (2.13). *Then the estimate*

$$[F'(x_0)]_{\mathrm{A}} \leq [F]_{\mathrm{A}} \tag{4.11}$$

is true. In particular, $F'(x_0)$ *is compact if* F *is.*

Proof. Let $M \subset X$ be bounded and choose $R > 0$ such that $M \subseteq B_R(X)$. Given $\varepsilon > 0$ we may find $\delta \in (0,1]$ such that

$$\|F(x_0 + h) - F(x_0) - F'(x_0)h\| \leq \frac{\varepsilon\|h\|}{2R}$$

for every $h \in X$ with $\|h\| \leq \delta$. So we have $F'(x_0)h = F(x_0+h) - F(x_0) + \omega(x_0; h)$ for all $h \in B_\delta(X)$, where $\|\omega(x_0; h)\| \leq \delta\varepsilon/2R$. Since $\delta R^{-1}M \subseteq B_\delta(X)$, we obtain

$$\begin{aligned}
\alpha(F'(x_0)(\delta R^{-1}M)) &= \alpha(F[\{x_0\} + \delta R^{-1}M] - \{F(x_0)\} + \omega(x_0; \delta R^{-1}M)) \\
&\leq \alpha(F[\{x_0\} + \delta R^{-1}M]) + \alpha(\omega(x_0; \delta R^{-1}M)) \\
&\leq [F]_\mathrm{A}\alpha(\delta R^{-1}M) + \frac{\delta\varepsilon}{R} \\
&= \frac{\delta}{R}[F]_\mathrm{A}\alpha(M) + \frac{\delta\varepsilon}{R} \\
&= \frac{\delta}{R}([F]_\mathrm{A}\alpha(M) + \varepsilon),
\end{aligned}$$

by Proposition 1.1 (b), (c), and (d). Therefore, since $F'(x_0)$ is linear,

$$\alpha(F'(x_0)M) = \frac{R}{\delta}\alpha(F'(x_0)(\delta R^{-1}M)) \leq [F]_\mathrm{A}\alpha(M) + \varepsilon.$$

The assertion now follows from the fact that $\varepsilon > 0$ is arbitrary. □

Interestingly, the inequality (4.11) may be strict. For example, in the space $X = c_0$ of all sequences converging to zero, consider the operator

$$F(x_1, x_2, x_3, \dots) = (x_1^2, x_2^2, x_3^2, \dots). \tag{4.12}$$

This operator is not compact, since it maps each basis element $e_k = (\delta_{k,n})_k$ into itself. In fact, by considering the spheres $S_r(X)$ for large r, it is not hard to see that $[F]_\mathrm{A} = \infty$ which means that F is as non-compact as it could be. On the other hand, the derivative $F'(x)$ at $x = (x_1, x_2, x_3, \dots) \in c_0$ which has the form

$$F'(x)h = F'(x_1, x_2, x_3, \dots)(h_1, h_2, h_3, \dots) = 2(x_1h_1, x_2h_2, x_3h_3, \dots) \quad (h \in c_0)$$

is compact. To see this, it suffices to prove the compactness of $F'(x)$ on the canonical basis elements e_k. In fact, for every $x \in c_0$ we have $F'(x)e_k \to \theta$, as $k \to \infty$, and so $F'(x) \in \mathfrak{KL}(X)$.

For further reference we now state a parallel result to Lemma 4.1 for so-called asymptotic derivatives. Recall that an operator $F\colon X \to Y$ is called *asymptotically linear* if there is an operator $L \in \mathfrak{L}(X, Y)$ such that

$$\lim_{\|x\|\to\infty} \frac{1}{\|x\|}\|F(x) - Lx\| = 0, \tag{4.13}$$

i.e., if $[F-L]_\mathrm{Q} = 0$. In this case the (unique) linear operator L is called the *asymptotic derivative* of F and denoted by $F'(\infty)$. Clearly, we have $\|F'(\infty)\| = [F]_\mathrm{Q}$.

Lemma 4.2. *Suppose that $F\colon X \to Y$ is asymptotically linear and satisfies $[F]_\mathrm{A} < \infty$, with $[F]_\mathrm{A}$ being the α-norm* (2.13). *Then the estimate*

$$[F'(\infty)]_\mathrm{A} \leq [F]_\mathrm{A} \tag{4.14}$$

is true. In particular, $F'(\infty)$ is compact if F is.

Proof. We write $F(x) = F'(\infty)x + \omega(x)$. Let $M \subset X$ be bounded and choose $r > 0$ such that $M \subseteq B_r(X)$. Suppose first that

$$\rho := \inf\{\|x\| : x \in M\} > 0. \tag{4.15}$$

Given $\varepsilon > 0$ we may find $R > 0$ such that

$$\|\omega(x)\| = \|F(x) - F'(\infty)x\| \leq \frac{\varepsilon\|x\|}{2r}$$

for every $x \in X$ with $\|x\| \geq R$. Given $t \geq R/\rho$, we have $\|tx\| = t\|x\| \geq t\rho \geq R$ and thus

$$\|\omega(tx)\| \leq \frac{\varepsilon t\|x\|}{2r} \leq \frac{t\varepsilon}{2}.$$

Consequently,

$$\alpha(\omega(tM)) \leq \operatorname{diam}(tM) \leq t\varepsilon,$$

where $\operatorname{diam} M = \sup\{\|x - y\| : x, y \in M\}$ denotes the diameter of M. We conclude that

$$\begin{aligned} t\alpha(F'(\infty)(M)) &= \alpha(F'(\infty)(tM)) \\ &\leq \alpha(F(tM)) + \alpha(\omega(tM)) \\ &\leq [F]_{\mathrm{A}}\alpha(tM) + t\varepsilon \\ &= t([F]_{\mathrm{A}}\alpha(M)) + \varepsilon. \end{aligned}$$

So we have

$$\alpha(F'(\infty)(M)) \leq [F]_{\mathrm{A}}\alpha(M) + \varepsilon,$$

provided that M satisfies (4.15).

Now let M be an arbitrary bounded subset of X. For $\rho > 0$ we put

$$M_\rho := M \cap B_\rho(X), \quad M^\rho := M \setminus B_\rho(X).$$

On the one hand, we have then

$$\alpha(F'(\infty)(M_\rho)) \leq \|F'(\infty)\|\alpha(M_\rho) \leq 2\|F'(\infty)\|\rho.$$

On the other hand, since M^ρ satisfies (4.15), we have

$$\alpha(F'(\infty)(M^\rho)) \leq [F]_{\mathrm{A}}\alpha(M^\rho) + \varepsilon \leq [F]_{\mathrm{A}}\alpha(M) + \varepsilon.$$

Combining these two estimates we obtain

$$\begin{aligned} \alpha(F'(\infty)(M)) &= \max\{\alpha(F'(\infty)(M_\rho)), \alpha(F'(\infty)(M^\rho))\} \\ &\leq \max\{2\|F'(\infty)\|\rho, [F]_{\mathrm{A}}\alpha(M) + \varepsilon\}. \end{aligned}$$

Since $\varepsilon > 0$ and $\rho > 0$ may be chosen arbitrarily small, we conclude that

$$\alpha(F'(\infty)(M)) \leq [F]_{\mathrm{A}}\alpha(M),$$

which gives (4.14). □

A similar example as (4.12) shows that the assertion of Lemma 4.2 cannot be conversed. For instance, given the scalar function

$$f(t) := \begin{cases} \sqrt{|t|}\,\mathrm{sign}\, t & \text{if } |t| \le 1, \\ t & \text{if } |t| > 1, \end{cases}$$

we may consider the operator

$$F(x_1, x_2, x_3, \dots) = (f(x_1), f(x_2), f(x_3), \dots)$$

again in $X = c_0$, say. Clearly, $[F'(\infty)]_{\mathrm{A}} = [I]_{\mathrm{A}} = 1$. On the other hand, the fact that F maps the sphere $S_{1/n^2}(X)$ onto the sphere $S_{1/n}(X)$ implies that $[F]_{\mathrm{A}} = \infty$.

4.3 The Neuberger spectrum

Now we discuss a spectrum for $F \in \mathfrak{C}^1(X)$ which was introduced by Neuberger in 1969. So, we suppose now that $F\colon X \to X$ admits at each point $x \in X$ a (Fréchet) derivative $F'(x)$ which depends continuously (in the operator norm) on x. We call the set

$$\rho_{\mathrm{N}}(F) = \{\lambda \in \mathbb{K} : \lambda I - F \text{ is bijective and } R(\lambda; F) \in \mathfrak{C}^1(X)\} \tag{4.16}$$

the *Neuberger resolvent set* and its complement

$$\sigma_{\mathrm{N}}(F) = \mathbb{K} \setminus \rho_{\mathrm{N}}(F) \tag{4.17}$$

the *Neuberger spectrum* of F. Thus, a point $\lambda \in \mathbb{K}$ belongs to $\rho_{\mathrm{N}}(F)$ if and only if $\lambda I - F$ is a *diffeomorphism* on X. Obviously, in case of a linear operator F, this again gives the familiar definition of the spectrum.

Since $\mathfrak{C}^1(X) \subseteq \mathfrak{C}(X)$, we have the trivial, though useful inclusions

$$\rho_{\mathrm{N}}(F) \subseteq \rho_{\mathrm{R}}(F), \quad \sigma_{\mathrm{N}}(F) \supseteq \sigma_{\mathrm{R}}(F) \tag{4.18}$$

for $F \in \mathfrak{C}^1(X)$. To illustrate this, let us consider again some of the above examples.

In Example 4.1 we have in case $X = \mathbb{R}$ now $\sigma_{\mathrm{N}}(F) = \mathbb{R}$ if n is even, and $\sigma_{\mathrm{N}}(F) = [0, \infty)$ if n is odd. On the other hand, in case $X = \mathbb{C}$ we have again $\sigma_{\mathrm{N}}(F) = \mathbb{C}$ no matter what n is.

In Example 4.2 we have $\sigma_{\mathrm{N}}(F) \supseteq [0, 1)$, by (4.18). However, for $\lambda = 1$ the map $F_1(x) = x - \arctan x$ has derivative zero at $x = 0$, and thus cannot be a diffeomorphism. On the other hand, the derivative of the map $F_\lambda(x) = \lambda x - \arctan x$ is strictly negative for $\lambda < 0$ and strictly positive for $\lambda > 1$. We conclude that $\sigma_{\mathrm{N}}(F) = [0, 1]$.

Example 4.3 is not applicable, since the map given there is not differentiable at any point $(z, w) \in \mathbb{C}^2$. This is not accidental. In fact, it turns out that the Neuberger spectrum shares a remarkable property with the linear spectrum: it is always nonempty if the underlying space is complex!

Theorem 4.1. *The spectrum* $\sigma_{\mathrm{N}}(F)$ *is nonempty in case* $\mathbb{K} = \mathbb{C}$.

We do not give Neuberger's proof of Theorem 4.1, since we will prove a slightly more general (and precise) result rather easily below. Since the Neuberger spectrum is defined for continuously differentiable operators, one should expect that it might be expressed through the (linear) spectra of the Fréchet derivatives $F'(x)$ of F. This is in fact true; moreover, this fact makes it possible to "calculate" the Neuberger spectrum rather explicitly.

To show this, we return to the definition of *properness* which we studied in Section 3.1. Given an operator $F \in \mathfrak{C}(X)$, let us denote by $\pi(F)$ the set of all $\lambda \in \mathbb{K}$ such that $\lambda I - F$ is *not* proper. For example, as already observed in Section 3.3, for a compact operator F with $[F]_{\mathrm{q}} = 0$ in an infinite dimensional space X we have $\pi(F) = \{0\}$.

Theorem 4.2. *For* $F \in \mathfrak{C}^1(X)$*, the formula*

$$\sigma_{\mathrm{N}}(F) = \pi(F) \cup \bigcup_{x \in X} \sigma(F'(x)) \tag{4.19}$$

holds, where $\sigma(L)$ *on the right-hand side of* (4.19) *denotes the usual spectrum* (1.5) *of a bounded linear operator* L. *In particular,* $\sigma_{\mathrm{N}}(F) \neq \emptyset$ *in case* $\mathbb{K} = \mathbb{C}$.

Proof. Suppose first that $\lambda \in \rho(F'(x))$ for each $x \in X$, and that $\lambda I - F$ is proper, i.e., $\lambda \notin \pi(F)$. From Theorem 3.2 (a) it follows then that $\lambda I - F$ is not only a (global) homeomorphism, but even a diffeomorphism.

Viceversa, suppose that $\lambda \in \rho_{\mathrm{N}}(F)$. Then $\lambda I - F$ is obviously proper, and $(\lambda I - F)'(x_0)$ exists and is an isomorphism for each $x_0 \in X$, by the chain rule. □

We illustrate Theorem 4.2 first by means of a simple operator in finite dimensions, and then by an application to a nonlinear integral equation of Hammerstein type.

Example 4.5. Consider the operator $F\colon \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$F(x, y, z) = (y - x^2, z + 2xy - 2x^3, 2x^2y - x^4 - y^2).$$

A simple calculation shows that the matrix $\lambda I - F'(x_0, y_0, z_0)$ has the form

$$\lambda I - F'(x_0, y_0, z_0) = \begin{pmatrix} \lambda + 2x_0 & 1 & 0 \\ 6x_0^2 - 2y_0 & \lambda - 2x_0 & 1 \\ 4x_0^3 - 4x_0y_0 & 2y_0^2 - 2x_0^2 & \lambda \end{pmatrix}.$$

The determinant of this matrix is $\det(\lambda I - F'(x_0, y_0, z_0)) \equiv \lambda^3$ for *every* $(x_0, y_0, z_0) \in \mathbb{R}^3$, which means that $\lambda = 0$ is the only eigenvalue of the derivative of F at any point. Moreover, $\lambda I - F$ is proper for $\lambda \neq 0$, since F is compact. On the other hand, F is not proper, because F maps the parabola $P = \{(x, x^2, 0) : x \in \mathbb{R}\}$ into $(0, 0, 0)$. So from (4.19) we conclude that

$$\sigma_{\mathrm{N}}(F) = \pi(F) \cup \{0\} = \{0\}$$

in this example. ♡

Example 4.6. In the space $X = C[0, 1]$, consider the *Hammerstein integral operator*

$$H(x)(s) = s^{\alpha} \int_0^1 t^{\beta} \sin x(t)\, dt \quad (0 \le s \le 1), \tag{4.20}$$

where $\alpha, \beta \ge 0$ are fixed. This operator may be written as product $H = KF$ of the (autonomous) Nemytskij operator

$$F(x)(t) = f(x(t)) \tag{4.21}$$

generated by the nonlinearity $f(u) = \sin u$ and the linear *Fredholm integral operator*

$$Ky(s) = \int_0^1 k(s, t)y(t)\, dt \tag{4.22}$$

generated by the (degenerate) kernel $k(s, t) = s^{\alpha}t^{\beta}$. It is clear that the operator (4.20) is compact and satisfies $[H]_{\mathrm{q}} = 0$, because it maps the whole space X into a bounded subset of the one-dimensional subspace spanned by the function $t \mapsto t^{\alpha}$. So $\pi(H) = \{0\}$, and we only have to calculate the spectrum of the derivative $F'(x)$ at $x \in X$.

Since the operator (4.22) is linear, the derivative of $H = KF$ at $x \in X$ is given by

$$H'(x)h(s) = s^{\alpha} \int_0^1 t^{\beta}[\cos x(t)]h(t)\, dt \quad (h \in X). \tag{4.23}$$

Computing the Neuberger spectrum $\sigma_{\mathrm{N}}(H)$ of H directly is not easy, but we may use formula (4.19) here. Having a degenerate kernel $k_x(s, t) = s^{\alpha}t^{\beta} \cos x(t)$, the linear integral operator (4.23) has at most one eigenvalue $\lambda_x \neq 0$. A trivial calculation shows that this eigenvalue is given by

$$\lambda_x = \int_0^1 t^{\alpha+\beta} \cos x(t)\, dt,$$

with corresponding eigenfunction $h(s) = s^{\alpha}$. Now, every such eigenvalue satisfies $|\lambda_x| \le (\alpha+\beta+1)^{-1}$. Conversely, for every $\tau \in [-1, 1]$ we have $\lambda_{\tau} := \tau/(\alpha+\beta+1) \in \sigma(H'(x_{\tau}))$ with $x_{\tau}(t) \equiv \arccos \tau$. This shows that

$$\sigma_{\mathrm{N}}(H) = \left[-\frac{1}{\alpha+\beta+1}, \frac{1}{\alpha+\beta+1}\right] \tag{4.24}$$

in this example. ♡

The following example shows that the Neuberger spectrum need not be closed even in the one-dimensional case, compare this with Example 4.2.

Example 4.7. In $X = \mathbb{R}$, define F on $[0, 1]$ by $F(x) := x$, and on all intervals of the form $[2k, 2k + 1]$ $(k = 1, 2, 3, \dots)$ by $F(x) := \frac{1}{2k}x + 4k^2 - 1$. Extend F on all intervals of the form $[2k - 1, 2k]$ as a C^1-function with strictly positive derivative, and on $(-\infty, 0)$ as an odd function.

Since $F\colon \mathbb{R} \to \mathbb{R}$ is a diffeomorphism, we have $0 \in \rho_{\mathrm{N}}(F)$. On the other hand, since F has constant slope $1/2k$ on $[2k, 2k + 1]$, we conclude that

$$\left\{1, \frac{1}{2}, \frac{1}{4}, \dots, \frac{1}{2k}, \dots\right\} \subseteq \sigma_{\mathrm{N}}(F),$$

and so $\sigma_{\mathrm{N}}(F)$ cannot be closed. ♡

4.4 Special classes of operators

If we impose additional conditions on the operators involved, more can be said about their spectra. First of all, we have a result on compact operators which is an exact analogue to a well-known result in linear spectral theory and may be proved as in Theorem 3.5.

Theorem 4.3. *Suppose that X is infinite dimensional, and $F \in \mathfrak{C}(X) \cap \mathfrak{K}(X)$. Then $0 \in \sigma_{\mathrm{R}}(F)$, and hence the Rhodius spectrum of F is nonempty. A similar result holds for the Neuberger spectrum $\sigma_{\mathrm{N}}(F)$ of an operator $F \in \mathfrak{C}^1(X) \cap \mathfrak{K}(X)$.*

As in the linear case, we call the number

$$r_{\mathrm{N}}(F) := \sup\{|\lambda| : \lambda \in \sigma_{\mathrm{N}}(F)\} \tag{4.25}$$

the *Neuberger spectral radius* of $F \in \mathfrak{C}^1(X)$. Theorem 4.2 gives a close connection between the Neuberger spectrum of a differentiable operator, on the one hand, and the classical spectra of all its derivatives, on the other. The following result is therefore not too surprising.

Theorem 4.4. *Let X be an infinite dimensional Banach space, and suppose that $F \in \mathfrak{C}^1(X)$ is compact and satisfies $[F]_{\mathrm{q}} = 0$. Then the spectrum $\sigma_{\mathrm{N}}(F)$ is bounded if and only if the spectra of all derivatives $\sigma(F'(x))$ are uniformly bounded. In this case the equality*

$$r_{\mathrm{N}}(F) = \sup_{x \in X} r(F'(x)) \tag{4.26}$$

holds, where $r(L)$ on the right-hand side of (4.26) *denotes the usual spectral radius* (1.8) *of a bounded linear operator L.*

Proof. For $F \in \mathfrak{C}^1(X) \cap \mathfrak{K}(X)$ with $[F]_{\mathrm{q}} = 0$ we have $\pi(F) = \{0\}$ and $0 \in \sigma(F'(x))$ for all $x \in X$, by Lemma 4.1. So from Theorem 4.2 we conclude that

$$\sigma_{\mathrm{N}}(F) = \{0\} \cup \bigcup_{x \in X} \sigma(F'(x)) = \bigcup_{x \in X} \sigma(F'(x))$$

which immediately implies the statement. □

Let us now consider operators F with $[F]_{\mathrm{B}} < \infty$, see (2.6). It follows directly from the definition of the spectral sets (2.29), (2.30) and (3.18) that $\sigma_{\mathrm{p}}(F) \subseteq \sigma_{\mathrm{b}}(F)$ and $\sigma_{\mathrm{q}}(F) \subseteq \sigma_{\mathrm{b}}(F)$ for any operator $F \in \mathfrak{B}(X)$. Now, assume that F is Fréchet differentiable at zero, i.e.,

$$F(h) - F'(\theta)h = \omega(h) \quad (h \in X), \tag{4.27}$$

where $\|\omega(h)\| = o(\|h\|)$ for $\|h\| \to 0$. (Recall that $F(\theta) = \theta$ for every $F \in \mathfrak{B}(X)$.) Then we have also $\sigma_{\mathrm{q}}(F'(\theta)) \subseteq \sigma_{\mathrm{b}}(F)$. In fact, for $\lambda \in \sigma_{\mathrm{q}}(F'(\theta))$ we can find a sequence $(e_n)_n$ in $S(X)$ such that $\|\lambda e_n - F'(\theta)e_n\| \to 0$ as $n \to \infty$. So for the elements $h_n := e_n/n$ we get

$$\lim_{n\to\infty} \frac{\|\lambda h_n - F(h_n)\|}{\|h_n\|} = \lim_{n\to\infty} \left\| (\lambda I - F'(\theta))e_n + \frac{\omega(h_n)}{\|h_n\|} \right\| = 0,$$

with ω as above, and so $[\lambda I - F]_{\mathrm{b}} = 0$. Consequently, we have proved the inclusion

$$\sigma_{\mathrm{p}}(F) \cup \sigma_{\mathrm{q}}(F) \cup \sigma_{\mathrm{q}}(F'(\theta)) \subseteq \sigma_{\mathrm{b}}(F). \tag{4.28}$$

We show now that we have even equality in (4.28) if F is in addition compact.

Proposition 4.1. *Let $F \in \mathfrak{B}(X)$ be compact, and suppose that the derivative $F'(\theta)$ of F at zero exists. Then the equality*

$$\sigma_{\mathrm{p}}(F) \cup \sigma_{\mathrm{q}}(F) \cup \sigma_{\mathrm{q}}(F'(\theta)) = \sigma_{\mathrm{b}}(F) \tag{4.29}$$

holds true.

Proof. Given $\lambda \in \sigma_{\mathrm{b}}(F)$, we distinguish three cases. First, suppose that there exists a sequence $(x_n)_n$ in $X \setminus \{\theta\}$ such that $\|x_n\| \to 0$ and

$$\frac{\|\lambda x_n - F(x_n)\|}{\|x_n\|} \to 0 \quad (n \to \infty). \tag{4.30}$$

Then we have, with $e_n := x_n/\|x_n\|$ and ω as in (4.27),

$$\begin{aligned} \|(\lambda I - F'(\theta))e_n\| &\le \frac{1}{\|x_n\|} \|(\lambda I - F)(x_n) + \omega(x_n)\| \\ &\le \frac{\|\lambda x_n - F(x_n)\|}{\|x_n\|} + \frac{\|\omega(x_n)\|}{\|x_n\|} \to 0, \end{aligned}$$

as $n \to \infty$, hence $\lambda \in \sigma_q(F'(\theta))$.

Second, suppose that there exists an unbounded sequence $(x_n)_n$ in X with (4.30). Then we immediately get $[\lambda I - F]_q = 0$, hence $\lambda \in \sigma_q(F)$.

Third, suppose that there exists a sequence $(x_n)_n$ in X such that $0 < c_1 \leq \|x_n\| \leq c_2 < \infty$ and (4.30) holds. Then

$$\frac{\|\lambda x_n - F(x_n)\|}{\|x_n\|} \geq \frac{1}{c_2}\|\lambda x_n - F(x_n)\|. \tag{4.31}$$

Since F is compact, we find a subsequence $(x_{n_k})_k$ of $(x_n)_n$ such that $F(x_{n_k}) \to y_*$, as $k \to \infty$, for some $y_* \in X$. Given $\varepsilon > 0$, by (4.31) we may choose $n' \in \mathbb{N}$ such that

$$\frac{1}{c_2}\|\lambda x_{n_k} - F(x_{n_k})\| \leq \frac{\varepsilon}{2} \quad (n_k \geq n').$$

Moreover, we find $n'' \in \mathbb{N}$ such that

$$\|F(x_{n_k}) - y_*\| \leq \frac{c_2\varepsilon}{2} \quad (n_k \geq n''),$$

since $F(x_{n_k}) \to y_*$. So for $n \geq \max\{n', n''\}$ we have

$$\|\lambda x_{n_k} - y_*\| - \frac{c_2\varepsilon}{2} \leq |\,\|\lambda x_{n_k} - y_*\| - \|F(x_{n_k}) - y_*\|\,| \leq \|\lambda x_{n_k} - F(x_{n_k})\| \leq \frac{c_2\varepsilon}{2},$$

hence $\|\lambda x_{n_k} - y_*\| \leq c_2\varepsilon$. Now we distinguish the cases $\lambda = 0$ and $\lambda \neq 0$. In the first case we have $\lambda \in \sigma_q(F'(\theta))$ if X is infinite dimensional, because $F'(\theta)$ is compact (see Lemma 4.1). Similarly, we have $\lambda \in \sigma_p(F)$ if X is finite dimensional, because we may then assume that $x_{n_k} \to x_*$ for some $x_* \in X \setminus \{\theta\}$ with $F(x_*) = \theta$. On the other hand, in case $\lambda \neq 0$ we know that $x_{n_k} \to y_*/\lambda$ as $k \to \infty$, and so $y_\lambda := y_*/\lambda$ satisfies

$$F(y_\lambda) = y_* = \lambda y_\lambda,$$

which shows that $\lambda \in \sigma_p(F)$. This completes the proof. □

We illustrate the equality with an example which we already considered before in a more general form.

Example 4.8. In $X = C[0, 1]$, consider the operator (4.20) for $\alpha = \beta + 1$, i.e.,

$$H(x)(s) = s^{\beta+1}\int_0^1 t^\beta \sin x(t)\,dt \quad (0 \leq s \leq 1). \tag{4.32}$$

Since $[H]_Q = 0$ and $F(x_n) \equiv \theta$ for $x_n(t) \equiv n\pi$, we immediately get $\sigma_q(H) = \{0\}$. This shows also that $0 \in \sigma_p(H)$.

To calculate the whole point spectrum $\sigma_p(H)$, observe first that the eigenvalue equation $H(x) = \lambda x$ with $\lambda \neq 0$ implies that $x(t) = ct^{\beta+1}$ for some $c \neq 0$, and thus

$$\lambda = \frac{1}{c}\int_0^1 t^\beta \sin(ct^{\beta+1})\,dt =: \psi(c). \tag{4.33}$$

Conversely, every function $x(t) = ct^{\beta+1}$ with $c \neq 0$ is certainly an eigenfunction of H corresponding to the eigenvalue $\lambda = \psi(c)$. We conclude that

$$\sigma_{\mathrm{p}}(H) = \{\psi(c) : c \in \mathbb{R} \setminus \{0\}\}. \tag{4.34}$$

This can be made more explicit. After the substitution $ct^{\beta+1} =: \tau$ the relation (4.33) becomes

$$\lambda = \psi(c) = \frac{1}{(\beta+1)c^2} \int_0^1 \sin \tau \, d\tau = \frac{1 - \cos c}{(\beta+1)c^2} \quad (c \neq 0). \tag{4.35}$$

Applying L'Hospital's rule to (4.35) one sees that the function ψ may be extended continuously to the whole real line by putting $\psi(0) := 1/2(\beta+1)$. Moreover, the mean value theorem implies that

$$0 \leq \psi(c) \leq \frac{1}{2(\beta+1)} \quad (-\infty < c < \infty).$$

The right endpoint $1/2(\beta+1)$, however, cannot be assumed for any non-zero c, because $\psi(c) = 1/2(\beta+1)$ implies

$$\frac{1}{2(\beta+1)} = \frac{1 - \cos c}{(\beta+1)c^2} = \frac{2\sin^2(c/2)}{(\beta+1)c^2},$$

hence $\sin(c/2) = c/2$ and so $c = 0$. Summarizing we have proved that

$$\sigma_{\mathrm{p}}(H) = [0, 1/2(\beta+1)). \tag{4.36}$$

It remains to calculate $\sigma_{\mathrm{q}}(H'(\theta))$. In Example 4.6 we have already seen that $H'(\theta)$ is the compact linear integral operator

$$H'(\theta)h(s) = s^{\beta+1} \int_0^1 t^{\beta} h(t) \, dt \quad (h \in X). \tag{4.37}$$

A simple calculation shows that $\sigma_{\mathrm{q}}(H) = \sigma(H) = \{0, 1/2(\beta+1)\}$. Indeed, $\lambda = 1/2(\beta+1)$ is an eigenvalue of (4.37) with eigenfunction $h(t) = t^{\beta+1}$, while $\lambda = 0$ belongs to the residual spectrum $\sigma_{\mathrm{r}}(H'(\theta))$ of (4.37), since the range of $H'(\theta)$ is not dense in X.

So from Proposition 4.1 we conclude that

$$\sigma_{\mathrm{b}}(H) = [0, 1/2(\beta+1)]. \tag{4.38}$$

We remark that a direct verification of (4.38) would have been more difficult. ♡

4.5 Notes, remarks and references

The *Rhodius spectrum* (4.5) has been introduced in [226], but it has not been used in the literature, as far as we know. This is probably due to its bad analytical properties. On the other hand, the *Neuberger spectrum* (4.17) seems to be useful in solvability of certain operator equations [204]. Theorem 4.1 is given in [204], Theorem 4.2 and Example 4.5 in [15], see also [8].

Example 4.3 which shows that the Rhodius spectrum may be empty is due to Georg and Martelli [130], as already observed in the previous chapter. We point out again that the operator F from Example 4.3 is not (complex) differentiable; this explains the exceptional role of Theorem 4.1 in nonlinear spectral theory. In fact, we shall see that all spectra we are going to study in subsequent chapters are empty for the operator from Example 4.3.

The differentiability results for the Nemytskij operator (4.9) between Lebesgue spaces are discussed in the book [161], see also [20], [164]. This shows that the smoothness properties of the operator F are by no means consequences of corresponding smoothness properties of the generating function f, in contrast to what is often tacitly assumed.

Much information on Fréchet differentiable operators may be found in the monograph [184]. In particular, our Lemma 4.1 is Proposition 6.5 in Section 3.6 of [184], while Lemma 4.2 is (a variant of) Lemma 4.2 in Section 4.4 of [184]. The notion of asymptotically linear operators seems to be due to Krasnosel'skij [161], an application to fixed point theorems in cones may be found in [3].

We point out that the chain of inequalities

$$[F]_{\mathrm{a}} \leq [F'(\infty)]_{\mathrm{a}} \leq [F'(\infty)]_{\mathrm{A}} \leq [F]_{\mathrm{A}}$$

was proved in [103] for an asymptotically linear operator F. Moreover, it was shown there that, if F is in addition coercive (see (2.5)), then

$$r_\kappa(F'(\infty)) \leq \lim_{n\to\infty} \sqrt[n]{[F^n]_{\mathrm{A}}},$$

where $r_\kappa(L)$ denotes any of the radii (1.74) of the essential spectra considered in Section 1.4.

Example 4.6 is taken from [15], the Examples 4.5 and 4.7 are of course elementary. One might ask if the set $\pi(F)$ defined before Theorem 4.2 is always closed, as the other spectral sets are. The following example which was communicated to the authors by Väth shows that the answer is negative.

Example 4.9. Let X be an infinite dimensional Banach space, and let $F\colon X \to X$ be defined by

$$F(x) = e^{-\|x\|}x.$$

Since $F^-(K) \subseteq \mathrm{co}\,(K \cup \{\theta\})$ for any set $K \subseteq X$, F is certainly proper, i.e., $0 \notin \pi(F)$. On the other hand, fix $\lambda \in (0, 1)$ and take $\rho := -\log\lambda$. Then $\|x\| = \rho$ implies

$\lambda x - F(x) = \lambda x - e^{-\rho}x = \theta$ which shows that $(\lambda I - F)^{-}(\{\theta\}) \supseteq S_\rho(X)$. So $\lambda I - F$ cannot be proper, i.e., $(0, 1) \subseteq \pi(F)$. ♡

Theorem 4.3 is of course very similar to Theorem 3.7 and may be proved in the same way. Theorem 4.4 is taken from [15], where the more general estimate

$$r_{\mathrm{N}}(F) \le \max\left\{[F]_{\mathrm{A}},\ \sup_{x\in X} r(F'(x))\right\} \tag{4.39}$$

was proved for $F \in \mathfrak{C}^1(X) \cap \mathfrak{A}(X)$, rather than $F \in \mathfrak{C}^1(X) \cap \mathfrak{K}(X)$. Proposition 4.1 is taken from the thesis [82] which contains more results on the Neuberger spectrum.

Differentiability of a nonlinear operator is, by its very nature, a local property. So it is not surprising that one could try to "localize" the spectrum of a differentiable operator in some sense. This has been done indeed by May [189] in the following way. Denoting $B_r(x) = \{y \in X : \|x - y\| \le r\}$, the *local resolvent set* $\rho(F; x)$ of F at $x \in X$ contains, by definition, all $\lambda \in \mathbb{K}$ such that there exist $\delta, \varepsilon > 0$ with the property that $(\lambda I - F)|_{B_\delta(x)}$ is injective, $(\lambda I - F)(B_\delta(x)) \supseteq B_\varepsilon(\lambda x - F(x))$, and the inverse operator

$$(\lambda I - F)|_{B_\delta(x)}^{-1} : B_\varepsilon(\lambda x - F(x)) \to B_\delta(x)$$

is Lipschitz continuous. The *local spectrum* $\sigma(F; x)$ of F at x is the complement $\mathbb{K} \setminus \rho(F; x)$. It is then proved in [189] that $\sigma(F; x)$ is always bounded, provided that F is locally Lipschitz at x. Moreover, if $F'(x)$ exists and is continuous at x, then $\sigma(F; x) \subseteq \sigma(F'(x))$, with equality in case of a finite dimensional space X. Spectra for globally Lipschitz continuous operators will be discussed in detail in the next chapter.

Another notion which is defined through the Fréchet derivative is that of *nonlinear Fredholm operators*. An operator $F \in \mathfrak{C}^1(X)$ is called *Fredholm* if the derivative $F'(x)$ is a linear Fredholm operator for every $x \in X$. Since the index of a Fredholm operator is constant on connected components, the number ind $F'(x)$ does not depend on $x \in X$. So we may call the number

$$\operatorname{ind} F := \operatorname{ind} F'(\theta)$$

the *index* of F. At this point one may introduce analogues to the essential spectra in Wolf's and Schechter's sense as in (1.67) and (1.68) by putting

$$\sigma_{\mathrm{ew}}(F) := \sigma_{\mathrm{ew}}(F'(\theta)), \quad \sigma_{\mathrm{es}}(F) := \sigma_{\mathrm{es}}(F'(\theta)).$$

As far as we know, however, such a theory of essential spectra has not been studied yet in the literature.

Chapter 5

The Kachurovskij and Dörfner Spectra

In this chapter we discuss a spectrum for Lipschitz continuous operators which was defined by Kachurovskij in 1969, as well as a spectrum for linearly bounded operators introduced by Dörfner in 1997. In contrast to the two spectra considered in the previous chapter, the Kachurovskij spectrum is always compact; however, it may be empty. In the last part of this chapter we study some continuity properties of nonlinear spectra, viewed as multivalued maps, and of the corresponding nonlinear resolvent operators.

5.1 Lipschitz continuous operators

Let X and Y be Banach spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. As in Section 2.1 we denote by $\mathfrak{Lip}(X, Y)$ the set of all Lipschitz continuous operators $F\colon X \to Y$; in particular, $\mathfrak{Lip}(X) := \mathfrak{Lip}(X, X)$. Equipped with the norm

$$\|F\|_{\mathrm{Lip}} = \|F(\theta)\| + [F]_{\mathrm{Lip}}, \tag{5.1}$$

where

$$[F]_{\mathrm{Lip}} = \sup_{x \neq y} \frac{\|F(x) - F(y)\|}{\|x - y\|} \tag{5.2}$$

as in (2.1), the set $\mathfrak{Lip}(X, Y)$ becomes a Banach space. It is easy to see that the space $\mathfrak{L}(X, Y)$ of all bounded linear operators from X into Y is a closed subspace of $\mathfrak{Lip}(X, Y)$, and (5.1) is the usual operator norm on $\mathfrak{L}(X, Y)$. This close connection between the spaces $\mathfrak{L}(X, Y)$ and $\mathfrak{Lip}(X, Y)$ may be the reason for the fact that nonlinear spectral theory is most advanced for Lipschitz continuous operators. In this section we collect some useful information on the operator class $\mathfrak{Lip}(X, Y)$.

Proposition 2.1 shows that the minimal Lipschitz constant (5.2) on the class $\mathfrak{Lip}(X)$ has similar properties as the usual operator norm on the algebra $\mathfrak{L}(X)$. The following important theorem shows that there is also an exact analogue to the fact that "small" perturbations of the identity are invertible. Here and in the sequel we call a bijection $F\colon X \to Y$ a *lipeomorphism* if both $F \in \mathfrak{Lip}(X, Y)$ and $F^{-1} \in \mathfrak{Lip}(Y, X)$. Thus, a bijection F is a lipeomorphism if and only if $[F]_{\mathrm{Lip}} < \infty$ and $[F]_{\mathrm{lip}} > 0$, with $[F]_{\mathrm{lip}}$ given by (2.2).

Proposition 5.1. *Let X be a Banach space and $F \in \mathfrak{Lip}(X)$ with $[F]_{\mathrm{Lip}} < 1$. Then $I - F$ is a lipeomorphism, and $(I - F)^{-1}(y)$ may be obtained, for any $y \in X$, as* *limit of the successive approximations*

$$x_0 = \theta,\ x_1 = F(x_0) + y, \ldots, x_{n+1} = F(x_n) + y. \tag{5.3}$$

Moreover, the estimate

$$[(I - F)^{-1}]_{\mathrm{Lip}} \leq \frac{1}{1 - [F]_{\mathrm{Lip}}} \tag{5.4}$$

holds true.

Proof. For fixed $z \in X$, define $F_z : X \to X$ as in (2.24). Since $[F_z]_{\mathrm{Lip}} = [F]_{\mathrm{Lip}} < 1$, the Banach contraction mapping principle implies that the equation $x - F(x) = z$ has a unique solution $x \in X$. This shows that $(I - F)^{-1}$ exists and satisfies

$$(I - F)^{-1}(z) = F((I - F)^{-1}(z)) + z \quad (z \in X). \tag{5.5}$$

The sequence (5.3) is nothing else but the usual successive approximations by iterates of the contraction F_z.

Now, from (5.5) we deduce that, for $y, z \in X$,

$$\begin{aligned}\|(I - F)^{-1}(y) - &(I - F)^{-1}(z)\| \\ &\leq \|F((I - F)^{-1}(y)) - F((I - F)^{-1}(z))\| + \|y - z\| \\ &\leq [F]_{\mathrm{Lip}} \|(I - F)^{-1}(y) - (I - F)^{-1}(z)\| + \|y - z\|,\end{aligned}$$

hence

$$\|(I - F)^{-1}(y) - (I - F)^{-1}(z)\| \leq \frac{1}{1 - [F]_{\mathrm{Lip}}} \|y - z\| \tag{5.6}$$

which proves (5.4). □

It is well known that, for bounded linear operators L, the condition $\|L^m\| < 1$ for some $m \in \mathbb{N}$ suffices for the existence of $(I - L)^{-1}$. Example 3.19 shows that this is not true for Lipschitz continuous nonlinear operators. Indeed, in that example we have $[F]_{\mathrm{Lip}} = 1$, but $[F^m]_{\mathrm{Lip}} = 0$ for all $m \geq 2$. Since the function $I - F$ is constant on $[1, 2]$, it cannot be invertible.

As in the linear case, we get the following important perturbation result as a corollary from Proposition 5.1.

Proposition 5.2. *Let X be a Banach space and $F : X \to X$ a lipeomorphism. Suppose that $G \in \mathfrak{Lip}(X)$ satisfies $[G]_{\mathrm{Lip}} < [F]_{\mathrm{lip}}$. Then $F + G$ is also a lipeomorphism and*

$$[(F + G)^{-1}]_{\mathrm{Lip}} \leq \frac{[F^{-1}]_{\mathrm{Lip}}}{1 - [G]_{\mathrm{Lip}}[F^{-1}]_{\mathrm{Lip}}} = \frac{1}{[F]_{\mathrm{lip}} - [G]_{\mathrm{Lip}}}. \tag{5.7}$$

In particular, the set of lipeomorphisms on X is open in the normed space $\mathfrak{Lip}(X)$.

Proof. Note that $F + G = (I + GF^{-1})F$, since composition on the right distributes over addition. From our hypothesis $[GF^{-1}]_{\text{Lip}} \le [G]_{\text{Lip}}[F]_{\text{lip}}^{-1} < 1$ (see Proposition 2.1 (b)) we conclude that $(I + GF^{-1})^{-1}$ exists and

$$[(I + GF^{-1})^{-1}]_{\text{Lip}} \le \frac{1}{1 - [G]_{\text{Lip}}[F^{-1}]_{\text{Lip}}},$$

by Proposition 5.1. Thus $(F + G)^{-1} = F^{-1}(I + GF^{-1})^{-1} \in \mathfrak{Lip}(X)$, and the assertions follow. □

In view of the similarity of the classes $\mathfrak{Lip}(X, Y)$ and $\mathfrak{L}(X, Y)$, one might think that almost all results on bounded linear operators carry over to Lipschitz continuous nonlinear operators. However, this is not true. For instance, the following example shows that the uniform boundedness principle is not true.

Example 5.1. Let $X = \mathbb{R}$ and

$$F(x) = \begin{cases} 0 & \text{if } -\infty < x \le 0, \\ \sqrt{x} & \text{if } 0 \le x \le 1, \\ 1 & \text{if } 1 \le x < \infty. \end{cases}$$

Let $(F_n)_n$ be a sequence of polynomials that converges uniformly on $[0, 1]$ to $\sqrt{x}$. Extending F_n to $\mathbb{R}$ by putting $F_n(x) \equiv 0$ for $x \le 0$ and $F_n(x) \equiv 1$ for $x \ge 1$ we have $F_n \in \mathfrak{Lip}(\mathbb{R})$ and $F_n(x) \to F(x)$ for all $x \in \mathbb{R}$. However, the sequence $([F_n]_{\text{Lip}})_n$ is not bounded, and $F \notin \mathfrak{Lip}(\mathbb{R})$. ♡

5.2 The Kachurovskij spectrum

Now we proceed as before and define a spectrum for Lipschitz continuous operators. Given $F \in \mathfrak{Lip}(X)$, we call the set

$$\rho_{\text{K}}(F) := \{\lambda \in \mathbb{K} : \lambda I - F \text{ is a lipeomorphism}\} \tag{5.8}$$

the *Kachurovskij resolvent set* of F and its complement

$$\sigma_{\text{K}}(F) := \mathbb{K} \setminus \rho_{\text{K}}(F) \tag{5.9}$$

the *Kachurovskij spectrum* of F. Thus, $\lambda \in \sigma_{\text{K}}(F)$ means that either $\lambda I - F$ is not invertible, or the inverse exists but is not Lipschitz continuous. In case of a bounded linear operator L, the equality $[L]_{\text{Lip}} = \|L\|$ shows that $\sigma_{\text{K}}(L)$ is just the usual spectrum (1.5).

We point out that the operator $\lambda I - F$ with $F : \mathbb{C}^2 \to \mathbb{C}^2$ as in Example 3.18 is a lipeomorphism for *each* $\lambda \in \mathbb{C}$. In fact, it is easy to see that the resolvent operator (3.17) is Lipschitz continuous. This shows that also the Kachurovskij spectrum may be empty. On the other hand, the Kachurovskij spectrum shares an important property with the spectrum of a bounded linear operator:

Theorem 5.1. *For $F \in \mathfrak{Lip}(X)$, the spectrum $\sigma_{\mathrm{K}}(F)$ is compact. Moreover, the Kachurovskij spectral radius*

$$r_K(F) := \sup\{|\lambda| : \lambda \in \sigma_{\mathrm{K}}(F)\} \tag{5.10}$$

satisfies the estimate

$$r_K(F) \leq [F]_{\mathrm{Lip}}. \tag{5.11}$$

Proof. Let $\lambda \in \rho_{\mathrm{K}}(F)$ and $|\mu - \lambda| < [R(\lambda; F)]_{\mathrm{Lip}}^{-1}$. Then we obtain from the estimate $[(\mu - \lambda)R(\lambda; F)]_{\mathrm{Lip}} < 1$ and (5.4) that

$$[(I - (\mu - \lambda)R(\lambda; F))^{-1}]_{\mathrm{Lip}} \leq \frac{1}{1 - |\mu - \lambda|\,[R(\lambda; F)]_{\mathrm{Lip}}}.$$

Since $F - \mu I = [I - (\mu - \lambda)R(\lambda; F)](F - \lambda I)$, it follows that $\mu \in \rho_{\mathrm{K}}(F)$. We have shown that $\rho_{\mathrm{K}}(F)$ is open, and hence $\sigma_{\mathrm{K}}(F)$ is closed.

For any $\lambda \in \mathbb{K}$ with $|\lambda| > [F]_{\mathrm{Lip}}$ we know from Proposition 5.2 that $\lambda I - F$ is a lipeomorphism, i.e., $\lambda \in \rho_{\mathrm{K}}(F)$. So we have proved the compactness of $\sigma_{\mathrm{K}}(F)$ and the estimate (5.11). □

Observe that we have proved even more: for every $\lambda \in \rho_{\mathrm{K}}(F)$, the resolvent operator $R(\lambda; F) = (\lambda I - F)^{-1}$ satisfies the estimate

$$[R(\lambda; F)]_{\mathrm{Lip}} \leq \frac{1}{|\lambda| - [F]_{\mathrm{Lip}}}. \tag{5.12}$$

This is of course analogous to the estimate (1.7) for bounded linear operators. For $\lambda = 1$ we get again the estimate (5.4).

The following Proposition 5.3 shows, in particular, that the Kachurovskij spectrum of a *compact* operator $F \in \mathfrak{Lip}(X)$ may be described very easily. This is similar to Proposition 1.4 (a) for linear operators.

Proposition 5.3. *For every $F \in \mathfrak{Lip}(X)$, the inclusion*

$$\partial\sigma_{\mathrm{K}}(F) \subseteq \sigma_{\mathrm{lip}}(F) \tag{5.13}$$

is true, where $\sigma_{\mathrm{lip}}(F)$ is the spectral set (2.28). *If in addition F is compact, then even*

$$\sigma_{\mathrm{K}}(F) = \sigma_{\mathrm{lip}}(F). \tag{5.14}$$

Proof. Given $\lambda \in \partial\sigma_{\mathrm{K}}(F)$ we may find a sequence $(\lambda_n)_n$ in $\rho_{\mathrm{K}}(F)$ such that $\lambda_n \to \lambda$, and hence $\lambda_n I - F \to \lambda I - F$ in $\mathfrak{Lip}(X)$, as $n \to \infty$. Since $\sigma_{\mathrm{K}}(F)$ is compact, we have $\mathrm{dist}(\lambda_n, \sigma_{\mathrm{K}}(F)) > 0$ for each n; moreover,

$$[\lambda_n I - F]_{\mathrm{Lip}} \geq \frac{1}{\mathrm{dist}(\lambda_n, \sigma_{\mathrm{K}}(F))}. \tag{5.15}$$

This implies, by Proposition 2.1 (a), that

$$[\lambda I - F]_{\text{lip}} = \lim_{n\to\infty} [\lambda_n I - F]_{\text{lip}} = \lim_{n\to\infty} \frac{1}{[\lambda_n I - F]_{\text{Lip}}} \leq \lim_{n\to\infty} \text{dist}(\lambda_n, \sigma_{\text{K}}(F)) = 0,$$

which proves the inclusion (5.13).

Now suppose that $F \in \mathfrak{Lip}(X)$ is compact. Without loss of generality we may assume that X is infinite dimensional. Then F cannot be proper, and so $[F]_{\text{lip}} = 0$, hence $0 \in \sigma_{\text{lip}}(F)$. This means that for each $\lambda \in \mathbb{K} \setminus \sigma_{\text{lip}}(F)$ we have $\lambda \neq 0$ and $[I - F/\lambda]_{\text{lip}} > 0$. Theorem 3.6 implies that $I - F$ is a homeomorphism. We conclude that $\lambda I - F = \lambda(I - F/\lambda)$ is a lipeomorphism for $\lambda \in \mathbb{K} \setminus \sigma_{\text{lip}}(F)$, and so (5.14) is proved. □

We illustrate Proposition 5.3 in the following Example 5.2. Afterwards we show by another example that the equality (5.14) may fail if F is not compact.

Example 5.2. Let X be a complex Banach space, $e \in S(X)$ fixed, and

$$F(x) := \|x\|e.$$

Obviously, F is compact with $[F]_{\text{Lip}} = 1$ and $[F]_{\text{lip}} = 0$, so that $\sigma_{\text{K}}(F) \subseteq \overline{\mathbb{D}}$, by Theorem 5.1. We show that $\sigma_{\text{lip}}(F) \supseteq \overline{\mathbb{D}}$, and hence

$$\sigma_{\text{K}}(F) = \sigma_{\text{lip}}(F) = \overline{\mathbb{D}}. \tag{5.16}$$

Clearly $0 \in \sigma_{\text{lip}}(F)$, since $[F]_{\text{lip}} = 0$. For $0 < |\lambda| \leq 1$ we put

$$\hat{x} := \frac{|\lambda| + 1}{2|\lambda|}\overline{\lambda}e, \quad \hat{y} := \frac{|\lambda| - 1}{2|\lambda|}\overline{\lambda}e$$

and obtain $F(\hat{x}) = \frac{1}{2}(1 + |\lambda|)e$ and $F(\hat{y}) = \frac{1}{2}(1 - |\lambda|)e$. Consequently,

$$[\lambda I - F]_{\text{lip}} \leq \frac{\|\lambda\hat{x} - \lambda\hat{y} - F(\hat{x}) + F(\hat{y})\|}{\|\hat{x} - \hat{y}\|} = |\lambda| - |\lambda| = 0$$

which implies the inclusion $\overline{\mathbb{D}} \setminus \{0\} \subseteq \sigma_{\text{lip}}(F)$, and hence (5.16). ♡

Example 5.3. Let X be the complex sequence space l_p $(1 \leq p < \infty)$, and define $F \in \mathfrak{Lip}(X)$ by

$$F(x_1, x_2, x_3, x_4, \ldots) := (\|x\|, x_1, x_2, x_3, \ldots). \tag{5.17}$$

We claim that

$$\sigma_{\text{K}}(F) = \overline{\mathbb{D}}_{2^{1/p}} = \{\lambda \in \mathbb{C} : |\lambda| \leq 2^{1/p}\} \tag{5.18}$$

and

$$\sigma_{\text{lip}}(F) = \overline{\mathbb{D}}_{2^{1/p}} \setminus \mathbb{D} = \{\lambda \in \mathbb{C} : 1 \leq |\lambda| \leq 2^{1/p}\}, \tag{5.19}$$

and so (5.13) is true, but (5.14) is not. We may write the operator (5.17) in the form $F(x) = \|x\|e + Lx$, where $e = (1, 0, 0, \dots)$ is the first basis element in X, and L is the linear right-shift operator (1.38). It is not hard to see that

$$[F]_{\text{lip}} = 1, \quad [F]_{\text{Lip}} = 2^{1/p} \tag{5.20}$$

(see Example 3.15 for the special case $p = 2$), and so $\sigma_{\text{lip}}(F) \subseteq \overline{\mathbb{D}}_{2^{1/p}} \setminus \mathbb{D}$, by (2.29). So we have to prove only the reverse inclusion.

Suppose first that $\lambda \in \mathbb{S}$, i.e., $|\lambda| = 1$. From (1.37) and Proposition 1.4 (a) we know that $\mathbb{S} = \partial\sigma(L) \subseteq \sigma_{\text{q}}(L)$, so we may find a sequence $(e_n)_n$ in $S(X)$ such that $\lambda e_n - Le_n \to \theta$ as $n \to \infty$. Setting $x_n := \frac{1}{2}e_n$ and $y_n := -\frac{1}{2}e_n$ yields

$$[\lambda I - F]_{\text{lip}} \le \|(\lambda I - F)(x_n) - (\lambda I - F)(y_n)\| = \|\lambda e_n - Le_n\| \to 0,$$

and so $\lambda \in \sigma_{\text{lip}}(F)$.

Now suppose that λ satisfies $1 < |\lambda| \le 2^{1/p}$. Then the sequence

$$x_\lambda := (\lambda^{-1}, \lambda^{-2}, \lambda^{-3}, \dots)$$

belongs to X with $\|x_\lambda\| = (|\lambda|^p - 1)^{-1/p}$. Define $G_\lambda \colon X \to X$ by

$$G_\lambda(x) := \|x\|x_\lambda. \tag{5.21}$$

We claim that

$$1 \in \sigma_{\text{lip}}(G_\lambda). \tag{5.22}$$

Indeed, choosing

$$\hat{x} := \frac{1}{2}(1 + \|x_\lambda\|)x_\lambda, \quad \hat{y} := \frac{1}{2}(1 - \|x_\lambda\|)x_\lambda$$

we obtain $\hat{x} - \hat{y} = G_\lambda(\hat{x}) - G_\lambda(\hat{y}) = \|x_\lambda\|x_\lambda$, and (5.22) follows. Now, from the simple algebraic identity

$$(\lambda I - L)(I - G_\lambda)(x) = (\lambda I - L)x - (\lambda I - L)G_\lambda(x) = \lambda x - Lx - \|x\|e = \lambda x - F(x)$$

we deduce, together with (5.22), that

$$[\lambda I - F]_{\text{lip}} \le \|\lambda I - L\|\, [I - G_\lambda]_{\text{lip}} = 0,$$

and so the equality (5.19) is proved.

For the proof of (5.18) we show that the operator $\lambda I - F$ is not onto for $0 < |\lambda| < 1$. Indeed, suppose that the equation $\lambda x - F(x) = e$ has a solution $\hat{x} \in X$ for such a λ. Then $\lambda\hat{x} - L\hat{x} = (1 + \|\hat{x}\|)e$ and hence, writing this in components,

$$\hat{x}_n = \frac{1 + \|\hat{x}\|}{\lambda^n} \quad (n = 1, 2, 3, \dots)$$

which yields the contradiction $1 + \|\hat{x}\| = 0$. So we see that $\mathbb{D} \setminus \{0\} \subseteq \sigma_{\text{K}}(F)$, and the assertion follows from the closedness of the Kachurovskij spectrum. ♡

We return now to other properties of the Kachurovskij spectrum (5.9). For instance, one could ask if there is also an analogue to the Gel'fand formula (1.9) for (5.10) with $\|\cdot\|$ replaced by $[\cdot]_{\mathrm{Lip}}$. Example 3.19 shows again that this is false. In fact, from $\sigma_{\mathrm{K}}(F) = [0, 1]$ it follows that $r_K(F) = 1$. On the other hand, we have already seen that $\|F^n\|_{\mathrm{Lip}} \equiv 0$ for all $n \geq 2$.

Now we show that the estimate (5.12) may be slightly improved. Given $F \in \mathfrak{Lip}(X)$ and $z \in X$, consider the translation F_z of F defined by (2.24). Clearly, $F_z \in \mathfrak{Lip}(X)$ and

$$[F_z]_{\mathrm{Lip}} = [F]_{\mathrm{Lip}}, \quad \|F\|_{\mathrm{Lip}} - \|z\| \leq \|F_z\|_{\mathrm{Lip}} \leq \|F\|_{\mathrm{Lip}} + \|z\|. \tag{5.23}$$

Lemma 5.1. *Let X be a Banach space and $F \in \mathfrak{Lip}(X)$ with*

$$\beta_m := \sup\{[(F_z)^m]_{\mathrm{Lip}} : z \in X\} < 1 \tag{5.24}$$

for some $m \in \mathbb{N}$. Then $I - F$ is invertible in $\mathfrak{Lip}(X)$ and

$$[(I - F)^{-1}]_{\mathrm{Lip}} \leq \frac{1 + [F]_{\mathrm{Lip}} + \cdots + [F]_{\mathrm{Lip}}^{m-1}}{1 - \beta_m}. \tag{5.25}$$

Proof. Note first that for each $z \in X$ we have $[I - (F_z)^m]^{-1} \in \mathfrak{Lip}(X)$ with

$$[(I - (F_z)^m)^{-1}]_{\mathrm{Lip}} \leq \frac{1}{1 - \beta_m},$$

by Proposition 5.1. Furthermore, if $z \in X$ then $(I - F)(x) = z$ is equivalent to $(F_z)^m(x) = x$. Thus it follows that $(I-F)(x) = z$ if and only if $x = [I-(F_z)^m]^{-1}(\theta)$. From this it follows in turn that $I - F$ is a bijection and $G(z) = [I - (F_z)^m]^{-1}(\theta)$ for all $z \in X$, where we have put $(I - F)^{-1} =: G$. Noting that

$$[I - (F_z)^m]^{-1}(\theta) - (F_z)^m[I - (F_z)^m]^{-1}(\theta) = [I - (F_z)^m][I - (F_z)^m]^{-1}(\theta) = \theta$$

we conclude that

$$\begin{aligned}
\|G(z) - G(w)\| &= \|[I - (F_z)^m]^{-1}(\theta) - [I - (F_w)^m]^{-1}(\theta)\| \\
&= \|(F_z)^m[I - (F_z)^m]^{-1}(\theta) - (F_w)^m[I - (F_w)^m]^{-1}(\theta)\| \\
&\leq \|(F_z)^m[I - (F_z)^m]^{-1}(\theta) - (F_w)^m[I - (F_z)^m]^{-1}(\theta)\| \\
&\quad + \|(F_w)^m[I - (F_z)^m]^{-1}(\theta) - (F_w)^m[I - (F_w)^m]^{-1}(\theta)\| \\
&\leq (1 + [F]_{\mathrm{Lip}} + \cdots + [F]_{\mathrm{Lip}}^{m-1})\|z - w\| \\
&\quad + [(F_w)^m]_{\mathrm{Lip}}\,\|G(z) - G(w)\|.
\end{aligned}$$

Since $[(F_w)^m]_{\mathrm{Lip}} \leq \beta_m < 1$, by assumption, we have that

$$\|G(z) - G(w)\| \leq \frac{1 + [F]_{\mathrm{Lip}} + \cdots + [F]_{\mathrm{Lip}}^{m-1}}{1 - \beta_m}\|z - w\|$$

which implies (5.25). □

As a consequence of Lemma 5.1, we obtain the following extension of (5.12).

Proposition 5.4. *Let X be a Banach space, $F \in \mathfrak{Lip}(X)$, and $\lambda \in \mathbb{K}$ with*

$$\beta_m(\lambda) := \sup\{[(\lambda^{-1} F_z)^m]_{\text{Lip}} : z \in X\} < 1 \tag{5.26}$$

for some $m \in \mathbb{N}$. Then $\lambda \in \rho(F)$ and

$$[R(\lambda; F)]_{\text{Lip}} \leq \frac{1 + |\lambda|^{-1}[F]_{\text{Lip}} + \cdots + |\lambda|^{-1}[F]_{\text{Lip}}^{m-1}}{|\lambda|(1 - \beta_m(\lambda))}. \tag{5.27}$$

Proof. Applying Lemma 5.1 to $I - F/\lambda$ we see that $(I - F/\lambda)^{-1} \in \mathfrak{Lip}(X)$ with

$$\begin{aligned} [(I - \lambda^{-1}F)^{-1}]_{\text{Lip}} &\leq \frac{1 + [\lambda^{-1}F]_{\text{Lip}} + \cdots + [\lambda^{-1}F]_{\text{Lip}}^{m-1}}{1 - \beta_m(\lambda)} \\ &= \frac{1 + |\lambda|^{-1}[F]_{\text{Lip}} + \cdots + |\lambda|^{-m+1}[F]_{\text{Lip}}^{m-1}}{1 - \beta_m(\lambda)}. \end{aligned}$$

Since $\lambda I - F = \lambda(I - F/\lambda)$, the assertion is seen to be true. □

Example 3.19 shows that Lemma 5.1 and Proposition 5.4 are false if one assumes, instead of (5.26), only that $[(\lambda^{-1}F)^m]_{\text{Lip}} < 1$ for some $m \in \mathbb{N}$. In fact, in that example we have $\beta_m = 1$ for all $m \in \mathbb{N}$.

Let us now calculate the Kachurovskij spectrum for some operators in the simplest case $X = \mathbb{C}$. To this end, we first state a trivial, though useful, lemma.

Lemma 5.2. *For $F \in \mathfrak{Lip}(\mathbb{C})$ we have*

$$\sigma_{\text{K}}(F) = \overline{\left\{\frac{F(z) - F(w)}{z - w} : z, w \in \mathbb{C},\ z \neq w\right\}}. \tag{5.28}$$

In particular, $\sigma_{\text{K}}(F) \neq \emptyset$ in case $X = \mathbb{C}$.

Proof. It is clear that the closed set $\sigma_{\text{K}}(F)$ (see Theorem 5.1) contains the set on the right-hand side of (5.28). Conversely, suppose that $\lambda \in \mathbb{C}$ does not belong to this set. Then we find $k > 0$ such that

$$\left|\lambda - \frac{F(z) - F(w)}{z - w}\right| \geq k \quad (z, w \in \mathbb{C},\ z \neq w).$$

But this implies that $|\lambda(z - w) - F(z) + F(w)| \geq k|z - w|$, hence $[\lambda I - F]_{\text{lip}} \geq k$; in particular, $\lambda I - F$ is injective.

It remains to show that $\lambda I - F$ is onto. Now, the invariance of domain theorem for continuous injective maps in finite dimensional spaces implies that the range $R(\lambda I - F)$ is open in $\mathbb{C}$. On the other hand, $\lambda I - F$ is a lipeomorphism between $\mathbb{C}$ and $R(\lambda I - F)$, and so $R(\lambda I - F)$ is complete and hence closed in $\mathbb{C}$. From the connectedness of $\mathbb{C}$ we conclude that $\lambda I - F$ is onto. □

The formula (5.28) may be used to calculate the Kachurovskij spectrum in case $X = \mathbb{C}$. We illustrate this by a series of examples.

Example 5.4. Let $F(z) = |z|$; then $\|F\|_{\rm Lip} = [F]_{\rm Lip} = 1$, hence $\sigma_{\rm K}(F) \subseteq \overline{\mathbb{D}}$, by (5.11). Now, for $|\lambda| \leq 1$ we have

$$\mathrm{Re}(\lambda z - F(z)) = \mathrm{Re}(\lambda z) - \|z\| \leq 0,$$

and thus $\lambda I - F$ cannot be onto. It follows that

$$\sigma_{\rm K}(F) = \overline{\mathbb{D}}$$

is the whole closed disc. ♡

Example 5.5. Let $F(z) = \mathrm{Re}\, z$; then $\|F\|_{\rm Lip} = [F]_{\rm Lip} = 1$, hence again $\sigma_{\rm K}(F) \subseteq \overline{\mathbb{D}}$. Since F is $\mathbb{R}$-linear and $F(0) = 0$, we have

$$\left\{ \frac{F(z) - F(w)}{z - w} : z, w \in \mathbb{C},\ z \neq w \right\} = \left\{ \frac{F(z)}{z} : z \in \mathbb{C},\ z \neq 0 \right\}. \tag{5.29}$$

Now, for $z = x + iy \neq 0$ we get

$$\left| \frac{F(z)}{z} - \frac{1}{2} \right|^2 = \left| \frac{x}{x + iy} - \frac{1}{2} \right|^2 = \left| \frac{x - iy}{2(x + iy)} \right|^2 = \frac{1}{4},$$

which shows that the set (5.29) is included in the circumference $\{\lambda \in \mathbb{C} : |\lambda - \frac{1}{2}| = \frac{1}{2}\}$. Conversely, if $\lambda = \mu + i\nu \neq 0$ satisfies $|\lambda - \frac{1}{2}| = \frac{1}{2}$, then $\mu^2 + \nu^2 = \mu$ so that

$$\frac{\mu}{\mu - i\nu} = \frac{\mu^2 + i\mu\nu}{\mu^2 + \nu^2} = \mu + i\nu \in \sigma_{\rm K}(F).$$

As $\sigma_{\rm K}(F)$ is closed we conclude that

$$\sigma_{\rm K}(F) = \{\lambda \in \mathbb{C} : |\lambda - \tfrac{1}{2}| = \tfrac{1}{2}\};$$

in particular, $[0, 1] \subset \sigma_{\rm K}(F)$. ♡

Example 5.6. Let $F(z) = |\mathrm{Re}\, z|$; then $\|F\|_{\rm Lip} = [F]_{\rm Lip} = 1$, so again $\sigma_{\rm K}(F) \subseteq \overline{\mathbb{D}}$. Using the obvious geometric fact that

$$-1 \leq \frac{|\mathrm{Re}\, z| - |\mathrm{Re}\, w|}{\mathrm{Re}\, z - \mathrm{Re}\, w} \leq 1$$

we see that every point λ in the set

$$\left\{ \frac{F(z) - F(w)}{z - w} : z, w \in \mathbb{C},\ z \neq w \right\}$$

has the form $\lambda = \tau\mu$ with μ belonging to the right-hand side of (5.29) and $|\tau| \leq 1$. A convexity and closedness argument shows then that

$$\sigma_{\rm K}(F) = \{\lambda \in \mathbb{C} : |\lambda - \tfrac{1}{2}| \leq \tfrac{1}{2}\} \cup \{\lambda \in \mathbb{C} : |\lambda + \tfrac{1}{2}| \leq \tfrac{1}{2}\};$$

in particular, $[-1, 1] \subset \sigma_{\rm K}(F)$. ♡

5.3 The Dörfner spectrum

A similar definition of a spectrum was introduced by Dörfner in 1997. Recall that $F \in \mathfrak{B}(X)$ means that $F\colon X \to X$ is *linearly bounded*, i.e.,

$$[F]_{\mathrm{B}} = \sup_{x \neq 0} \frac{\|F(x)\|}{\|x\|} < \infty, \tag{5.30}$$

see (2.6). Obviously, the linear space $\mathfrak{B}(X)$ with norm (5.30) is a Banach space, and $[F]_{\mathrm{B}} \leq [F]_{\mathrm{Lip}}$ for $F \in \mathfrak{Lip}(X)$ with $F(\theta) = \theta$. Putting $\mathfrak{M}(X) = \mathfrak{B}(X)$ in (4.1) and (4.3) we arrive at the *Dörfner resolvent set*

$$\rho_{\mathrm{D}}(F) = \{\lambda \in \mathbb{K} : \lambda I - F \text{ is bijective and } R(\lambda; F) \in \mathfrak{B}(X)\} \tag{5.31}$$

and the *Dörfner spectrum*

$$\sigma_{\mathrm{D}}(F) = \mathbb{K} \setminus \rho_{\mathrm{D}}(F). \tag{5.32}$$

Thus, a point $\lambda \in \mathbb{K}$ belongs to $\rho_{\mathrm{D}}(F)$ if and only if $\lambda I - F$ is a homeomorphism on X satisfying the two-sided estimate

$$c\|x\| \leq \|\lambda x - F(x)\| \leq C\|x\| \quad (x \in X) \tag{5.33}$$

for some $C, c > 0$. Since $\mathfrak{Lip}_0(X) \subseteq \mathfrak{B}(X) \subseteq \mathfrak{C}(X)$, we have the trivial, though useful inclusions

$$\rho_{\mathrm{K}}(F) \subseteq \rho_{\mathrm{D}}(F) \subseteq \rho_{\mathrm{R}}(F), \quad \sigma_{\mathrm{K}}(F) \supseteq \sigma_{\mathrm{D}}(F) \supseteq \sigma_{\mathrm{R}}(F) \tag{5.34}$$

for $F \in \mathfrak{Lip}_0(X)$.

In spite of the apparent similarity of the classes $\mathfrak{Lip}(X, Y)$ and $\mathfrak{B}(X, Y)$, the properties of the corresponding spectra are quite different. Thus, in contrast to the Kachurovskij spectrum, the Dörfner spectrum may be unbounded or not closed, as the following two examples show.

Example 5.7. Let $X = \mathbb{R}$, $F(x) \equiv 0$ for $x \leq 1$, and $F(x) = \sqrt{x-1}$ for $x \geq 1$. Then $F \notin \mathfrak{Lip}(X)$, but $F \in \mathfrak{B}(X)$ with $[F]_{\mathrm{B}} = \frac{1}{2}$. Obviously, $0 \in \sigma_{\mathrm{D}}(F)$, since F is not onto. For $\lambda > 0$, the operator $F_\lambda(x) = \lambda x - F(x)$ is strictly decreasing on $[1, 1 + 1/4\lambda^2)$ and strictly increasing on $(1 + 1/4\lambda^2, \infty)$. Consequently, $(0, \infty) \subseteq \sigma_{\mathrm{D}}(F)$. On the other hand, for $\lambda < 0$, the operator $F_\lambda(x) = \lambda x - F(x)$ is strictly decreasing on the whole axis with $F_\lambda(x) \to \mp\infty$ as $x \to \pm\infty$, and thus a homeomorphism. Moreover, the estimate (5.33) holds for $\lambda < 0$ with $c = |\lambda|$ and $C = |\lambda| + 1$. We conclude that $\sigma_{\mathrm{D}}(F) = [0, \infty)$ in this example. ♡

Example 5.8. Let $X = \mathbb{R}$, and let $(\alpha_n)_n$ be a strictly increasing sequence in $[0, 1]$ with $\alpha_0 = 0$ and $\alpha_n \to 1$ as $n \to \infty$. Put

$$c_n = \sum_{k=1}^{n} \frac{\alpha_k - \alpha_{k-1}}{k}, \quad c = \sum_{k=1}^{\infty} \frac{\alpha_k - \alpha_{k-1}}{k}.$$

Define F on $[0, \infty)$ by

$$F(x) = \begin{cases} c_n + \dfrac{x - \alpha_n}{n+1} & \text{if } \alpha_n \le x \le \alpha_{n+1}, \\ cx & \text{if } x \ge 1, \end{cases}$$

and extend F as an odd function to the whole real axis. It is clear that $0 \in \rho_{\mathrm{D}}(F)$ since F is bijective with $c|x| \le |F(x)| \le |x|$ for all $x \in \mathbb{R}$. On the other hand, the function $\frac{1}{n}I - F$ is not injective for any $n \in \mathbb{N}$, being constant on the interval $[\alpha_n, \alpha_{n+1}]$. We conclude that $\{1, \frac{1}{2}, \frac{1}{3}, \dots\} \subseteq \sigma_{\mathrm{D}}(F)$, and so $\sigma_{\mathrm{D}}(F)$ is not closed. ♡

Interestingly, the successive approximations (5.3) need not converge any more for $F \in \mathfrak{B}(X)$ and $|\lambda| > [F]_{\mathrm{B}}$. In fact, the resolvent operator $R(\lambda; F)$ in Example 5.7 does not exist for $\lambda > \frac{1}{2}$, although $[F]_{\mathrm{B}} = \frac{1}{2}$. Of course, an estimate of the form (5.11) for the *Dörfner spectral radius*

$$r_D(F) = \sup\{|\lambda| : \lambda \in \sigma_{\mathrm{D}}(F)\} \tag{5.35}$$

with $[F]_{\mathrm{Lip}}$ replaced by $[F]_{\mathrm{B}}$ is not true either.

One of our main goals in this and the preceding chapter was to compare the existing nonlinear spectra from the viewpoint of the properties they have in common with the linear spectrum. Some of these properties are collected in the following Table 5.1 for the Rhodius, Neuberger, Kachurovskij and Dörfner spectra; the numbers in brackets refer to the corresponding theorem or example.

Table 5.1

spectrum	$\neq \emptyset$	closed	bounded	compact
$\sigma_{\mathrm{R}}(F)$	no (Ex. 3.18)	no (Ex. 4.1)	no (Ex. 4.1)	no (Ex. 4.1)
$\sigma_{\mathrm{N}}(F)$	yes (Th. 4.1)	no (Ex. 4.7)	no (Ex. 4.1)	no (Ex. 4.1)
$\sigma_{\mathrm{K}}(F)$	no (Ex. 3.18)	yes (Th. 5.1)	yes (Th. 5.1)	yes (Th. 5.1)
$\sigma_{\mathrm{D}}(F)$	no (Ex. 3.18)	no (Ex. 5.8)	no (Ex. 5.7)	no (Ex. 5.7)

Imposing additional conditions on the operator F we get again more properties of the spectrum. We confine ourselves to stating the following analogue to Theorem 4.3.

Theorem 5.2. *Suppose that X is infinite dimensional and $F \in \mathfrak{Lip}(X)$ is compact. Then $0 \in \sigma_{\mathrm{K}}(F)$, and hence the Kachurovskij spectrum of F is nonempty. An analogous statement holds for the Dörfner spectrum $\sigma_{\mathrm{D}}(F)$ of a compact operator $F \in \mathfrak{B}(X)$.*

5.4 Restricting the Kachurovskij spectrum

The fact that each row in Table 5.1 contains at least some "no" is disappointing: it means that, as we stated in the Introduction, the requirement on the nonlinear spectra to share as many properties as possible with the linear spectrum is not fulfilled for any of the spectra discussed so far. The point is that, in each of these spectra, we tried to cover the largest possible class of operators having a certain regularity property. It turns out, however, that one may overcome this difficulty by simply restricting the Kachurovskij spectrum from the class $\mathfrak{Lip}(X)$ to the class $\mathfrak{Lip}(X) \cap \mathfrak{C}^1(X)$. To this end, we first need a technical lemma.

Lemma 5.3. *Suppose that $F\colon X \to X$ is a lipeomorphism which admits a Fréchet-derivative $F'(x_0)$ at some point $x_0 \in X$. Then $F'(x_0)$ is a linear isomorphism on X.*

Proof. Define $G \in \mathfrak{Lip}(X)$ by $G(h) = F(x_0 + h) - F(x_0)$; then $[G]_{\mathrm{Lip}} = [F]_{\mathrm{Lip}}$, $[G]_{\mathrm{lip}} = [F]_{\mathrm{lip}}$, and $G(\theta) = \theta$. Moreover, G is differentiable at θ with $G'(\theta) = F'(x_0)$. We claim that

$$\|F'(x_0)h\| \geq c\|h\| \quad (h \in X), \tag{5.36}$$

where $c = [G]_{\mathrm{lip}}$. In fact, for $h \in X$ with $\|h\| = 1$ we have

$$\|F'(x_0)h\| = \lim_{\tau\to 0} \frac{\|F'(x_0)(\tau h)\|}{\|\tau h\|} = \lim_{\tau\to 0} \frac{\|G(\tau h)\|}{\|\tau h\|} \geq [G]_{\mathrm{lip}},$$

since $\|F'(x_0)h - G(h)\| = o(\|h\|)$ as $h \to \theta$. From (5.36) it follows that $F'(x_0)$ is invertible on its range $R(F'(x_0))$ with bounded inverse.

The estimate (5.36) also shows that the range $R(F'(x_0))$ is closed. Indeed, let $(h_k)_k$ be a sequence in X such that $y_k := F'(x_0)h_k \to y_*$ as $k \to \infty$. Then (5.36) implies that $(h_k)_k$ is Cauchy, hence $h_k \to h_*$ for some $h_* \in X$ with $F'(x_0)h_* = y_*$, by continuity.

Finally, we claim that $R(F'(x_0))$ is dense in X. To see this, fix $\varepsilon > 0$ and $w \in X$ with $\|w\| = 1$. Choose $\delta > 0$ such that $\|F'(x_0)h - G(h)\| \leq \varepsilon\|h\|$ for $\|h\| \leq \delta$, and put $v := G^{-1}(\delta c w)$ with $c = [G]_{\mathrm{lip}}$ as before. Then $\|v\| \leq [G^{-1}]_{\mathrm{Lip}}\delta c\|w\| = \delta$, and thus $\|F'(x_0)v - G(v)\| \leq \varepsilon\|v\| \leq \varepsilon\delta$. Consequently,

$$\|w - F'(x_0)(v/\delta c)\| = \frac{1}{\delta c}\,\|G(v) - F'(x_0)v\| \leq \frac{\varepsilon}{c}.$$

Thus we have proved that, given any $y \in X \setminus \{\theta\}$, we can find an $x_y \in X$ such that

$$\left\| \frac{y}{\|y\|} - F'(x_0)x_y \right\| \leq [G^{-1}]_{\mathrm{Lip}}\,\varepsilon,$$

and so

$$\|y - F'(x_0)(\|y\|x_y)\| \leq \|y\|\,[G^{-1}]_{\mathrm{Lip}}\,\varepsilon.$$

We conclude that $R(F'(x_0)) = X$, and hence $F'(x_0)$ is a bijection with bounded inverse. □

Lemma 5.3 implies that the inclusions

$$\rho_{\mathrm{K}}(F) \subseteq \rho_{\mathrm{N}}(F), \quad \sigma_{\mathrm{N}}(F) \subseteq \sigma_{\mathrm{K}}(F) \tag{5.37}$$

are true for $\mathfrak{Lip}(X) \cap \mathfrak{C}^1(X)$. To see this, fix $x \in X$. From $\lambda \in \sigma(F'(x))$ it follows that $\lambda I - F'(x)$ is not bijective, and hence $\lambda I - F$ cannot be a lipeomorphism, by Lemma 5.3. This shows that

$$\bigcup_{x \in X} \sigma(F'(x)) \subseteq \sigma_{\mathrm{K}}(F).$$

Moreover, $\lambda \in \rho_{\mathrm{K}}(F)$ certainly implies that $\lambda I - F$ is proper, and hence $\lambda \notin \pi(F)$. We conclude that $\pi(F) \subseteq \sigma_{\mathrm{K}}(F)$, and the assertion follows from Theorem 4.2.

The harmless inclusions (5.37) make it possible to consider a class of nonlinear operators for which all the requirements are finally met. In fact, the Kachurovskij spectrum $\sigma_{\mathrm{K}}(F)$, restricted to $F \in \mathfrak{Lip}(X) \cap \mathfrak{C}^1(X)$, is both nonempty in case $\mathbb{K} = \mathbb{C}$, by Theorem 4.1, and compact, by Theorem 5.1. Moreover, it certainly reduces to the familiar spectrum in the linear case. So we have finally achieved our goal to find a spectrum which has the usual properties in common with the linear spectrum, but is also defined for a reasonably large class of nonlinear operators. A simple, but nontrivial example of an operator in the class $\mathfrak{Lip}(X) \cap \mathfrak{C}^1(X)$ is given by the integral operator (4.17).

5.5 Semicontinuity properties of spectra

We have seen in Example 3.20 that the mapping spectrum (3.11) is not upper semicontinuous on $\mathfrak{C}(X)$ with respect to any of the seminorms introduced in Section 2.1. Now we study the same problem for the Kachurovskij spectrum on $\mathfrak{Lip}(X)$ and the Dörfner spectrum on $\mathfrak{B}(X)$. We start with a general discussion of multivalued maps.

Let $\mathfrak{M}$ be a linear space over $\mathbb{K}$ with seminorm p, and let $\sigma : \mathfrak{M} \to 2^{\mathbb{K}}$ be a multivalued map on $\mathfrak{M}$, where $2^{\mathbb{K}}$ denotes the family of all nonempty subsets of $\mathbb{K}$. For $F \in \mathfrak{M}$ and $\delta > 0$ we denote by $U_\delta(F)$ the p-neighbourhood

$$U_\delta(F) = \{G \in \mathfrak{M} : p(G - F) < \delta\} \tag{5.38}$$

of F. Recall that σ is called *closed* if the graph of σ is closed in $\mathfrak{M} \times \mathbb{K}$, i.e., $\lambda_n \in \sigma(F_n)$, $\lambda_n \to \lambda$ and $p(F_n - F) \to 0$ imply that $\lambda \in \sigma(F)$. Obviously, σ is closed if and only if, for every $F \in \mathfrak{M}$ and $\lambda \in \mathbb{K} \setminus \sigma(F)$, there exist $\delta > 0$ and $V_\lambda \subset \mathbb{K}$ open such that $\lambda \in V_\lambda$ and $\sigma(U_\delta(F)) \cap V_\lambda = \emptyset$.

It is easy to see that every upper semicontinuous map with closed values is closed, but not vice versa. For example, the map $\sigma : \mathbb{R} \to 2^{\mathbb{R}}$ defined by $\sigma(t) = [-1/t, 1/t]$ for $t \neq 0$ and $\sigma(0) = 0$ is closed, but not upper semicontinuous at 0. The following lemma provides a simple condition which implies the upper semicontinuity of a closed map.

Lemma 5.4. *Let $(\mathfrak{M}, p)$ be a seminormed linear space and $\sigma : \mathfrak{M} \to 2^{\mathbb{K}}$ a closed map with bounded values. If*

$$\sup_{\lambda \in \sigma(F)} |\lambda| \leq p(F) \quad (F \in \mathfrak{M}), \tag{5.39}$$

then σ is upper semicontinuous.

Proof. Let $F \in \mathfrak{M}$, and let $V \subseteq \mathbb{K}$ be open with $\sigma(F) \subseteq V$. Choose $\eta > 0$ with $\sigma(U_\eta(F)) \setminus V \neq \emptyset$. For $G \in U_\eta(F)$ we have

$$\sup_{\lambda \in \sigma(G)} |\lambda| \leq p(G) \leq p(F) + \eta.$$

Consequently, $\sigma(U_\eta(F))$ is bounded, and so the set $C := \overline{\sigma(U_\eta(F))} \setminus V$ is compact.

Fix $\lambda \in C$. Since $\lambda \notin \sigma(F)$ and the map σ is closed, we find $\delta(\lambda) > 0$ and $V_\lambda \subseteq \mathbb{K}$ open with $\lambda \in V_\lambda$ and $\sigma(U_{\delta(\lambda)}(F)) \cap V_\lambda = \emptyset$. Obviously, $\{V_\lambda : \lambda \in C\}$ is an open covering of C. From the compactness of C we get $C \subseteq V_{\lambda_1} \cup \cdots \cup V_{\lambda_m}$ for suitable $\lambda_1, \ldots, \lambda_m \in C$. Putting $\delta := \min\{\eta, \delta(\lambda_1), \ldots, \delta(\lambda_m)\}$ we see that $\sigma(U_\delta(F)) \subseteq V$ as claimed. □

We point out that Lemma 5.4 even holds for arbitrary metric spaces X in the following form: *A multivalued map $\sigma : X \to 2^{\mathbb{K}}$ is upper semicontinuous if it has a closed graph and is locally bounded.* We will use Lemma 5.4 in this form in Chapter 8. Now we are going to apply Lemma 5.4 to the space $\mathfrak{M} = \mathfrak{Lip}(X)$ equipped with the seminorm $p(F) = [F]_{\text{Lip}}$.

Theorem 5.3. *The multivalued map $\sigma_K : \mathfrak{Lip}(X) \ni F \mapsto \sigma_K(F) \in 2^{\mathbb{K}}$ is closed and upper semicontinuous.*

Proof. Let $(F_n)_n$ and $(\lambda_n)_n$ be sequences with $\lambda_n \in \sigma_{\text{K}}(F_n)$, $[F_n - F]_{\text{Lip}} \to 0$, and $\lambda_n \to \lambda$. Then

$$[(\lambda_n I - F_n) - (\lambda I - F)]_{\text{Lip}} \leq |\lambda_n - \lambda| + [F_n - F]_{\text{Lip}} \to 0 \quad (n \to \infty),$$

and so the sequence $(\lambda_n I - F_n)_n$ tends to $\lambda I - F$ in the seminorm (2.1). We have to show that $\lambda \in \sigma_{\text{K}}(F)$.

Suppose that $\lambda \in \rho_{\text{K}}(F)$, hence $[\lambda I - F]_{\text{lip}} > 0$. Choose n so large that

$$\max\{[F_n - F]_{\text{Lip}}, |\lambda_n - \lambda|\} < \frac{1}{2}[\lambda I - F]_{\text{lip}}.$$

Applying Proposition 2.1 (c) we get

$$[(\lambda_n - \lambda)I + F - F_n]_{\text{Lip}} \leq [F_n - F]_{\text{Lip}} + |\lambda_n - \lambda| < [\lambda I - F]_{\text{Lip}}.$$

Consequently, $\lambda_n I - F_n = ((\lambda_n - \lambda)I + F - F_n) + (\lambda I - F)$ is a lipeomorphism, by Proposition 5.2, contradicting the hypothesis $\lambda_n \in \sigma_K(F_n)$. So we have shown that the map σ_{K} is closed.

The upper semicontinuity of σ_K is a simple consequence of Lemma 5.4. Indeed, the required estimate (5.39) is nothing else but a reformulation of (5.11). □

As a particular case we get from Theorem 5.3 the upper semicontinuity of the spectrum of a bounded linear operator, see Theorem 1.1 (i).

Now we try to study the same question for the Dörfner spectrum (5.32) on the operator class $\mathfrak{B}(X)$. It turns out that a parallel result for the Dörfner spectrum is not true. This may be seen by means of the following example.

Example 5.9. Let X and F be defined as in Example 5.8, and put $F_n(x) = F(x) - \frac{1}{n}x$. We already know that $F \in \mathfrak{B}(X)$ with $[F]_B = 1$. Since F is a homeomorphism with $[F]_b = c > 0$ we have $0 \notin \sigma_D(F)$. On the other hand, every $\lambda \in \mathbb{R}$ with $c \leq |\lambda| \leq 1$ is certainly an eigenvalue of F. Moreover, every scalar $\lambda_n^{\pm} = \pm\frac{1}{n}$ belongs to $\sigma_D(F)$, since $F_n(x)$ is constant on the interval $[\alpha_n, \alpha_{n+1}]$.

Obviously, we have $[F - F_n]_B \to 0$ as $n \to \infty$. So, if we take $\lambda_n \equiv 0$, we have $\lambda_n \in \sigma_D(F_n)$, $\lambda_n \to 0$, but $0 \notin \sigma_D(F)$. This shows that the multivalued map σ_D is *neither closed nor upper semicontinuous* on $\mathfrak{B}(\mathbb{R})$. So, in contrast to the Kachurovskij spectrum (5.8), the Dörfner spectrum (5.32) may “blow up” when F changes continuously. ♡

5.6 Continuity properties of resolvent operators

In this section we consider the nonlinear resolvent operator

$$R(\lambda; F) = (\lambda I - F)^{-1} \tag{5.40}$$

for $\lambda \in \rho_K(F)$ or $\lambda \in \rho_D(F)$. In particular, we will be interested in continuity properties of the maps $(\lambda, x) \mapsto R(\lambda; F)(x)$ and $\lambda \mapsto R(\lambda; F)$.

We start with two identities for the resolvent operator (5.40) which are similar to those given in Theorem 1.1 (a) for the linear resolvent operator.

Proposition 5.5. *The resolvent operator* (5.40) *has the following properties.*

(a) *The resolvent identities*

$$R(\lambda; F) - R(\mu; F) = R(\lambda; F)(\mu - \lambda)R(\mu; F) \quad (\lambda, \mu \in \rho_K(F)) \tag{5.41}$$

and

$$R(\lambda; F) - R(\lambda; G) = R(\lambda; F)(F - G)R(\lambda; G) \quad (\lambda \in \rho_K(F) \cap \rho_K(G)) \tag{5.42}$$

are true for $F, G \in \mathfrak{Lip}(X)$.

(b) $\lambda \in \rho_K(F)$ *and* $|\mu - \lambda| < [R(\lambda; F)]_{\mathrm{Lip}}^{-1}$ *imply that also* $\mu \in \rho_K(F)$ *with*

$$R(\mu; F) = R(\lambda; F)[I - (\mu - \lambda)R(\lambda; F)]^{-1} \tag{5.43}$$

and

$$\|R(\mu;F)(x)-R(\lambda;F)(x)\| \le \frac{\|R(\lambda;F)(x)\|\,[R(\lambda;F)]_{\mathrm{Lip}}}{1-|\mu-\lambda|\,[R(\lambda;F)]_{\mathrm{Lip}}}\,|\mu-\lambda|. \quad (5.44)$$

(c) *The map* $(\lambda, x) \mapsto R(\lambda; F)(x)$ *is continuous from* $\rho_{\mathrm{K}}(F) \times X$ *into* X.

Proof. The two identities (5.41) and (5.42) are proved exactly like their linear analogues (1.10) and (1.11). The equality (5.43) follows from the identity $F - \mu I = [I-(\mu-\lambda)R(\lambda;F)](F-\lambda I)$ which we already used in the proof of Theorem 5.1.

To prove (5.44), fix $x \in X$ and $\mu \in \mathbb{K}$ with $|\mu-\lambda| < [R(\lambda;F)]_{\mathrm{Lip}}^{-1}$. A straightforward calculation yields

$$\begin{aligned}&\|R(\mu;F)(x)-R(\lambda;F)(x)\|\\ &\quad = \|R(\lambda;F)[I-(\mu-\lambda)R(\lambda;F)]^{-1}(x)-R(\lambda;F)(x)\|\\ &\quad \le [R(\lambda;F)]_{\mathrm{Lip}}\,\|[1-(\mu-\lambda)R(\lambda;F)]^{-1}(x)-x\|.\end{aligned}$$

But

$$\|[I-(\mu-\lambda)R(\lambda;F)]^{-1}(x)-x\| \le \frac{\|R(\lambda;F)(x)\|}{1-|\mu-\lambda|\,[R(\lambda;F)]_{\mathrm{Lip}}}|\mu-\lambda|,$$

and substituting this estimate in the preceding inequality establishes (5.44).

It remains to prove (c). So fix $(\lambda, x), (\mu, y) \in \rho_{\mathrm{K}}(F) \times X$ with $|\mu-\lambda| < [R(\lambda;F)]_{\mathrm{Lip}}^{-1}$. Then

$$\begin{aligned}&\|R(\lambda;F)(x)-R(\mu;F)(y)\|\\ &\quad \le \|R(\lambda;F)(x)-R(\mu;F)(x)\| + \|R(\mu;F)(x)-R(\mu;F)(y)\|\\ &\quad \le |\mu-\lambda|\,\|R(\lambda;F)(x)\|\,\frac{[R(\lambda;F)]_{\mathrm{Lip}}}{1-|\mu-\lambda|\,[R(\lambda;F)]_{\mathrm{Lip}}} + [R(\lambda;F)]_{\mathrm{Lip}}\,\|x-y\|\end{aligned}$$

which shows that $R(\mu;F)(y) \to R(\lambda;F)(x)$ in X as $(\mu, y) \to (\lambda, x)$ in $\rho_{\mathrm{K}}(F) \times X$, and so we are done. □

The following example shows that the map $\lambda \mapsto R(\lambda; F)$ is in general *not* continuous from $\rho_{\mathrm{K}}(F)$ into $\mathfrak{Lip}(X)$. This is of course in sharp contrast to Theorem 1.1 (d).

Example 5.10. Let $X = \mathbb{R}$ and define $F \in \mathfrak{Lip}(X)$ by

$$F(x) = \begin{cases} x & \text{if } x \le 2,\\ 2 & \text{if } x > 2.\end{cases}$$

Obviously, $[F]_{\mathrm{Lip}} = 1$ and $\sigma_{\mathrm{K}}(F) = [0, 1]$. The resolvent operator (5.40) has the form

$$R(\lambda;F)(x) = \begin{cases} \frac{x}{\lambda-1} & \text{if } x \le 2(1-\lambda),\\ \frac{x-2}{\lambda} & \text{if } x > 2(1-\lambda)\end{cases}$$

for $\lambda < 0$, and

$$R(\lambda; F)(x) = \begin{cases} \frac{x-2}{\lambda} & \text{if } x < 2(1-\lambda), \\ \frac{x}{\lambda-1} & \text{if } x \geq 2(1-\lambda) \end{cases}$$

for $\lambda > 1$. We show that the map $\lambda \mapsto R(\lambda; F)$ is discontinuous at $\lambda = 2$. Indeed, for $\lambda_n := 2 - 1/n$, $x_n := -2 - 1/n$, and $y_n := -2 - 1/2n$ we obtain the estimate

$$\begin{aligned}
&[R(\lambda_n; F) - R(2; F)]_{\mathrm{Lip}} \\
&\quad \geq \frac{\|[R(\lambda_n; F) - R(2; F)](x_n) - [R(\lambda_n; F) - R(2; F)](y_n)\|}{\|x_n - y_n\|} \\
&\quad = \left| -\frac{3n+1}{2n(n+1)} + \frac{7n+1}{4n(n+1)} \right| |2n| \\
&\quad = \frac{2n(n-1)}{4n(n+1)} \quad (n \in \mathbb{N}),
\end{aligned}$$

which shows that

$$\limsup_{n\to\infty} [R(\lambda_n; F) - R(2; F)]_{\mathrm{Lip}} \geq \frac{1}{2}.$$

So $[R(\lambda_n; F) - R(2; F)]_{\mathrm{Lip}} \not\to 0$ as $n \to \infty$. ♡

5.7 Notes, remarks and references

Lipschitz continuous operators are of course widely used in nonlinear analysis; we just recall the Banach–Caccioppoli contraction mapping theorem and its various generalizations. Most of the material on Lipschitz continuous operators presented here may be found in the monograph [184]. The operators which we call lipeomorphisms are sometimes called *bi-Lipschitz maps* in the literature.

The Banach space $\mathfrak{Lip}_0(X)$ of all Lipschitz continuous operators $F\colon X \to X$ satisfying $F(\theta) = \theta$ is quite similar to the algebra $\mathfrak{L}(X)$ of all bounded linear operators on X, and this might be the reason why nonlinear spectral theory has been considered by some authors for such operators. However, $\mathfrak{Lip}(X)$ is *not* a Banach algebra, since the distributive law $F(G+H) = FG + FH$ for addition and composition on the left fails. On the other hand, the distributive law $(F+G)H = FH + GH$ is true, and we used this in the proof of Proposition 5.2.

We point out that in [79] one may find several invertibility results for Lipschitz continuous operators in the spirit of Chapter 3. Loosely speaking, these results apply to operators which are not "too far from being linear". More precisely, suppose that $F \in \mathfrak{Lip}(X, Y)$ maps θ into θ, and assume in addition that there exists a linear *isomorphism* $L \in \mathfrak{L}(X, Y)$ such that $[F - L]_{\mathrm{Lip}} < \lfloor L \rfloor$, where $\lfloor L \rfloor$ denotes the inner norm (1.79) of L. Then the equation $F(x) = y$ is equivalent to the equation $x + L^{-1}(F - L)x = L^{-1}y$, and the latter one may be solved uniquely by means of the Banach–Caccioppoli contraction mapping principle.

Brouwer's theorem on the invariance of domain in finite-dimensional spaces may be found in any book on topological methods of nonlinear analysis, e.g. [73]. Observe that this theorem actually implies the more general result whose proof goes exactly like our surjectivity proof for $\lambda I - F$ in Lemma 5.2: *if X is finite-dimensional and $F \in \mathfrak{Lip}(X)$, then $\sigma_K(F) = \sigma_{lip}(F)$, where $\sigma_{lip}(F)$ is given by* (2.28). We have proved this for infinite dimensional spaces in Proposition 5.3. A detailed study of such theorems for operators F satisfying $[F]_A \leq 1$ and $[F]_Q < \infty$, by means of degree theoretical methods, may be found in [217].

The characteristics (2.1) and (2.2) which are fundamental in this chapter have been studied in [240] and [181]. Apart from these characteristics, the author of [240] also defines, for $F \in \mathfrak{Lip}(\mathbb{C})$, the characteristics

$$M[F] := \lim_{\varepsilon \downarrow 0} \frac{[I + \varepsilon F]_{\mathrm{Lip}} - 1}{\varepsilon} = \lim_{\omega \to \infty} [F + \omega I]_{\mathrm{Lip}} - \omega \tag{5.45}$$

and

$$m[F] := \lim_{\varepsilon \downarrow 0} \frac{[I + \varepsilon F]_{\mathrm{lip}} - 1}{\varepsilon} = \lim_{\omega \to \infty} [F + \omega I]_{\mathrm{lip}} - \omega \tag{5.46}$$

which have similar properties as those given in Proposition 2.1. If $L \in \mathfrak{L}(\mathbb{C})$ is a matrix, then $M[L]$ is nothing else but the so-called *logarithmic norm* of L introduced by Dahlquist [69].

The following invertibility result which is similar to Proposition 5.1 may be found in [240] (see also [214], [215]).

Proposition 5.6. *Let X be a Banach space and $F \in \mathfrak{Lip}(X)$ with $M[F] < 1$. Then $I - F$ is a lipeomorphism, and the estimate*

$$[(I - F)^{-1}]_{\mathrm{Lip}} \leq \frac{1}{1 - M[F]}$$

holds true.

Several statements and effective bounds for the characteristics $M[F]$ and $m[F]$ may be found in the survey [69].

The *Kachurovskij spectrum* (5.9) was introduced, apparently, by Kachurovskij [156], and later studied in [82], [181], [270]. It seems that the authors of [181] and [270] were unaware of Kachurovskij's paper [156]. Theorem 5.1 may be found in [181] or [15], Proposition 5.3 in [M]. The "scalar" Examples 5.4–5.6 are all taken from [MW]; they show that, even for very simple scalar functions $F: \mathbb{C} \to \mathbb{C}$, it is impossible to guess the explicit form of the spectrum $\sigma_K(F)$.

In [181] the authors show that $\sigma_K(F) \neq \emptyset$ for $X = \mathbb{C}$ and pose the question whether or not this is also true in case $\dim X \geq 2$. Example 3.18 gives a negative answer to this question.

As mentioned before, there is no analogue to the Gel'fand formula (1.9) for the spectral radius (5.10), see Example 3.19. However, Dörfner [82] discusses a class of operators $F \in \mathfrak{Lip}(X)$ for which one has at least an upper estimate of the form

$$r_K(F) \leq \limsup_{n\to\infty} [F^n]_{\mathrm{Lip}}^{1/n} \tag{5.47}$$

which is of course better than (5.11). Following [82] we call an operator F *iteration-invariant* if there exists some $n_0 \in \mathbb{N}$ such that

$$\sup_{z\in X} [(\mu F + z)^n]_{\mathrm{Lip}} \leq |\mu|^n [F^n]_{\mathrm{Lip}} \quad (\mu \in \mathbb{K},\ n \geq n_0). \tag{5.48}$$

For example, if $f : \mathbb{R} \to \mathbb{R}$ is a globally Lipschitz continuous C^1-function with some additional properties, then the Nemytskij operator (4.21) generated by f is iteration-invariant on the space $C[0, 1]$ of continuous functions, as was shown in [82]. Moreover, every iteration-invariant operator F belonging to $\mathfrak{Lip}(X)$ satisfies the estimate (5.47).

A detailed discussion of the properties of the Kachurovski resolvent set $\rho_{\mathrm{K}}(F)$, the spectrum $\sigma_{\mathrm{K}}(F)$, and the resolvent operator $R(\lambda; F) = (\lambda I - F)^{-1}$ may be found in Chapter 1 of the survey [70]. In particular, the closedness and the estimate (5.12) are proved there. Moreover, in [70] it is shown that, as a consequence of the resolvent identity (5.43), the differentiability of the map $x \mapsto R(\lambda; F)(x)$ implies the differentiability of the map $\lambda \mapsto R(\lambda; F)(x)$ with

$$\frac{\partial}{\partial\lambda} R(\lambda; F)(x) = -\frac{\partial}{\partial x} R(\lambda; F)(x) R(\lambda; F)(x).$$

As application, the author of [70] considers a Cauchy problem for an abstract (i.e., Banach space valued) function and calculates the corresponding Kachurovskij spectrum and resolvent operator. Perturbation theorems for Lipschitz continuous operators in the spirit of Kato's book [158] may be found in [228], as well as results on resolvent operators which are similar to [70].

The *Dörfner spectrum* (5.32) was introduced and studied in the thesis [82]. In view of its "bad" analytic properties, however, this spectrum will be hardly used in the study of nonlinear problems. Table 5.1 may be found, together with some more examples, in [13]; Subsection 5.4 is also taken from [13].

All results of Section 5.5 may be found in the recent paper [11], the results and examples of Section 5.6 in Chapter 3 of the book [184]. We will come back to semicontinuity properties of other spectra in the next two chapters. More precisely, we will show that all spectra which have been studied in the literature are upper semicontinuous.

Estimates for the nonlinear resolvent operator (5.40) with respect to special metrics may be found in the paper [255]. As we have seen in Example 5.10, the map which associates to each $\lambda \in \rho_{\mathrm{K}}(F)$ the resolvent operator $R(\lambda; F)$ for fixed $F \in \mathfrak{Lip}_0(X)$ is *not* continuous in the norm $[\,\cdot\,]_{\mathrm{Lip}}$. Interestingly, Singhof [239] has shown in another context that this map *is* in fact continuous from $\rho_{\mathrm{D}}(F)$ into $\mathfrak{B}(X)$ with norm (2.6). To explain this strange phenomenon requires some technical preparation.

Let $F\colon X \to Y$ be a continuous operator with $F(\theta) = \theta$, and denote by

$$\Gamma(F) = \{(x, y) \in X \times Y : y = F(x)\} \tag{5.49}$$

its *graph*. This is a closed subset of $X \times Y$; here we equip $X \times Y$ with the norm $\|(x, y)\| := (\|x\|^2 + \|y\|^2)^{1/2}$, which has the advantage that $X \times Y$ becomes a Hilbert space whenever X and Y are Hilbert spaces.

In [238], [239] the author introduces two topologies on $\mathfrak{C}(X, Y)$ which may be defined as follows. Given $F \in \mathfrak{C}(X, Y)$ and $y \in X$, we denote by $\tilde{F}_y$ the operator shift defined by

$$\tilde{F}_y(x) := F(x + y) - F(y); \tag{5.50}$$

so $\tilde{F}_y \equiv F$ for all $y \in X$ if and only if F is additive. Moreover, for $R > 0$ we put

$$\Gamma_R(F) := \{(x, F(x)) \in \Gamma(F) : \|(x, F(x))\| \leq R\}. \tag{5.51}$$

For $F, G \in \mathfrak{C}(X, Y)$ let

$$d'_S(F, G) := \sup_{R>0} \sup \left\{\tfrac{1}{R} \operatorname{dist}((x, F(x)), \Gamma(G)) : (x, F(x)) \in \Gamma_R(F)\right\}, \tag{5.52}$$

$$d_S(F, G) := \max\{d'_S(F, G), d'_S(G, F)\}, \tag{5.53}$$

$$D'_S(F, G) := \sup_{R>0} \sup_{y\in X} \sup \left\{\tfrac{1}{R} \operatorname{dist}((x, F(x)), \Gamma(\tilde{G}_y)) : (x, F(x)) \in \Gamma_R(\tilde{F}_y)\right\}, \tag{5.54}$$

and

$$D_S(F, G) := \max\{D'_S(F, G), D'_S(G, F)\}. \tag{5.55}$$

The two topologies induced by the characteristics (5.53) and (5.55) may be defined in the following way. Let us call a set $U \subseteq \mathfrak{C}(X, Y)$ open in the *Singhof d-topology* if for every $F \in U$ one may find $\delta > 0$ such that $G \in U$ for all $G \in \mathfrak{C}(X, Y)$ with $d_S(F, G) < \delta$. The *Singhof D-topology* is defined similarly by using D_S instead of d_S. It is not hard to see that $\mathfrak{C}(X, Y)$ becomes in fact a topological Hausdorff space with these topologies. The following proposition gives an interesting relation with the characteristics (2.1) and (2.6) which were crucial in the definition of the Kachurovskij and Dörfner spectra in this chapter.

Proposition 5.7. *The following holds true*:

(a) *The set* $\mathfrak{Lip}_0(X, Y)$ *is open in* $\mathfrak{C}(X, Y)$ *with the Singhof D-topology.*

(b) *The set* $\mathfrak{B}(X, Y)$ *is open in* $\mathfrak{C}(X, Y)$ *with the Singhof d-topology.*

(c) *The Singhof D-topology induces on* $\mathfrak{Lip}_0(X, Y)$ *the same topology as the characteristic* $[\cdot]_{\mathrm{Lip}}$ *defined in* (2.1). *Moreover, for every* $F, G \in \mathfrak{C}(X, Y)$ *with* $F - G \in \mathfrak{Lip}_0(X, Y)$ *one has* $D_S(F, G) \leq [F - G]_{\mathrm{Lip}}$.

(d) *The Singhof d-topology induces on* $\mathfrak{B}(X, Y)$ *the same topology as the characteristic* $[\cdot]_{\mathrm{B}}$ *defined in* (2.6). *Moreover, for every* $F, G \in \mathfrak{C}(X, Y)$ *with* $F - G \in \mathfrak{B}(X, Y)$ *one has* $d_S(F, G) \leq [F - G]_{\mathrm{B}}$.

This proposition shows that, roughly speaking, the quasimetric (5.53) corresponds to the characteristic $[\cdot]_{\mathrm{B}}$, while the quasimetric (5.55) corresponds to the characteristic $[\cdot]_{\mathrm{Lip}}$. Moreover, it is shown in [239] that, for invertible continuous operators F and G, the equality

$$d_S(F^{-1}, G^{-1}) = d_S(F, G) \tag{5.56}$$

is true. As a consequence, one may prove that

$$d_S(R(\lambda; F), R(\mu; F)) \leq 2(1 + |\lambda|^2)|\lambda - \mu| \tag{5.57}$$

for μ sufficiently close to λ. However, neither (5.56) nor (5.57) is true for D_S instead of d_S, and this is precisely the reason for the fact that the resolvent operator $R(\lambda; F)$ depends continuously on λ in $\mathfrak{B}(X)$, but not in $\mathfrak{Lip}(X)$.

A particular continuity property of spectra was studied in [241], [242]. Suppose that $F\colon X \to X$ is a continuous operator in a Banach space X satisfying $F(\theta) = \theta$ and having a compact Fréchet derivative $L := F'(\theta)$ at zero. Assume that the spectral radius $r(L)$ of L is a simple pole of the resolvent map $\lambda \mapsto R(\lambda; L)$, and so the nullspace $N(r(L)I - L)$ of the operator $r(L)I - L$ is finite dimensional. Then the point spectrum $\sigma_{\mathrm{p}}(F)$ of F is called *continuous at* $r(L)$ if $(r(L) - \delta, r(L) + \delta) \subseteq \sigma_{\mathrm{p}}(F)$ for some $\delta > 0$. Sufficient conditions for this, together with some applications to Hammerstein integral equations, are given in [241], [242].

Chapter 6

The Furi–Martelli–Vignoli Spectrum

In this chapter we discuss a spectrum for nonlinear operators in Banach spaces which was introduced in 1978 by Furi, Martelli, and Vignoli. This spectrum is based on the notion of stable solvability of operators, a nonlinear analogue to surjectivity, and has several surprising applications of topological character. In fact, it can be thought of as a truly nonlinear tool for solving new problems, as well as for re-interpreting in a new terminology some well-known results of nonlinear analysis. Some applications of the Furi–Martelli–Vignoli spectrum to boundary value problems will be given in Chapter 12. In the last section we discuss a certain modification of the Furi–Martelli–Vignoli spectrum which builds on the notion of so-called strictly stable solvability of operators, a reinforcement of stable solvability.

6.1 Stably solvable operators

Before introducing the Furi–Martelli–Vignoli spectrum, we have to recall some classes of special operators in this and the following section. We call a continuous operator $F\colon X \to Y$ *stably solvable* if, given any compact operator $G\colon X \to Y$ with

$$[G]_{\mathrm{Q}} = \limsup_{\|x\|\to\infty} \frac{\|G(x)\|}{\|x\|} = 0,$$

the equation $F(x) = G(x)$ has a solution $x \in X$. Every stably solvable operator is certainly surjective (choose $G(x) \equiv y$), but not vice versa. For example, the operator $F(x) = x/\sqrt{1+|x|}$ is a homeomorphism on the real line, but not stably solvable, as may be seen by taking $G(x) = F(x) + 1$. For *linear* operators, however, stable solvability just reduces to surjectivity:

Lemma 6.1. *Let X and Y be Banach spaces and $L \in \mathfrak{L}(X, Y)$. Then L is stably solvable if and only if L is onto.*

Proof. We only have to prove that a surjective operator is stably solvable. So let $L \in \mathfrak{L}(X, Y)$ be onto. From Michael's selection theorem it follows that we may find a continuous function $s\colon Y \to X$ such that $s(y) \in L^{-1}y$ for all $y \in Y$. Moreover, s may be chosen such that $\|s(y)\| \le M\|y\|$ for some $M = M(L) > 0$. In particular, if L is a linear isomorphism we may choose $s(y) = L^{-1}y$ and $M = \|L^{-1}\|$.

Now let $G\colon X \to Y$ be compact with $[G]_{\mathrm{Q}} = 0$. Then the map $Gs\colon Y \to Y$ is also compact and satisfies $[Gs]_{\mathrm{Q}} = 0$. From Theorem 2.2 it follows that Gs has a fixed point $y \in Y$. Consequently, $x = s(y)$ satisfies $Lx = Ls(y) = y = G(s(y)) = G(x)$, and so L is stably solvable. □

A somewhat stronger version of Lemma 6.1 will be proved in Proposition 6.6 below.

If $F\colon X \to Y$ is stably solvable, it is sometimes necessary to get solutions of the equation $F(x) = G(x)$ if $G\colon X \to Y$ is compact, but not necessarily quasibounded. Here the following continuation principle is very useful.

Proposition 6.1. *Let $F\colon X \to Y$ be stably solvable, and further suppose that $H\colon X \times [0, 1] \to Y$ is continuous and compact and such that $H(x, 0) = \theta$ for any $x \in X$. Let*

$$S = \{x \in X : F(x) = H(x, t) \text{ for some } t \in [0, 1]\}, \tag{6.1}$$

and assume that the set $F(S)$ is bounded in Y. Then the equation $F(x) = H(x, 1)$ has a solution $\hat{x} \in X$.

Proof. Choose $r > 0$ such that $\overline{F(S)}$ is contained in the open ball $B_r^o(Y)$, and define $\pi\colon X \to [0, 1]$ by

$$\pi(x) = \frac{\operatorname{dist}(F(x), Y \setminus B_r(Y))}{\operatorname{dist}(F(x), \overline{F(S)}) + \operatorname{dist}(F(x), Y \setminus B_r(Y))},$$

where

$$\operatorname{dist}(z, M) = \inf\{\|z - x\| : x \in M\}$$

denotes the distance of the point z from the set M. Define $G\colon X \to Y$ by

$$G(x) = \begin{cases} H(x, \pi(x)) & \text{if } \|H(x, \pi(x))\| \le r, \\ r\dfrac{H(x, \pi(x))}{\|H(x, \pi(x))\|} & \text{if } \|H(x, \pi(x))\| > r. \end{cases}$$

Since G is compact, $[G]_{\mathrm{Q}} = 0$, and F is stably solvable, there exists $\hat{x} \in X$ such that $F(\hat{x}) = G(\hat{x})$. Now, $\|F(\hat{x})\| > r$ would imply $\pi(\hat{x}) = 0$, and hence $F(\hat{x}) = H(\hat{x}, 0) = \theta$, a contradiction. So, we know that $\|F(\hat{x})\| \le r$, hence $F(\hat{x}) = H(\hat{x}, \pi(\hat{x}))$. Since $0 \le \pi(\hat{x}) \le 1$, we conclude that $\hat{x} \in S$. It follows that $\pi(\hat{x}) = 1$, i.e., $F(\hat{x}) = H(\hat{x}, 1)$ as claimed. □

The following three examples show how one may generate stably solvable operators from others.

Example 6.1. Let X, Y and Z be Banach spaces, and let $F\colon X \to Y$ be stably solvable. Suppose that $G\colon Z \to X$ is continuous and surjective with a quasibounded right inverse $\tilde{G}\colon X \to Z$. Then $FG\colon Z \to Y$ is stably solvable. In fact, let $H\colon Z \to Y$ be compact with $[H]_Q = 0$. Since F is stably solvable and $H\tilde{G}\colon X \to Y$ is compact with $[H\tilde{G}]_Q \leq [H]_Q[\tilde{G}]_Q = 0$, the equation $F(x) = H(\tilde{G}(x))$ has a solution $\tilde{x} \in X$. But then $\tilde{y} = \tilde{G}(\tilde{x}) \in Z$ is a solution of the equation $FG(y) = H(y)$. ♡

Example 6.2. Let X, Y and Z be Banach spaces, and let $F\colon X \to Y$ be stably solvable. Suppose that $G\colon Y \to Z$ is continuous and surjective with a quasibounded right inverse $\hat{G}\colon Z \to Y$. Then $GF\colon X \to Z$ is stably solvable. In fact, let $H\colon X \to Z$ be compact with $[H]_Q = 0$. Clearly, the equation $F(x) = \hat{G}(H(x))$ has a solution $\hat{x} \in X$. Applying now G to both sides of this equation, we see that $\hat{x}$ is also a solution of the equation $G(F(x)) = H(x)$. ♡

Example 6.3. Let X, Y and Z be Banach spaces, and let $F\colon X \to Y$ and $G\colon Z \to X$ be continuous. If G is quasibounded and $FG\colon Z \to Y$ is stably solvable then F is stably solvable. In fact, let $H\colon X \to Y$ be compact with $[H]_Q = 0$. Since $FG\colon Z \to Y$ is stably solvable, the equation $F(G(z)) = H(G(z))$ has a solution $\tilde{z} \in Z$. But then $\tilde{x} = G(\tilde{z}) \in X$ is a solution of the equation $F(x) = H(x)$. ♡

To see how the abstract reasoning in these examples works, consider again the operator $F(x) = \|x\|x$ from Example 2.33. Since $[F^{-1}]_Q = 0$, from Example 6.1 (or 6.2) we conclude that F is stably solvable; this is not easy to prove directly.

One may also deduce Lemma 6.1 from Example 6.1 (or 6.2). In fact, suppose that $L = G\colon Y \to X$ is linear, bounded, and onto, and so L has a quasibounded right inverse. Since $I = F\colon X \to X$ is stably solvable, the same is true for $IL = L\colon Y \to X$.

The following result on stably solvable operators will be needed later and seems to be of independent interest.

Lemma 6.2. *Let $F \in \mathfrak{C}(X)$ be stably solvable, $B \subseteq X$ a closed subset, and $H\colon B \to X$ a continuous operator. Assume that $H(B)$ is bounded and*

$$F^{-}(\overline{\mathrm{co}}\, H(B)) \subseteq B. \tag{6.2}$$

Moreover, suppose that the equality

$$\alpha(F(M)) = \alpha(H(M)) \quad (M \subseteq B) \tag{6.3}$$

implies the precompactness of M. Then the equation $F(x) = H(x)$ has a solution $\hat{x} \in X$.

Proof. We construct a sequence $(x_n)_n$ as follows. Take $x_0 \in X$ arbitrary and choose $x_n \in F^{-}(H(x_{n-1}))$ for $n \in \mathbb{N}$; we have then $F(x_n) = H(x_{n-1})$ for all n. The set $A = \{x_0, x_1, x_2, \dots\}$ satisfies $F(A) = \{F(x_0)\} \cup H(A)$, by construction, and hence is precompact, by (6.3). Moreover, $H(A)$ is bounded, since $H(B)$ is bounded and $A \subseteq B$.

Denote by A' the set of all cluster points of A. We claim that $A' \subseteq F^-(H(A'))$. In fact, for $x \in A'$ we find a subsequence $(x_{n_k})_k$ of $(x_n)_n$ such that $x_{n_k} \to x$, hence also $H(x_{n_k-1}) = F(x_{n_k}) \to F(x)$ as $k \to \infty$. Let $y \in A'$ be a cluster point of $(x_{n_k-1})_k$; clearly, $H(y) = F(x)$.

Consider the family

$$\mathfrak{M} = \{M : M \supseteq A',\ \overline{M} = M,\ F^-(\overline{\mathrm{co}}\, H(M)) \subseteq M\}.$$

This family is nonempty, since $B \in \mathfrak{M}$. Denote by M_0 the intersection of all $M \in \mathfrak{M}$. On the one hand, since

$$F^-(\overline{\mathrm{co}}\, H(M_0)) \subseteq F^-(\overline{\mathrm{co}}\, H(M)) \subseteq M$$

for all $M \in \mathfrak{M}$, we have $F^-(\overline{\mathrm{co}}\, H(M_0)) \subseteq M_0$. On the other hand, the set $M_1 := F^-(\overline{\mathrm{co}}\, H(M_0))$ is closed and satisfies both

$$M_1 = F^-(\overline{\mathrm{co}}\, H(M_0)) \supseteq F^-(\overline{\mathrm{co}}\, H(A')) \supseteq A'$$

and

$$F^-(\overline{\mathrm{co}}\, H(M_1)) \subseteq F^-(\overline{\mathrm{co}}\, H(M_0)) = M_1.$$

Therefore $M_1 \in \mathfrak{M}$, $M_1 \supseteq M_0$, and $F^-(\overline{\mathrm{co}}\, H(M_0)) = M_0$. This equality implies that

$$\alpha(\overline{\mathrm{co}}\, H(M_0)) = \alpha(H(M_0)) = \alpha(F(M_0)),$$

and thus M_0 is compact, by (6.3). By Dugundji's extension theorem we find a continuous operator $G\colon X \to X$ with $G|_{M_0} = H|_{M_0}$ and $G(X) \subseteq \overline{\mathrm{co}}\, H(M_0)$. Since $G(X)$ is precompact and $[G]_{\mathrm{Q}} = 0$, by construction, the equation $F(x) = G(x)$ has a solution $\hat{x} \in X$. But from $F(\hat{x}) \in \overline{\mathrm{co}}\, H(M_0)$ it follows that $\hat{x} \in F^-(\overline{\mathrm{co}}\, H(M_0)) = M_0$. We conclude that $F(\hat{x}) = G(\hat{x}) = H(\hat{x})$, and the proof is complete. □

We point out that Lemma 6.2 contains both Darbo's fixed point theorem (see Theorem 2.1) and Sadovskij's fixed point theorem (see Theorem 2.3) as special cases; this may be seen by choosing $F = I$ and $B \subset X$ closed, bounded, and convex. We will use Lemma 6.2 later to establish an important Rouché type perturbation result (Lemma 6.4) for a class of operators which we study in the next section.

We consider now a generalization of stably solvable operators. For $k \geq 0$, let us call an operator $F \in \mathfrak{C}(X, Y)$ *k-stably solvable* if, given any continuous operator $G\colon X \to Y$ with $[G]_{\mathrm{A}} \leq k$ and $[G]_{\mathrm{Q}} \leq k$, the equation $F(x) = G(x)$ has a solution $x \in X$. Obviously, 0-stably solvable operators are just stably solvable in the sense introduced above. Moreover, every k-stably solvable operator is certainly also k'-stably solvable for $k' < k$. This motivates the following definition. For $F \in \mathfrak{C}(X, Y)$, we call the number

$$\mu(F) = \inf\{k : k \geq 0,\ F \text{ is not } k\text{-stably solvable}\} \tag{6.4}$$

the *measure of stable solvability* of F. (Here and in what follows we put $\inf \emptyset := \infty$.) We call an operator $F \in \mathfrak{C}(X, Y)$ *strictly stably solvable* if $\mu(F) > 0$, i.e., F is k-stably solvable for some positive k.

Observe that Theorem 2.2 states precisely that the identity I in any Banach space is k-stably solvable at least for $k < 1$. On the other hand, I is not 1-stably solvable, as may be seen by choosing $G(x) = x + x_0$ with $x_0 \neq \theta$. Consequently, we have $\mu(I) = 1$ in every Banach space.

In the special case $X = \mathbb{R}$, the characteristic (6.4) strongly degenerates because

$$\mu(F) = \begin{cases} \infty & \text{if } F \text{ is onto,} \\ 0 & \text{if } F \text{ is not onto.} \end{cases}$$

In fact, if F is not onto, we may simply choose $G(x) \equiv y$ with $y \in Y \setminus R(F)$.

The following Proposition 6.2 is an analogue to Proposition 6.1 for k-stably solvable operators.

Proposition 6.2. *Let $F: X \to Y$ be k-stably solvable, and suppose further that $H: X \times [0, 1] \to Y$ is continuous with $[H(\cdot, 0)]_Q < 1$ and*

$$\alpha(H(M \times [0, 1])) \leq k\alpha(M) \quad (M \subset X \textit{ bounded}).$$

Let S be defined as in (6.1)*, and assume that $F(S)$ is bounded in Y. Then the equation $F(x) = H(x, 1)$ has a solution $\hat{x} \in X$.*

Proof. Let $\pi(x)$ and $G(x)$ be defined as in the proof of Proposition 6.1. Since now $[G]_A \leq k$, $[G]_Q = 0$, and F is k-stably solvable, there exists $\hat{x} \in X$ such that $F(\hat{x}) = G(\hat{x})$. Now, $\|F(\hat{x})\| > r$ would imply $\pi(\hat{x}) = 0$, hence $\|F(\hat{x})\| = \|H(\hat{x}, 0)\| \leq r$, a contradiction. So we have $\|F(\hat{x})\| \leq r$. But this implies that $F(\hat{x}) = H(\hat{x}, \pi(\hat{x}))$, hence $\hat{x} \in S$. It follows that $\pi(\hat{x}) = 1$, i.e., $F(\hat{x}) = H(\hat{x}, 1)$ as claimed. □

The following highly nontrivial example, which was discovered quite recently by Furi, shows that there exist stably solvable operators which are not strictly stably solvable.

Example 6.4. Consider the operator F of Example 2.33, i.e.,

$$F(x) = \|x\|x$$

in the Banach space $X = C[0, 1]$. We already know that F is a homeomorphism which is stably solvable (see the Remark after Example 6.3). Now let $\rho_4: B(X) \to S(X)$ be the retraction for $u = 4$ which we constructed in Example 2.35. From (2.23) it follows then that $[\rho_4]_A \leq 2$. For each $n \in \mathbb{N}$, denote by F_n the operator

$$F_n(x) = \begin{cases} F\left(\frac{1}{n}\rho_4(nx)\right) & \text{if } \|x\| \leq \frac{1}{n}, \\ F(x) & \text{if } \|x\| > \frac{1}{n}. \end{cases}$$

It is easy to see that F_n is continuous and maps the space X onto the exterior of the open ball $B^o_{1/n^2}(X)$. Moreover, we certainly have $[F - F_n]_Q = 0$ for each n.

We claim that $[F - F_n]_A \leq 3/n$. Indeed, denoting $M_n := M \cap B_{1/n}(X)$ for any bounded $M \subset X$ we obtain

$$\begin{aligned}\alpha((F - F_n)(M)) &= \alpha((F - F_n)(M_n)) \\ &\leq \alpha(F(M_n)) + \alpha(F_n(M_n)) \\ &\leq \frac{1}{n}\alpha(M) + \frac{1}{n}\alpha\left(\frac{1}{n}\rho_4(nM_n)\right) \\ &\leq \frac{1}{n}\alpha(M) + \frac{1}{n}[\rho_4]_A\alpha(M) \leq \frac{3}{n}\alpha(M).\end{aligned}$$

Now we show that F is not strictly stably solvable. Suppose that $\mu(F) > 0$, and fix $k \in (0, \mu(F))$. By definition, for each continuous operator G with $[G]_A \leq k$ and $[G]_Q \leq k$, the equation $F(x) = G(x)$ has a solution $\hat{x} \in X$. In particular, we may choose $G = F - F_n$, where $kn \geq 3$, since then $[G]_A \leq \frac{3}{n} \leq k$ and $[G]_Q = 0$, by what we have proved before. But any solution $\hat{x} \in X$ of $F(x) = G(x)$ for this G would satisfy $F_n(\hat{x}) = \theta$ which is impossible by $F_n(X) \subseteq X \setminus B^o_{1/n^2}(X)$. ♡

6.2 FMV-regular operators

There is a subclass of stably solvable operators which is so important in nonlinear spectral theory that it merits a special name. Let us call a stably solvable operator $F \in \mathfrak{C}(X, Y)$ *FMV-regular* (where FMV stands for Furi–Martelli–Vignoli, of course) if both $[F]_q > 0$ and $[F]_a > 0$ (see (2.4) and (2.17)). A trivial example of an FMV-regular operator is the identity I. More generally, the following important result is true for linear operators:

Theorem 6.1. *Let X and Y be Banach spaces and $L \in \mathfrak{L}(X, Y)$. Then L is FMV-regular if and only if L is an isomorphism.*

Proof. Let L be FMV-regular. Then L is injective, since $[L]_q > 0$, and surjective, by Lemma 6.1. Conversely, let L be a linear isomorphism. Then $[L]_q > 0$ and also $[L]_a > 0$, by Proposition 2.5 (j) and (i). Again from Lemma 6.1 it follows that L is stably solvable, and so we are done. □

The following Lemma 6.3 provides a connection between FMV-regular and strictly stably solvable operators which we will need in Section 6.6.

Lemma 6.3. *Every FMV-regular operator is strictly stably solvable. More precisely, the estimate*

$$\mu(F) \geq \min\{[F]_q, [F]_a\} \tag{6.5}$$

holds.

Proof. Fix k with $k < [F]_q$ and $k < [F]_a$; we have to show that the equation $F(x) = G(x)$ has a solution $\hat{x} \in X$ for any $G \in \mathfrak{C}(X, Y)$ with $[G]_Q \leq k$ and $[G]_A \leq k$.

Choose real numbers b and c such that $[G]_Q < b < c < [F]_q$, hence $\|G(x)\| \leq b\|x\|$ and $\|F(x)\| \geq c\|x\|$ for $\|x\| \geq r$ with suitable $r > 0$. Since $\alpha(G(B_r(X))) \leq [G]_A \alpha(B_r(X)) < \infty$, the set $G(B_r(X))$ is bounded by some constant $R > 0$, and so $\|G(x)\| \leq R + b\|x\|$ for all $x \in X$. Put

$$\rho := \frac{R}{c - b},$$

where we may assume without loss of generality that $\rho \geq r$. We apply Lemma 6.2 with $B = B_\rho(X)$. It is clear that $G(B_\rho(X))$ is bounded because $G(B_\rho(X)) \subseteq B_{R+b\rho}(Y)$. Fix $x \in X$ with $F(x) \in B_{R+b\rho}(Y)$. Now, $\|x\| > \rho$ would imply

$$R + b\rho \geq \|F(x)\| \geq c\|x\| > c\rho,$$

contradicting our choice of ρ. Thus, we have proved that

$$F^-(\text{co}\, G(B_\rho(X))) \subseteq F^-(\text{co}\, B_{R+b\rho}(Y)) = F^-(B_{R+b\rho}(Y)) \subseteq B_\rho(X).$$

Moreover, for all $M \subseteq B_\rho(X)$ with $\alpha(M) > 0$ we have, by assumption,

$$\alpha(F(M)) \geq [F]_a \alpha(M) > [G]_A \alpha(M) \geq \alpha(G(M)).$$

Thus, all hypotheses of Lemma 6.2 are satisfied, and so the equation $F(x) = G(x)$ has a solution $\hat{x} \in X$. □

The assumption on F to be FMV-regular in Lemma 6.3 means, in particular, that $[F]_a > 0$. Example 6.4 shows that Lemma 6.3 is false without this assumption. In fact, we have already seen in Example 2.33 that the operator $F(x) = \|x\|x$ satisfies $[F]_a = 0$ in any infinite dimensional space.

The following perturbation results of Rouché type for FMV-regular and k-stably solvable operators are important for deriving certain topological properties of the spectra we will discuss in the sequel.

Lemma 6.4. *Let $F, G \in \mathfrak{C}(X, Y)$. Then the following is true*:

(a) *If F is k-stably solvable with $k \geq [G]_A$ and $k \geq [G]_Q$, then $F + G$ is k'-stably solvable for* $0 \leq k' \leq k - \max\{[F]_A, [F]_Q\}$.

(b) *If F is FMV-regular with $[F]_a > [G]_A$ and $[F]_q > [G]_Q$, then $F + G$ is also FMV-regular.*

Proof. To prove (a) fix $H \in \mathfrak{C}(X, Y)$ with $[H]_A \leq k - [G]_A$ and $[H]_Q \leq k - [G]_Q$; we have to show that the equation $F(x) + G(x) = H(x)$ has a solution $\hat{x} \in X$. But $[H - G]_A \leq [H]_A + [G]_A \leq k$ and $[H - G]_Q \leq [H]_Q + [G]_Q \leq k$. Since F is k-stably solvable, the equation $F(x) = (H - G)(x)$ has a solution $\hat{x} \in X$.

The assertion (b) is a consequence of (a). Indeed, since F is k-stably solvable for $k = \min\{[F]_a, [F]_q\}$ by Lemma 6.3, part (a) implies that $F + G$ is stably solvable as well. Also, $[F + G]_a \geq [F]_a - [G]_A > 0$ and $[F + G]_q \geq [F]_q - [G]_Q > 0$, by Proposition 2.2 (c) and Proposition 2.4 (d), and so we are done. □

Using the characteristic (6.4) we may express the statement of Lemma 6.4 (a) in a more suggestive way as Rouché type inequality

$$\mu(F + G) \geq \mu(F) - \max\{[G]_A, [G]_Q\}. \tag{6.6}$$

Now we discuss two important applications where FMV-regular operators occur quite naturally. Let X be an infinite dimensional Banach space, and let X_0 be a closed subspace of X of finite codimension. Recall that we write $B(X)$ for the closed unit ball and $S(X)$ for the unit sphere in X.

An operator $F\colon X \to X_0$ is called a *compact vector field* if the operator $I - F\colon X \to X$ is continuous and compact. A non-vanishing compact vector field $F\colon S(X) \to X_0 \setminus \{\theta\}$ is called *essential* if any extension of F to a compact vector field $\hat{F}\colon B(X) \to X_0$ vanishes at some interior point of $B(X)$.

To characterize essential compact vector fields in terms of FMV-regular operators we have to introduce some notation. Given a compact vector field $F\colon S(X) \to X_0$, define $F_1\colon X \to X_0$ as in (3.10) by

$$F_1(x) = \begin{cases} \|x\| F\left(\frac{x}{\|x\|}\right) & \text{if } x \neq \theta, \\ \theta & \text{if } x = \theta. \end{cases} \tag{6.7}$$

Clearly, the vector field F_1 is 1-homogeneous and coincides with F on $S(X)$. We call the operator F_1 the *homogeneous extension* of F.

Proposition 6.3. *Let $F\colon S(X) \to X_0 \setminus \{\theta\}$ be a non-vanishing compact vector field. Then F is essential if and only if its homogeneous extension* (6.7) *is FMV-regular.*

Proof. Assume that F_1 is FMV-regular, and let $\hat{F}\colon B(X) \to X_0$ be a compact vector field such that $\hat{F}|_{S(X)} = F$. Define $G \in \mathfrak{C}(X, X_0)$ by

$$G(x) = \begin{cases} F_1(x) - \hat{F}(x) & \text{if } \|x\| \leq 1, \\ \theta & \text{if } \|x\| > 1. \end{cases}$$

Since $[G]_A = [G]_Q = 0$ and F_1 is stably solvable , we find $\hat{x} \in B(X)$ such that $F_1(\hat{x}) = G(\hat{x})$, i.e., $\hat{F}(\hat{x}) = \theta$, which shows that F is essential.

Conversely, assume that $F\colon S(X) \to X_0$ is essential, and let $G\colon X \to X_0$ be compact with $[G]_Q = 0$. We have to prove the solvability of the equation $F_1(x) = G(x)$ in X. To this end, we distinguish two cases. First, assume that G has bounded support, i.e., $G(x) \equiv 0$ for $\|x\| \geq R$ with suitable $R > 0$. Define $\hat{F}\colon B(X) \to X_0$ by

$$\hat{F}(x) = F_1(x) - \frac{1}{R} G(Rx).$$

Clearly, $\hat{F}$ is a compact vector field with $\hat{F}|_{S(X)} = F$. Since F is essential, we have $\hat{F}(\hat{x}) = \theta$ for some $\hat{x} \in B(X)$. But this implies that $F_1(R\hat{x}) = RF_1(\hat{x}) = G(R\hat{x})$. So we have proved that F_1 is stably solvable; the fact that $[F_1]_{\mathrm{q}} > 0$ and $[F_1]_{\mathrm{a}} > 0$ is obvious, since F is non-vanishing.

Now, if the support of G is not bounded, we replace G by the operator

$$G_n(x) = d_n(\|x\|)G(x),$$

where

$$d_n(t) = \begin{cases} 1 & \text{if } 0 \le t \le n, \\ 2 - \frac{1}{n}t & \text{if } n \le t \le 2n, \\ 0 & \text{if } t \ge 2n. \end{cases} \tag{6.8}$$

By what we just proved, we find a sequence $(\hat{x}_n)_n$ such that $F_1(\hat{x}_n) = G_n(\hat{x}_n)$. If $\|\hat{x}_n\| \le n$ for some n, we have $F(\hat{x}_n) = G_n(\hat{x}_n) = G(\hat{x}_n)$, and the proof is finished. On the other hand, if $\|\hat{x}_n\| > n$ for all n, the sequence $(\hat{x}_n)_n$ is unbounded and

$$\frac{\|F_1(\hat{x}_n)\|}{\|\hat{x}_n\|} = d_n(\|\hat{x}_n\|)\frac{\|G(\hat{x}_n)\|}{\|\hat{x}_n\|} \le \frac{\|G(\hat{x}_n)\|}{\|\hat{x}_n\|} \to 0 \quad (n \to \infty).$$

But this contradicts the fact that $[F_1]_{\mathrm{q}} > 0$, and so the proof is complete. □

The next Proposition 6.4 may be regarded as some continuation principle for stably solvable and k-stably solvable operators.

Proposition 6.4. *Suppose that $F_0 \colon X \to Y$ is k_0-stably solvable for some $k_0 \ge 0$, and $H \colon X \times [0, 1] \to Y$ is continuous with $H(x, 0) \equiv \theta$,*

$$\alpha(H(M \times [0, 1])) \le k\alpha(M) \quad (M \subset X \text{ bounded}),$$

and

$$\sup_{0 \le t \le 1} \limsup_{\|x\| \to \infty} \frac{\|H(x, t)\|}{\|x\|} \le k$$

for some $k \le k_0$. Then the operator $F_1 := F_0 + H(\cdot\,, 1)$ is k_1-stably solvable for $k_1 \le k_0 - k$.

Proof. The assertion is a direct consequence of Lemma 6.4 (a), applied to the operator $G = H(\cdot\,, 1)$. □

We remark that a similar continuation principle as that given in Proposition 6.4 may be proved under the hypothesis that the set (6.1) be bounded. A certain "local version" of Proposition 6.4 will be proved in Section 7.1 in the next chapter.

6.3 The FMV-spectrum

Now we are ready to define the Furi–Martelli–Vignoli spectrum. Given $F \in \mathfrak{C}(X)$, we call the set

$$\rho_{\mathrm{FMV}}(F) = \{\lambda \in \mathbb{K} : \lambda I - F \text{ is FMV-regular}\} \tag{6.9}$$

the *Furi–Martelli–Vignoli resolvent set* and its complement

$$\sigma_{\mathrm{FMV}}(F) = \mathbb{K} \setminus \rho_{\mathrm{FMV}}(F) \tag{6.10}$$

the *Furi–Martelli–Vignoli spectrum* (or *FMV-spectrum*, for short) of F. Intuitively speaking, if a point $\lambda \in \mathbb{K}$ belongs to $\sigma_{\mathrm{FMV}}(F)$, then the operator $\lambda I - F$ is characterized by some lack of surjectivity, properness, or boundedness. By Theorem 6.1, for a linear operator this gives precisely the familiar spectrum.

Let us first study some topological properties of the spectrum $\sigma_{\mathrm{FMV}}(F)$ as we have done before for the other spectra. First of all, we remark that $\sigma_{\mathrm{FMV}}(F) = \emptyset$ for the operator F given in Example 3.18, and thus also the FMV-spectrum may be trivial. On the other hand, the spectrum $\sigma_{\mathrm{FMV}}(F)$ has a nice property which the Rhodius, Neuberger, and Dörfner spectra do not have in general:

Theorem 6.2. *The spectrum $\sigma_{\mathrm{FMV}}(F)$ is closed.*

Proof. Fix $\lambda \in \rho_{\mathrm{FMV}}(F)$, and let $0 < \delta < \min\{[\lambda I - F]_{\mathrm{a}}, [\lambda I - F]_{\mathrm{q}}\}$. We apply Lemma 6.4 (b) to show that $\mu \in \rho_{\mathrm{FMV}}(F)$ for $|\mu - \lambda| < \delta$. In fact, from

$$[(\mu - \lambda)I]_{\mathrm{A}} = |\mu - \lambda| < [\lambda I - F]_{\mathrm{a}}, \quad [(\mu - \lambda)I]_{\mathrm{Q}} = |\mu - \lambda| < [\lambda I - F]_{\mathrm{q}}$$

it follows that $\mu I - F = (\lambda I - F) + (\mu - \lambda)I$ is FMV-regular. This shows that λ is an interior point of $\rho_{\mathrm{FMV}}(F)$, and thus $\rho_{\mathrm{FMV}}(F)$ is open in $\mathbb{K}$. □

In the following theorem we consider a class of operators for which the FMV-spectrum is even compact.

Theorem 6.3. *Suppose that $F \in \mathfrak{C}(X)$ satisfies $[F]_{\mathrm{A}} < \infty$ and $[F]_{\mathrm{Q}} < \infty$. Then the spectrum $\sigma_{\mathrm{FMV}}(F)$ is bounded, hence compact.*

Proof. First of all, for $\lambda \in \mathbb{K}$ with $|\lambda| > [F]_{\mathrm{Q}}$ and $|\lambda| > [F]_{\mathrm{A}}$ we have $[\lambda I - F]_{\mathrm{q}} \geq |\lambda| - [F]_{\mathrm{Q}} > 0$ and $[\lambda I - F]_{\mathrm{a}} \geq |\lambda| - [F]_{\mathrm{A}} > 0$. We claim that $\lambda I - F$ is stably solvable for such λ. So, let $G : X \to X$ be compact with $[G]_{\mathrm{Q}} = 0$. Then the operator $H := (F + G)/\lambda$ satisfies $[H]_{\mathrm{A}} < 1$ and $[H]_{\mathrm{Q}} < 1$, and so the assertion follows from Theorem 2.2. □

Observe that we have actually proved that the *Furi–Martelli–Vignoli spectral radius*

$$r_{\mathrm{FMV}}(F) = \sup\{|\lambda| : \lambda \in \sigma_{\mathrm{FMV}}(F)\} \tag{6.11}$$

satisfies the upper estimate

$$r_{\mathrm{FMV}}(F) \le \max\{[F]_{\mathrm{A}}, [F]_{\mathrm{Q}}\}. \tag{6.12}$$

Later (see Example 6.10) we will show that the spectrum $\sigma_{\mathrm{FMV}}(F)$ may be unbounded if the right-hand side of (6.12) is infinite.

The following example shows that there is no analogue to the Gel'fand formula (1.9) for the FMV-spectrum.

Example 6.5. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0 & \text{if } x \le 1, \\ x - 1 & \text{if } 1 \le x \le 2, \\ 1 & \text{if } x \ge 2. \end{cases}$$

Consider the operator F given by

$$F(x_1, x_2, x_3, \dots) = (f(x_1), f(x_2), f(x_3), \dots)$$

in the space $X = l_\infty$ with the supremum norm. Since $f^2(x) \equiv 0$ we certainly have $F^n(x) \equiv \theta$ for $n \ge 2$. Now, the set $M = \{(x_1, x_2, x_3, \dots) \in X : 1 \le x_j \le 2\}$ is not precompact, so $\alpha(M) > 0$. On the other hand, the operator $I - F$ maps M into the point $(1, 1, 1, \dots)$, and so

$$[I - F]_{\mathrm{a}} = \inf_{\alpha(M)>0} \frac{\alpha((I - F)(M))}{\alpha(M)} = 0.$$

We conclude that $1 \in \sigma_{\mathrm{FMV}}(F)$, and thus neither the Gel'fand formula (1.9) nor the spectral mapping theorem (Theorem 1.1 (h)) may be true for the FMV-spectrum. ♡

Nevertheless, the FMV-spectrum has another pleasant feature: it is upper semicontinuous on a certain class of continuous operators. Here the most suitable class is the intersection $\mathfrak{A}(X) \cap \mathfrak{Q}(X)$; in fact, Theorem 6.3 shows that the multivalued map $F \mapsto \sigma_{\mathrm{FMV}}(F)$ is compact-valued on this class. Let us denote by p_{AQ} the seminorm

$$p_{AQ}(F) = \max\{[F]_{\mathrm{A}}, [F]_{\mathrm{Q}}\} \tag{6.13}$$

on $\mathfrak{A}(X) \cap \mathfrak{Q}(X)$. This seminorm defines a (locally convex) topology on $\mathfrak{A}(X) \cap \mathfrak{Q}(X)$ which we call the *FMV-topology* in the sequel.

Theorem 6.4. *The multivalued map $\sigma_{\mathrm{FMV}} : \mathfrak{A}(X) \cap \mathfrak{Q}(X) \to 2^{\mathbb{K}}$ which associates to each F its FMV-spectrum is upper semicontinuous in the FMV-topology.*

Proof. We apply Lemma 5.4 and show that the map σ_{FMV} is closed. So choose a sequence $(F_n)_n$ in $\mathfrak{A}(X) \cap \mathfrak{Q}(X)$ and a sequence $(\mu_n)_n$ in $\mathbb{K}$ such that

$$\mu_n \in \sigma_{\mathrm{FMV}}(F_n), \quad \mu_n \to \mu, \quad p_{AQ}(F - F_n) \to 0.$$

Since

$$p_{AQ}((\mu_n I - F_n) - (\mu I - F)) \leq |\mu_n - \mu| + p_{AQ}(F - F_n) \to 0 \quad (n \to \infty), \quad (6.14)$$

the sequence $(\mu_n I - F_n)_n$ converges in the FMV-topology to $\mu I - F$. But from Theorem 6.2 it follows then that $\mu \in \sigma_{\mathrm{FMV}}(F)$, and so σ_{FMV} has a closed graph. To apply Lemma 5.4, it remains to observe that (5.39) is satisfied with $p = p_{AQ}$, by (6.12), and so the proof is complete. □

The results of this section show that the FMV-spectrum has rather nice properties. However, there is a deplorable drawback: *in contrast to all the nonlinear spectra considered so far, the FMV-spectrum in general does not contain the point spectrum* (3.18). This may be illustrated by the following very simple example.

Example 6.6. Let $X = \mathbb{R}$ and $F(x) = \sqrt{|x|}$ as in Example 3.16. We already know that $\sigma_{\mathrm{p}}(F) = \mathbb{R} \setminus \{0\}$. However, $0 \in \sigma_{\mathrm{FMV}}(F)$, since F is not onto, and from (6.12) we conclude that $\sigma_{\mathrm{FMV}}(F) = \{0\}$. Thus, the FMV-spectrum may even be *disjoint* from the point spectrum. ♡

Example 6.6 is of course in sharp contrast to the familiar spectrum of a bounded linear operator, where the point spectrum is an important part of the whole spectrum. In the next chapter we will discuss another spectrum which is quite similar to the FMV-spectrum but contains the eigenvalues.

6.4 Subdivision of the FMV-spectrum

The definition of the FMV-spectrum suggests the following natural decomposition. Given $F \in \mathfrak{C}(X)$, let us put

$$\sigma_\delta(F) = \{\lambda \in \mathbb{K} : \lambda I - F \text{ is not stably solvable}\} \quad (6.15)$$

and

$$\sigma_\pi(F) = \sigma_{\mathrm{a}}(F) \cup \sigma_{\mathrm{q}}(F), \quad (6.16)$$

where $\sigma_{\mathrm{q}}(F)$ is as in (2.29) and $\sigma_{\mathrm{a}}(F)$ as in (2.31). So, by definition, we have the decomposition

$$\sigma_{\mathrm{FMV}}(F) = \sigma_\delta(F) \cup \sigma_\pi(F) = \sigma_\delta(F) \cup \sigma_{\mathrm{a}}(F) \cup \sigma_{\mathrm{q}}(F). \quad (6.17)$$

Of course, we have to be careful with the notation (6.15), because we have already used the symbol $\sigma_\delta(L)$ in (1.51). Fortunately, a comparison with the corresponding decomposition of the linear spectrum in Section 1.3 gives what we expect.

Lemma 6.5. *Let X be a Banach space and $L \in \mathfrak{L}(X)$. Then $\sigma_\delta(L)$ coincides with the defect spectrum* (1.51)*, and $\sigma_\pi(L)$ coincides with the approximate point spectrum* (1.50).

Proof. The first assertion is an immediate consequence of Lemma 6.1, while the second assertion has already been observed in Section 2.4. □

In view of Lemma 6.5 we shall retain the name of *defect spectrum* for $\sigma_\delta(F)$ and *approximate point spectrum* for $\sigma_\pi(F)$ also in the case when F is nonlinear. The following theorem gives useful information on the "position" of the approximate point spectrum in the whole spectrum.

Proposition 6.5. *The subspectrum* (6.16) *is closed and contains the boundary of the FMV-spectrum, i.e., we have*

$$\partial\sigma_{\mathrm{FMV}}(F) \subseteq \sigma_\pi(F). \tag{6.18}$$

Proof. We show that $\sigma_{\mathrm{FMV}}(F) \setminus \sigma_\pi(F)$ is an open subset of $\mathbb{K}$. Indeed, fix $\lambda \in \sigma_{\mathrm{FMV}}(F)$ with $\lambda \notin \sigma_\pi(F)$. From Theorem 2.4 we know that the subspectrum $\sigma_\pi(F)$ is closed. Therefore it suffices to show that there exists $\delta > 0$ such that $\mu I - F$ is not stably solvable for $|\lambda - \mu| < \delta$.

If this is not true, we find a sequence $(\lambda_n)_n$ with $\lambda_n \to \lambda$ such that $\lambda_n I - F$ is stably solvable for all n. Since $\lambda I - F$ is not stably solvable, by assumption, there exists a compact operator with $[G]_{\mathrm{Q}} = 0$ such that $\lambda x - F(x) \neq G(x)$ for all $x \in X$. On the other hand, the stable solvability of the operators $\lambda_n I - F$ implies the existence of a sequence $(x_n)_n$ with $\lambda_n x_n - F(x_n) = G(x_n)$.

We claim that the sequence $(x_n)_n$ is bounded. In the opposite case we would have $\|x_n\| \to \infty$, hence

$$\begin{aligned}\frac{\|\lambda x_n - F(x_n)\|}{\|x_n\|} &\leq \frac{\|\lambda x_n - \lambda_n x_n\|}{\|x_n\|} + \frac{\|\lambda_n x_n - F(x_n)\|}{\|x_n\|} \\ &= |\lambda - \lambda_n| + \frac{\|G(x_n)\|}{\|x_n\|} \to 0 \quad (n \to \infty).\end{aligned}$$

But this means that $[\lambda I - F]_{\mathrm{q}} = 0$, contradicting our choice $\lambda \notin \sigma_\pi(F)$.

Now, the boundedness of $(x_n)_n$ implies that

$$\|\lambda x_n - F(x_n) - G(x_n)\| = \|\lambda x_n - \lambda_n x_n\| \leq |\lambda - \lambda_n|\, \|x_n\| \to 0 \quad (n \to \infty).$$

From this and the fact that $[\lambda I - F - G]_{\mathrm{a}} = [\lambda I - F]_{\mathrm{a}} > 0$ it follows that there exists a subsequence $(x_{n_k})_k$ of $(x_n)_n$ with $x_{n_k} \to \hat{x}$. By continuity, $\hat{x}$ is a solution of the equation $\lambda x - F(x) - G(x) = \theta$. This contradiction shows that our assumption was false, and so the above assertion is true.

Now let $\lambda \in \partial\sigma_{\mathrm{FMV}}(F)$, and assume that $\lambda \notin \sigma_\pi(F)$. Then $\lambda \in \sigma_{\mathrm{FMV}}(F) \setminus \sigma_\pi(F)$ which is open in $\sigma_{\mathrm{FMV}}(F)$, contradicting $\lambda \in \partial\sigma_{\mathrm{FMV}}(F)$. □

Proposition 6.5 generalizes a well-known fact from linear spectral theory which states that, for $L \in \mathfrak{L}(X)$, the spectra $\sigma_\delta(L)$ and $\sigma_{\mathrm{q}}(L)$ are both closed, and the boundary of the whole spectrum is included in $\sigma_{\mathrm{q}}(L)$, see Proposition 1.4 (a). However, we do not know whether or not the defect spectrum $\sigma_\delta(F)$ is also closed for nonlinear F.

From Proposition 2.5 (i) it follows that for $L \in \mathfrak{L}(X)$ we actually have $\sigma_\pi(L) = \sigma_q(L)$. Thus, in this case we may replace (6.18) by the sharper inclusion

$$\partial\sigma_{FMV}(L) \subseteq \sigma_q(L), \tag{6.19}$$

which is exactly Proposition 1.4 (a). However, the following example shows that the inclusion (6.19) need not be true for nonlinear operators.

Example 6.7. In $X = C[0, 1]$, let $\rho_4 : B(X) \to S(X)$ be the same retraction as in Example 6.4 (or Example 2.35). Define $F \in \mathfrak{C}(X)$ by

$$F(x) = \begin{cases} \rho_4(x) & \text{if } \|x\| \leq 1, \\ x & \text{if } \|x\| > 1. \end{cases}$$

Clearly, $[F]_Q = 1$ and $[F]_A \leq 2$, and thus from (6.12) it follows that the spectrum $\sigma_{FMV}(F)$ is compact. It is not hard to see that $\sigma_q(F) = \{1\}$. Consequently, the inclusion $\partial\sigma_{FMV}(F) \subseteq \sigma_q(F)$ would imply that $\sigma_{FMV}(F) = \{1\}$. On the other hand, we have $0 \in \sigma_\delta(F) \subseteq \sigma_{FMV}(F)$, since F is not onto. ♡

The following theorem provides some further information on the "topological interaction" between the subspectrum $\sigma_\pi(F)$ and the whole spectrum $\sigma_{FMV}(F)$. A natural boundedness condition for the FMV-spectrum has been given in Theorem 6.3. For a closed set $\Sigma \subseteq \mathbb{K}$, we denote as before by $c_\infty[\Sigma]$ the unbounded connected component of $\mathbb{K} \setminus \Sigma$.

Theorem 6.5. *For $F \in \mathfrak{C}(X)$, the following is true:*

(a) *If λ and μ belong to the same component of $\mathbb{K} \setminus \sigma_\pi(F)$, then $\lambda I - F$ and $\mu I - F$ are either both FMV-regular or both not FMV-regular.*

(b) *If $\mathbb{K} = \mathbb{C}$ and $\sigma_{FMV}(F)$ is bounded, then $c_\infty[\sigma_\pi(F)] \subseteq \rho_{FMV}(F)$.*

(c) *If $\mathbb{K} = \mathbb{R}$, $\sigma_{FMV}(F)$ is bounded from above, and λ belongs to the right component $c_\infty[\sigma_\pi(F)]$, then $\lambda \in \rho_{FMV}(F)$; the same holds if $\sigma_{FMV}(F)$ is bounded from below and λ belongs to the left component $c_\infty[\sigma_\pi(F)]$.*

Proof. It is not hard to see that both (b) and (c) follow from (a); so we only have to prove (a). Let C denote the connected component of $\mathbb{K} \setminus \sigma_\pi(F)$ which contains λ and μ, and put

$$C_0 := \{\nu \in C : \nu I - F \text{ is FMV-regular}\}.$$

Suppose that $\lambda \in C_0$; then $C_0 \neq \emptyset$. Since the boundary of C_0 relative to C is empty, by Proposition 6.5, the set C_0 is both open and closed in C. From the connectedness of C it follows that $C_0 = C$, hence $\mu \in C_0$. □

The following lemma illustrates again the usefulness of the seminorm (6.13) and will be important in Section 6.6.

Lemma 6.6. *Let $F, \tilde{F}: X \to X$ two operators such that $p_{AQ}(F - \tilde{F}) = 0$. Then $\sigma_{\mathrm{q}}(F) = \sigma_{\mathrm{q}}(\tilde{F})$, $\sigma_{\mathrm{a}}(F) = \sigma_{\mathrm{a}}(\tilde{F})$, and $\sigma_\delta(F) = \sigma_\delta(\tilde{F})$, and hence $\sigma_{\mathrm{FMV}}(F) = \sigma_{\mathrm{FMV}}(\tilde{F})$.*

Proof. Clearly, $[F - \tilde{F}]_{\mathrm{Q}} = 0$ implies $\sigma_{\mathrm{q}}(F) = \sigma_{\mathrm{q}}(\tilde{F})$, by Proposition 2.2 (d), while $[F - \tilde{F}]_{\mathrm{A}} = 0$ implies $\sigma_{\mathrm{a}}(F) = \sigma_{\mathrm{a}}(\tilde{F})$, by Proposition 2.4 (e). But the condition $p_{AQ}(F - \tilde{F}) = 0$ implies as well that $F - \tilde{F}$ is a compact operator of quasinorm 0. Thus F is stably solvable if and only if $\tilde{F}$ is stably solvable. □

6.5 Special classes of operators

In this section we show that, as one may expect, the theory of the FMV-spectrum becomes richer if we restrict the class of operators under consideration. First of all, it is easy to prove the following analogue to Theorems 4.3 and 5.2.

Theorem 6.6. *Suppose that X is infinite dimensional and $F \in \mathfrak{C}(X)$ is compact; then $0 \in \sigma_{\mathrm{FMV}}(F)$, and hence $\sigma_{\mathrm{FMV}}(F)$ is nonempty.*

In the next three theorems we give a more precise description of the subspectra $\sigma_{\mathrm{q}}(F)$, $\sigma_{\mathrm{a}}(F)$ and $\sigma_\pi(F)$. In particular, Theorem 6.8 provides an interesting sufficient condition under which $\sigma_{\mathrm{q}}(F)$ is nonempty. For a closed set $\Sigma \subseteq \mathbb{K}$ we denote now by $c_0[\Sigma]$ the connected component of $\mathbb{K} \setminus \Sigma$ containing 0.

Theorem 6.7. *Suppose that X is infinite dimensional and $F \in \mathfrak{C}(X)$ is compact. Then the following is true:*

(a) $\sigma_{\mathrm{a}}(F) = \{0\}$, *hence* $\sigma_\pi(F) = \{0\} \cup \sigma_{\mathrm{q}}(F)$.

(b) *F is not onto; in particular,* $0 \in \sigma_\delta(F)$.

(c) *Either* $0 \in \sigma_{\mathrm{q}}(F)$, *or* $c_0[\sigma_{\mathrm{q}}(F)] \subseteq \sigma_\delta(F)$.

(d) *If* $\sigma_{\mathrm{FMV}}(F) \neq \mathbb{K}$, *then* $\sigma_{\mathrm{q}}(F) \neq \emptyset$.

(e) *If* $0 \notin \sigma_{\mathrm{q}}(F)$ *and* $\sigma_{\mathrm{FMV}}(F)$ *is bounded, then* $c_0[\sigma_{\mathrm{q}}(F)]$ *is bounded; consequently,* $\sigma_{\mathrm{q}}(F)$ *contains a positive and a negative value.*

(f) *If* $\mathbb{K} = \mathbb{C}$ *and* $\sigma_{\mathrm{FMV}}(F)$ *is bounded, then* $c_\infty[\sigma_\pi(F)] \cap \sigma_{\mathrm{FMV}}(F) = \emptyset$.

Proof. The assertion (a) follows immediately from the equality $[\lambda I - F]_{\mathrm{a}} = |\lambda|$ (see Proposition 2.4 (i)), while (b) follows from Theorem 3.5.

To see that (c) is true, observe that (a) and the condition $0 \notin \sigma_{\mathrm{q}}(F)$ imply that 0 is an isolated point of $\sigma_\pi(F)$. Therefore it suffices to show that $\lambda I - F$ is not onto for λ small enough. In fact, assume that the set $c_0[\sigma_{\mathrm{q}}(F)] \setminus \sigma_\delta(F)$ is nonempty. Since this set has no boundary in $c_0[\sigma_q(F)]$, by Proposition 6.5, it is both open and closed in $c_0[\sigma_q(F)]$. But $c_0[\sigma_q(F)]$ is connected, by definition, and so $\sigma_\delta(F) = \emptyset$.

Now, to show that $\lambda I - F$ is not onto for λ small enough, assume that this is not the case. Then there exists a sequence $(\lambda_n)_n$ in $\mathbb{K}$ converging to zero such that $\lambda_n I - F$ is onto for all n. Fix $a \in (0, \frac{1}{2}[F]_q)$, and choose $R > 0$ such that $\|F(x)\| \geq 2a\|x\|$ for $\|x\| \geq R$. Taking $b := 2aR$ we have then $\|F(x)\| \geq 2a\|x\| - b$ for all $x \in X$. Consequently, for $|\lambda| \leq a$ we have $\|\lambda x - F(x)\| \geq a\|x\| - b$.

Fix $y \in B(X)$. By assumption, we find a sequence $(x_n)_n$ in X such that $\lambda_n x_n - F(x_n) = y$ for all n. Without loss of generality we may assume that $|\lambda_n| \leq a$ for all n, and thus

$$1 \geq \|y\| = \|\lambda_n x_n - F(x_n)\| \geq a\|x_n\| - b,$$

hence $x_n \in B_r(X)$ with $r := (1 + b)/a$. Since F is compact and $\lambda_n x_n \to \theta$, we conclude that $F(x_n) \to -y$ as $n \to \infty$. But $y \in B(X)$ was arbitrary, and so we see that $B(X) \subseteq \overline{F(B_r(X))}$ which is impossible since $F(B_r(X))$ is precompact. This completes the proof of (c).

If $0 \in \sigma_q(F)$ then (d) is proved. On the other hand, if $0 \notin \sigma_q(F)$, it follows from (c) that 0 is an interior point of $\sigma_{\text{FMV}}(F)$. But (6.18) and (a) show that

$$\partial\sigma_{\text{FMV}}(F) \subseteq \sigma_\pi(F) = \{0\} \cup \sigma_q(F).$$

The assertion follows now from the fact that $\partial\sigma_{\text{FMV}}(F) \neq \emptyset$ since $0 \in \sigma_{\text{FMV}}(F)$ and $\sigma_{\text{FMV}}(F) \neq \mathbb{K}$.

Observe that (e) is an immediate consequence of (c). To prove (f), put $C_\infty := c_\infty[\sigma_\pi(F)]$ and $C := C_\infty \setminus \sigma_{\text{FMV}}(F)$; we have to show that $C = C_\infty$. Since the relative boundary of C with respect to C_∞ is empty, by Proposition 6.5, the set C is both open and closed in C_∞. From the connectedness of C_∞ it follows that either $C = C_\infty$ or $C = \emptyset$. But the latter is impossible if $\sigma_{\text{FMV}}(F) = C_\infty \setminus C$ is bounded. □

Observe that Theorem 6.7 (f) implies the following alternative on the "size" of the subspectrum $\sigma_q(F)$: If $\mathbb{K} = \mathbb{C}$ then either $\sigma_q(F) = \emptyset$ or $0 \in \sigma_q(F)$ or $\sigma_q(F)$ is infinite. This is false in case $\mathbb{K} = \mathbb{R}$. For example, for the operator $F(x_1, x_2, x_3, \dots) := (\|x\|, x_1, x_2, \dots)$ (see Example 3.15) in the real sequence space $X = l_2$ we have $\sigma_q(F) = \{\pm\sqrt{2}\}$.

We illustrate Theorem 6.7 by a simple example. This example shows, in particular, how both cases in Theorem 6.7 (c) may occur, and how this gives information on the subspectrum $\sigma_\delta(F)$ which is most difficult to calculate in practice.

Example 6.8. Let X be an infinite dimensional complex Banach space, $e \in X$ with $\|e\| = 1$, and

$$F_\alpha(x) := \|x\|^\alpha e \quad (\alpha > 0). \tag{6.20}$$

It is clear that $\sigma_a(F_\alpha) = \{0\}$ and $0 \in \sigma_\delta(F_\alpha)$ for any $\alpha > 0$. Concerning $\sigma_q(F_\alpha)$ and $\sigma_\delta(F_\alpha)$, we distinguish three cases.

1st case: $\alpha < 1$. Then $[F_\alpha]_Q = 0$, and so $\sigma_{FMV}(F_\alpha)$ can only contain 0, by (6.12). Indeed, $\sigma_q(F_\alpha) = \{0\}$ since $[F_\alpha]_Q = 0$, and $\sigma_\delta(F_\alpha) = \{0\}$ since F_α is not onto.

2nd case: $\alpha = 1$. This is the most interesting case. Here $\sigma_\delta(F_1) \subseteq \sigma_{FMV}(F_1)$ must be contained in the closure $\overline{\mathbb{D}}$ of the complex unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, again by (6.12). It is easy to see that $\sigma_q(F_1) = \mathbb{S} = \{z \in \mathbb{C} : |z| = 1\}$, hence $c_0[\sigma_q(F_1)] = \mathbb{D}$ and so

$$\mathbb{D} \subseteq \sigma_\delta(F_1) \subseteq \overline{\mathbb{D}},$$

by Theorem 6.7 (c). We claim that the operator $\lambda I - F_1$ is not onto for $|\lambda| = 1$, and so $\sigma_\delta(F_1) = \overline{\mathbb{D}}$. In fact, the element e does not belong to the range of $\lambda I - F_1$ for $|\lambda| = 1$. To see this, observe that the equality $\lambda x = \|x\|e + e$ for some $x \in X$ would imply $\|x\| = \|\lambda x\| = \|x\| + 1$.

Observe that here we have $c_\infty[\sigma_\pi(F_1)] = c_\infty[\{0\} \cup \mathbb{S}] = \mathbb{C} \setminus \overline{\mathbb{D}}$, in accordance with Theorem 6.7 (f).

3rd case: $\alpha > 1$. Here the estimate (6.12) does not provide any information on the size of the spectrum, since $[F_\alpha]_Q = \infty$ for $\alpha > 1$. Nevertheless, it is easy to see directly that $\sigma_q(F_\alpha) = \emptyset$. Indeed, suppose that $\lambda \in \sigma_q(F_\alpha)$. Then there exists a sequence $(x_n)_n$ with $\|x_n\| \to \infty$ and

$$\big|\, |\lambda| - \|x_n\|^{\alpha-1} \big| \leq \frac{\|\lambda x_n - \|x_n\|^\alpha e\|}{\|x_n\|} \to 0 \quad (n \to \infty)$$

which is absurd. So, $\sigma_q(F) = \emptyset$, hence $c_0[\sigma_q(F_\alpha)] = \mathbb{C}$. From Theorem 6.7 (c) we conclude that $\sigma_\delta(F_\alpha) = \mathbb{C}$.

We may summarize our results as follows. The spectrum $\sigma_{FMV}(F_\alpha)$ consists only of zero if F_α has sublinear growth ($\alpha < 1$), coincides with the closed unit disc if F_α has linear growth ($\alpha = 1$), and fills the whole complex plane if F_α has superlinear growth ($\alpha > 1$). This is of course what one could expect for a reasonable "nonlinear" spectrum. ♡

Theorem 6.8. *Suppose that X is infinite dimensional and $F \in \mathfrak{C}(X)$ satisfies*

$$0 \leq [F]_A < [F]_q \leq [F]_Q < \infty. \tag{6.21}$$

Then the subspectrum $\sigma_q(F)$ is nonempty.

Proof. Suppose first that $[F]_A < 1 < [F]_q$, i.e., F is α-contractive. Choose $n_0 \in \mathbb{N}$ such that $\|F(x)\| > \|x\|$ for $\|x\| \geq n_0$, and define $F_n : S_n(X) \to S_n(X)$ by

$$F_n(x) := n\frac{F(x)}{\|F(x)\|} \quad (n \geq n_0).$$

Then $[F_n|_{S_n(X)}]_A = [F]_A < 1$. By Theorem 2.5, we find $x_n \in S_n(X)$ with $F_n(x_n) = x_n$, hence

$$F(x_n) = \frac{\|F(x_n)\|}{n} F_n(x_n) = \frac{\|F(x_n)\|}{n} x_n = \lambda_n x_n$$

with $\lambda_n := \|F(x_n)\|/n$. Obviously, $\|x_n\| \to \infty$. Moreover, since

$$|\lambda_n| = \frac{\|F(x_n)\|}{\|x_n\|} \leq [F]_Q < \infty,$$

the sequence $(\lambda_n)_n$ is bounded. Passing to a subsequence, if necessary, we may therefore assume that $\lambda_n \to \lambda$ as $n \to \infty$. Obviously, the limit λ belongs to $\sigma_q(F)$.

Now suppose the (6.21) holds. Fix $c \in ([F]_A, [F]_q)$ and put $F_c(x) := \frac{1}{c}F(x)$. Then $[F_c]_A = \frac{1}{c}[F]_A < 1$ and $[F_c]_q = \frac{1}{c}[F]_q > 1$, and so we find $\lambda \in \sigma_q(F_c)$, by what we have proved before. But then $c\lambda \in \sigma_q(F)$, and so we are done. □

Example 3.18 shows that Theorem 6.8 is false in finite dimensions; in fact, in that example we have $[F]_A = 0$, $[F]_q = [F]_Q = 1$, and $\sigma_q(F) = \emptyset$. Similarly, Example 6.8 shows (in case $\alpha > 1$) that Theorem 6.8 also fails if F is not quasibounded.

One special class of operators which has interesting properties from the viewpoint of the FMV-spectrum is that of asymptotically linear operators (see Section 4.2). The following theorem gives a rather complete picture for such operators. Recall that $r_\kappa(L)$ denotes the spectral radius (1.74) of any of the essential spectra of $L \in \mathfrak{L}(X)$ considered in Chapter 1 (which is actually independent of the choice of the essential spectrum). In particular, we will use Schechter's essential resolvent set $\rho_{es}(L)$ and essential spectrum $\sigma_{es}(L)$ in the following theorem.

Theorem 6.9. *Let $F \in \mathfrak{C}(X)$ be an asymptotically linear operator with asymptotic derivative $F'(\infty)$. Then the following holds*:

(a) *The equality $\sigma_q(F) = \sigma_q(F'(\infty))$ is true.*

(b) *$\sigma_q(F)$ contains the boundary $\partial\sigma(F'(\infty))$ of $\sigma(F'(\infty))$, and hence is nonempty.*

(c) *If $\lambda \in \mathbb{K} \setminus \sigma_q(F)$ satisfies $|\lambda| > r_{es}(F'(\infty))$, then $\lambda \in \rho(F'(\infty))$.*

(d) *If $\lambda \in \sigma_q(F)$ satisfies $|\lambda| > r_{es}(F'(\infty))$, then $\lambda \in \sigma_p(F'(\infty))$.*

(e) *The estimates $|\lambda| \geq \|F'(\infty)\|$ and $|\lambda| > [F]_A$ imply that $\lambda \in \rho_{FMV}(F)$); in particular, $r_{FMV}(F) \leq \max\{[F]_A, \|F'(\infty)\|\}$.*

(f) *If $F - F'(\infty)$ is compact, then $\sigma_{FMV}(F) = \sigma(F'(\infty))$.*

Proof. The assertion (a) follows immediately from the definition of the asymptotic derivative. From (a) and Proposition 1.4 (a) (or (6.19)) we get

$$\partial\sigma(F'(\infty)) \subseteq \sigma_q(F'(\infty)) = \sigma_q(F),$$

which proves (b). To prove (c), observe first that from the condition $|\lambda| > r_{es}(F'(\infty))$ it follows that either $\lambda \in \sigma_p(F'(\infty))$ or $\lambda \in \rho(F'(\infty))$. Now, $\lambda \in \sigma_p(F'(\infty))$ would imply that $\lambda \in \sigma_q(F'(\infty))$, and so $\lambda \in \sigma_q(F)$, by (a), contradicting our hypothesis.

Let us now prove (d). Since $\lambda \in \sigma_q(F) = \sigma_q(F'(\infty))$, we may choose a sequence $(e_n)_n$ in $S(X)$ such that $\|\lambda e_n - F'(\infty)e_n\| \to 0$ as $n \to \infty$. The condition $|\lambda| > r_{es}(F'(\infty))$ implies $\lambda \in \rho_{es}(F'(\infty))$ (see (1.68)), i.e., $\lambda I - F'(\infty)$ is a Fredholm operator of index zero. Consequently, $[\lambda I - F'(\infty)]_a > 0$, by Proposition 1.2 (k), and so $\lambda I - F'(\infty)$ is proper on $S(X)$, by Proposition 2.4 (b). This means that we have $e_{n_k} \to e \in S(X)$, as $k \to \infty$, for some suitable subsequence $(e_{n_k})_k$ of $(e_n)_n$. But then $F'(\infty)e = \lambda e$, and so $\lambda \in \sigma_p(F'(\infty))$ as claimed.

To see that (e) is true, observe first that $\lambda \notin \sigma_q(F)$, by (a), and $\lambda \notin \sigma_a(F)$, by (2.35), since $|\lambda| > [F]_A$. So it remains to show that $\lambda \notin \sigma_\delta(F)$. To this end, let $G: X \to X$ be compact with $[G]_Q = 0$. Then the operator H defined by $H(x) := (F(x) + G(x))/\lambda$ satisfies $[H]_A = [F]_A/|\lambda| < 1$ and

$$[H]_Q = \frac{[F]_Q}{|\lambda|} = \frac{[F'(\infty)]_Q}{|\lambda|} = \frac{\|F'(\infty)\|}{|\lambda|} < 1,$$

by assumption. From Theorem 2.2 we conclude that the operator H has a fixed point in X; this fixed point solves then the equation $\lambda x - F(x) = G(x)$. So we have shown that $\lambda I - F$ is stably solvable, i.e., $\lambda \notin \sigma_\delta(F)$. The estimate for the FMV-spectral radius of F is an immediate consequence.

Assertion (f) is a trivial consequence of Lemma 6.6, since both $[F - F'(\infty)]_A = 0$ and $[F - F'(\infty)]_Q = 0$. □

6.6 The AGV-spectrum

We show now how strictly stably solvable operators may be used to define a certain variant of the FMV-spectrum. Recall that $F: X \to Y$ is called strictly stably solvable if $\mu(F) > 0$ (see (6.4)), i.e., F is k-stably solvable for some $k > 0$. The following Proposition 6.6 characterizes strictly stable solvability for linear operators.

Proposition 6.6. *Let $L: X \to Y$ be bounded and linear. If L is strictly stably solvable, then L is onto. Conversely, suppose that L is onto, and assume that the canonical quotient map $\nu: X \to X/N(L)$ has a continuous right inverse* (*not necessarily linear*) *$\tau: X/N(L) \to X$ satisfying*

$$p_{AQ}(\tau) < \infty, \tag{6.22}$$

where p_{AQ} is defined as in (6.13). *Then L is strictly stably solvable.*

Proof. The first statement has already been proved in Lemma 6.1. Let L be onto, and define $L_0: X/N(L) \to Y$ by the equality $L_0\nu(x) := Lx$. By construction, L_0 is then a linear bijection. We claim that

$$\mu(L) \geq \frac{[\![L_0]\!]}{p_{AQ}(\tau)},$$

where $[\![L_0]\!]$ denotes the inner norm (1.79) of L_0. In fact, fix $k < [\![L_0]\!] / p_{AQ}(\tau)$, and consider any $G: X \to Y$ with $[G]_A \leq k$ and $[G]_Q \leq k$. Then the continuous map $H = \tau L_0^{-1} G: X \to X$ satisfies $[H]_A < 1$ and $[H]_Q < 1$ and thus has a fixed point $x \in X$, by Theorem 2.2. But $x = \tau(L_0^{-1}G(x))$ implies $\nu(x) = L_0^{-1}G(x)$, and so $Lx = L_0\nu(x) = G(x)$. □

We point out that a map τ as required in Proposition 6.6 exists (even linear) if the nullspace $N(L)$ is topologically complemented. This holds, in particular, if L is one-to-one, or if X is a Hilbert space. We do not know, however, whether or not one may always find τ satisfying (6.22).

In view of the spectrum defined below, it seems desirable to find a definition of "regularity" such that at least linear "regular" maps are one-to-one. Let us call an operator $F \in \mathfrak{C}(X, Y)$ *AGV-regular* if both $[F]_q > 0$ and $\mu(F) > 0$. The following Proposition 6.7 together with Example 6.9 show that this is strictly weaker than FMV-regularity.

Proposition 6.7. *Every FMV-regular operator is AGV-regular.*

Proof. Let $F: X \to Y$ be FMV-regular. Then Lemma 6.3 implies that F is k-stably solvable for $k < \min\{[F]_a, [F]_q\}$, and so F is strictly stably solvable, hence AGV-regular. □

Example 6.9. Let $X = C[0, 1]$ be the space of all continuous real functions on $[0, 1]$ with the maximum norm. Fix some $c \in (0, 1)$, and define $F: X \to X$ by

$$F(x)(t) = \begin{cases} x(ct) & \text{if } \|x\| \leq c, \\ x(t\|x\|) & \text{if } c < \|x\| < 1, \\ x(t) & \text{if } \|x\| \geq 1. \end{cases} \tag{6.23}$$

We claim that $[F]_a = 0$, but F is strictly stably solvable and $[F]_q > 0$. This implies, of course, that F is AGV-regular, but not FMV-regular.

Indeed, $[F]_a = 0$ follows from the fact that F maps the (non-precompact) set

$$M_c = \{x \in X : \|x\| \leq c,\ x(t) \equiv 0 \text{ for } 0 \leq t \leq c\}$$

into θ. We claim that F is k-stably solvable for $k < \frac{1}{4}$. To see this, let $G \in \mathfrak{C}(X)$ be given with $[G]_A \leq k < \frac{1}{4}$ and $[G]_Q \leq k < \frac{1}{4}$; we have to prove that the equation $F(x) = G(x)$ has a solution. Define a map $H: X \to X$ by

$$H(x)(t) = \begin{cases} G(x)(t/c) & \text{if } \|x\| \leq c \text{ and } t \leq c, \\ G(x)(t/\|x\|) & \text{if } c < \|x\| < 1 \text{ and } t \leq \|x\|, \\ G(x)(1) & \text{if } c < \|x\| < 1 \text{ and } t > \|x\|, \\ G(x)(1) & \text{if } \|x\| \leq c \text{ and } t > c, \\ G(x)(t) & \text{if } \|x\| \geq 1. \end{cases} \tag{6.24}$$

A comparison of (6.23) and (6.24) shows that any fixed point x of H solves the equation $F(x) = G(x)$. To show the existence of such a fixed point, we use Theorem 2.2. First of all, we have $[H]_Q = [G]_Q < 1$. Furthermore, to see that $[H]_A < 1$, recall that, for any bounded $M \subseteq X$, we have

$$\frac{1}{2}\omega(M) \leq \alpha(M) \leq 2\omega(M), \tag{6.25}$$

where

$$\omega(M) = \inf_{\delta>0} \sup_{x\in M} \sup_{|t-s|\le\delta} |x(t) - x(s)|$$

denotes the modulus of continuity of M. In particular, given $M \subseteq X$ and $\varepsilon > 0$, we find some $\delta > 0$ such that $|t - s| \le \delta$ implies

$$|G(x)(t) - G(x)(s)| \le 2\alpha(G(M)) + \varepsilon \quad (x \in M).$$

Hence, the relation $|t - s| \le c\delta$ implies, by $|t - s| \le \delta$ and $|t - s|/c \le \delta$, that

$$|H(x)(t) - H(x)(s)| \le 2\alpha(G(M)) + \varepsilon \quad (x \in M).$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $\omega(H(M)) \le 2\alpha(G(M))$, and so $\alpha(H(M)) \le 4\alpha(G(M))$ which implies that $[H]_A \le 4[G]_A \le 4k < 1$, as desired. ♡

Now we define a modification of the FMV-spectrum. For $F \in \mathfrak{C}(X)$ we call the set

$$\rho_{AGV}(F) = \{\lambda \in \mathbb{K} : \lambda I - F \text{ is AGV-regular}\} \tag{6.26}$$

the *Appell–Giorgieri–Väth resolvent set* and its complement

$$\sigma_{AGV}(F) = \mathbb{K} \setminus \rho_{AGV}(F) \tag{6.27}$$

the *Appell–Giorgieri–Väth spectrum* (or *AGV-spectrum*, for short) of F. This spectrum seems to be simpler than the FMV-spectrum (6.10), since in the definition of AGV-regularity we got rid of the somewhat artificial compactness assumption which appears in the definition of FMV-regular operators.

Similarly as before, we have the decomposition

$$\sigma_{AGV}(F) = \sigma_\mu(F) \cup \sigma_q(F), \tag{6.28}$$

where we have put

$$\sigma_\mu(F) = \{\lambda \in \mathbb{K} : \mu(\lambda I - F) = 0\}, \tag{6.29}$$

and $\sigma_q(F)$ is as in (2.29). As an immediate consequence of Proposition 6.7 we get the following trivial, though useful, inclusions

$$\rho_{AGV}(F) \supseteq \rho_{FMV}(F), \quad \sigma_{AGV}(F) \subseteq \sigma_{FMV}(F). \tag{6.30}$$

Moreover, for the operator F in Example 6.9 we have $0 \in \sigma_{FMV}(F)$ but $0 \notin \sigma_{AGV}(F)$, and so strict inequality may occur in (6.30).

The following Theorems 6.10 and 6.11 show that also the spectrum (6.28) reduces to the familiar one in case of linear operators, and that for nonlinear operators the AGV-spectrum has the same "nice" properties as the FMV-spectrum.

Theorem 6.10. *Let $L\colon X \to Y$ be bounded and linear. Then $\lambda \in \rho_{AGV}(L)$ if and only if $\lambda I - L$ is an isomorphism.*

Proof. If $\lambda I - L$ is an isomorphism, then $\lambda I - L$ is FMV-regular, by Theorem 6.1, and so also AGV-regular, by Proposition 6.7. Conversely, if $\lambda I - L \in \rho_{\mathrm{AGV}}(L)$, then $\lambda I - L$ is strictly stably solvable and thus onto, by Proposition 6.6. Moreover, $[\lambda I - L]_{\mathrm{q}} > 0$ implies that $N(\lambda I - L) = \{\theta\}$, and thus $\lambda I - L$ is injective. □

Theorem 6.11. *The spectrum $\sigma_{\mathrm{AGV}}(F)$ is closed. Moreover, if F satisfies $p_{AQ}(F) < \infty$, with p_{AQ} as in* (6.13)*, then $\sigma_{\mathrm{AGV}}(F)$ is also bounded, hence compact. Finally, the multivalued map $\sigma_{\mathrm{AGV}} : \mathfrak{A}(X) \cap \mathfrak{Q}(X) \to 2^{\mathbb{K}}$ which associates to each F its AGV-spectrum is upper semicontinuous in the FMV-topology.*

Proof. Fix $\lambda \in \rho_{\mathrm{AGV}}(F)$, and let $0 < \delta < \min\{\mu(\lambda I - F), [\lambda I - F]_{\mathrm{q}}\}$. We claim that $\mu \in \rho_{\mathrm{AGV}}(F)$ for $|\mu - \lambda| < \delta$. In fact, from $[(\mu - \lambda)I]_{\mathrm{Q}} = |\mu - \lambda| < [\lambda I - F]_{\mathrm{q}}$ it follows that $\mu \notin \sigma_{\mathrm{q}}(F)$. Moreover, (6.6) implies that

$$\mu(\mu I - F) \geq \mu(\lambda I - F) - |\lambda - \mu| > \mu(\lambda I - F) - \delta > 0,$$

hence $\mu \in \rho_{\mathrm{AGV}}(F)$. This shows that λ is an interior point of $\rho_{\mathrm{AGV}}(F)$, and thus $\rho_{\mathrm{AGV}}(F)$ is open in $\mathbb{K}$.

The second and third assertion of Theorem 6.11 are direct consequences of the inclusions (6.30). □

If we define the *AGV-spectral radius* of F by

$$r_{\mathrm{AGV}}(F) = \sup\{|\lambda| : \lambda \in \sigma_{\mathrm{AGV}}(F)\}, \tag{6.31}$$

we get from (6.12) and (6.30) the trivial estimate

$$r_{\mathrm{AGV}}(F) \leq \max\{[F]_{\mathrm{A}}, [F]_{\mathrm{Q}}\}. \tag{6.32}$$

We make some comments on Theorem 6.11. As we have seen, both spectra $\sigma_{\mathrm{FMV}}(F)$ and $\sigma_{\mathrm{AGV}}(F)$ are always closed. However, it is not known if this is true for the subspectrum $\sigma_\delta(F)$ which constitutes the "characteristic ingredient" of the FMV-spectrum. On the other hand, it follows immediately from Lemma 6.4 (a) that the subspectrum $\sigma_\mu(F)$ which constitutes the "characteristic ingredient" of the AGV-spectrum, is in fact closed. Moreover, it is not hard to see that the multivalued map $\sigma_\mu : \mathfrak{A}(X) \cap \mathfrak{Q}(X) \to 2^{\mathbb{K}}$ which associates to each F the subspectrum (6.29) is upper semicontinuous in the FMV-topology. From this point of view, the AGV-spectrum has "nicer topological properties" than the FMV-spectrum. On the other hand, the FMV-spectrum has the boundary property (6.18) which is quite useful for localizing $\sigma_{\mathrm{FMV}}(F)$ in the complex plane. An analogous boundary property for the AGV-spectrum is not known (and is probably not true). From this point of view, the FMV-spectrum has "nicer geometrical properties" than the AGV-spectrum.

We close with an example which shows that the AGV-spectrum (and hence also the FMV-spectrum, by (6.30)) may be unbounded. From the upper estimate (6.12) it follows that such an example must involve an operator F satisfying $[F]_{\mathrm{A}} = \infty$ or $[F]_{\mathrm{Q}} = \infty$.

Example 6.10. Let X and F be defined as in (4.12) over $\mathbb{K} = \mathbb{R}$, i.e.,

$$F(x_1, x_2, x_3, \dots) = (x_1^2, x_2^2, x_3^2, \dots).$$

For any $\lambda \in \mathbb{R} \setminus \{0\}$, the operator equation

$$\lambda x - F(x) = (\lambda x_1, \lambda x_2, \lambda x_3, \dots) - (x_1^2, x_2^2, x_3^2, \dots) = (\lambda^2, 0, 0, \dots)$$

is not solvable, since the scalar equation $x_1^2 - \lambda x_1 + \lambda^2 = 0$ has no real solution. This shows that $\lambda I - F$ is not surjective, and so $\mathbb{R} \setminus \{0\} \subseteq \sigma_\mu(F)$. Since the AGV-spectrum is closed, we conclude that $\sigma_{\mathrm{AGV}}(F) = \sigma_{\mathrm{FMV}}(F) = \mathbb{R}$. ♡

Let us return once more to the problem of eigenvalues. As we have seen in Example 6.6, the FMV-spectrum (and so also the AGV-spectrum, by (6.30)) may be disjoint to the classical point spectrum $\sigma_{\mathrm{p}}(F)$, *provided that we define the point spectrum in the naive sense of* (3.18). Instead of considering this as a drawback of the FMV-spectrum, however, one could adapt the definition of the term "eigenvalue" to make it compatible with the definition of the FMV-spectrum. So, let us call $\lambda \in \mathbb{K}$ an *asymptotic eigenvalue* of $F : X \to X$ if there exists an unbounded sequence $(x_n)_n$ in X such that

$$\frac{\|\lambda x_n - F(x_n)\|}{\|x_n\|} \to 0 \quad (n \to \infty). \tag{6.33}$$

Of course, the set of all asymptotic eigenvalues of F is then nothing else but the subspectrum $\sigma_{\mathrm{q}}(F)$ defined in (2.29). Consequently, *the set of all asymptotic eigenvalues is contained in the AGV-spectrum* (6.27) *and the FMV-spectrum* (6.10), by definition. For linear operators L we have the trivial inclusion

$$\sigma_{\mathrm{p}}(L) \subseteq \sigma_{\mathrm{q}}(L), \tag{6.34}$$

but for nonlinear operators F there is of course no relation between $\sigma_{\mathrm{p}}(F)$ and $\sigma_{\mathrm{q}}(F)$. For instance, for the operator F in Example 6.6 we have

$$\sigma_{\mathrm{p}}(F) \cap \sigma_{\mathrm{q}}(F) = (\mathbb{R} \setminus \{0\}) \cap \{0\} = \emptyset.$$

This shows that the subspectrum (2.29) is more natural as a counterpart of the FMV-spectrum than the classical point spectrum (3.18).

We illustrate (6.34) by means of the three linear operators contained in Example 1.5 in the space $X = l_p$ $(1 \le p \le \infty)$.

First, for the operator (1.41), i.e.,

$$L(x_1, x_2, x_3, x_4, \dots) = (x_1, \tfrac{1}{2}x_2, \tfrac{1}{3}x_3, \tfrac{1}{4}x_4, \dots)$$

we get

$$\sigma_{\mathrm{p}}(L) = \{1, \tfrac{1}{2}, \tfrac{1}{3}, \dots\}, \quad \sigma_{\mathrm{q}}(L) = \{0, 1, \tfrac{1}{2}, \tfrac{1}{3}, \dots\}.$$

In fact, 0 does not belong to the point spectrum of L, since L is injective. On the other hand, for the canonical basis element e_k in X we have $\|e_k\| = 1$ and $\|L(e_k)\| = \frac{1}{k} \to 0$, which shows that $0 \in \sigma_{\mathrm{q}}(L)$, and so we have strict inequality in (6.34).

Second, for the operator (1.42), i.e.,

$$L(x_1, x_2, x_3, x_4, \dots) = (x_2, \tfrac{1}{2}x_3, \tfrac{1}{3}x_4, \tfrac{1}{4}x_5, \dots)$$

we get

$$\sigma_{\mathrm{p}}(L) = \{0\} = \sigma_{\mathrm{q}}(L).$$

Third, for the operator (1.44), i.e.,

$$L(x_1, x_2, x_3, x_4, \dots) = (0, x_1, \tfrac{1}{2}x_2, \tfrac{1}{3}x_3, \dots)$$

we get

$$\sigma_{\mathrm{p}}(L) = \emptyset, \quad \sigma_{\mathrm{q}}(L) = \{0\},$$

and so we have again strict inequality in (6.34).

Another notion of eigenvalue which is "compatible" with the FMV-spectrum is as follows. Let us call $\lambda \in \mathbb{K}$ an *unbounded eigenvalue* of $F\colon X \to X$ if there exists an unbounded sequence $(x_n)_n$ in X such that

$$F(x_n) = \lambda x_n \quad (n = 1, 2, 3, \dots), \tag{6.35}$$

i.e., $x_n \in N(\lambda I - F)$ for all n. Moreover, we call the set

$$\sigma_{\mathrm{p}}^0(F) := \{\lambda \in \mathbb{K} : \lambda \text{ unbounded eigenvalue of } F\} \tag{6.36}$$

the *unbounded point spectrum* of F (by "abuse de langage"). We have then the obvious inclusions

$$\sigma_{\mathrm{p}}^0(F) \subseteq \sigma_{\mathrm{p}}(F), \quad \sigma_{\mathrm{p}}^0(F) \subseteq \sigma_{\mathrm{q}}(F) \tag{6.37}$$

for general F, and

$$\sigma_{\mathrm{p}}^0(L) = \sigma_{\mathrm{p}}(L) \subseteq \sigma_{\mathrm{q}}(L) \tag{6.38}$$

for $L \in \mathfrak{L}(X)$. Although the second inclusion in (6.37) may be strict, the following theorem shows that $\sigma_{\mathrm{p}}^0(F)$ is not "too far" from $\sigma_{\mathrm{q}}(F)$.

Proposition 6.8. *The graph of the multivalued map $F \mapsto \sigma_{\mathrm{q}}(F)$ is the closure of the graph of the multivalued map $F \mapsto \sigma_{\mathrm{p}}^0(F)$ in the FMV-topology. Moreover, the representation*

$$\begin{aligned}\sigma_{\mathrm{q}}(F) = \{\lambda \in \mathbb{K} : &\text{ there exist sequences } (F_n)_n \text{ and } (\lambda_n)_n \text{ such that}\\ &p_{AQ}(F_n - F) \to 0,\ \lambda_n \to \lambda,\ \text{and } \lambda_n \in \sigma_{\mathrm{p}}^0(F_n) \text{ for all } n \in \mathbb{N}\}\\ = \{\lambda \in \mathbb{K} : &\text{ there exists } G \text{ such that } p_{AQ}(G - F) = 0 \text{ and } \lambda \in \sigma_{\mathrm{p}}^0(G)\}.\end{aligned} \tag{6.39}$$

is true, where p_{AQ} denotes the seminorm (6.13) *which generates the FMV-topology.*

Proof. Fix $\lambda \in \sigma_{\mathrm{q}}(F)$, and let $(x_n)_n$ be an unbounded sequence satisfying (6.33). Passing to a subsequence, if necessary, we may assume that $\|x_m - x_n\| > 2$ for $m \neq n$. Put $\eta_n(x) := \max\{1 - \|x - x_n\|, 0\}$, so that $\eta_n(x_m) = \delta_{mn}$, and define $G\colon X \to X$ by

$$G(x) := F(x) + \sum_{n=1}^{\infty} \eta_n(x)(\lambda x_n - F(x_n)).$$

Then $G(x_n) = \lambda x_n$ for all $n \in \mathbb{N}$, and so $\lambda \in \sigma_{\mathrm{p}}^0(G)$.

We claim that $p_{AQ}(G - F) = 0$. In fact,

$$[G - F]_{\mathrm{Q}} \leq 2 \limsup_{n\to\infty} \frac{\|\lambda x_n - F(x_n)\|}{\|x_n\|} = 0.$$

Moreover, $[G - F]_{\mathrm{A}} = 0$, because $G - F$ maps every bounded subset of X into a bounded subset of a finite dimensional space. This proves one inclusion in (6.39).

It is not hard to see that the graph of the multivalued map σ_{q} is closed. Indeed, let $(\lambda_n)_n$ and $(F_n)_n$ be sequences such that $\lambda_n \to \lambda$ in $\mathbb{K}$, $F_n \to F$ in the FMV-topology, and $\lambda_n \in \sigma_{\mathrm{q}}(F_n)$ for all n. Then

$$[\lambda I - F]_{\mathrm{Q}} \leq [(\lambda - \lambda_n)I]_{\mathrm{Q}} + [F - F_n]_{\mathrm{Q}} \leq |\lambda - \lambda_n| + p_{AQ}(F - F_n) \to 0$$

as $n \to \infty$, and so $\lambda \in \sigma_{\mathrm{q}}(F)$. By the second inclusion in (6.37), the graph of σ_{q} thus contains the closure of the graph of σ_{p}^0, and the converse inclusion follows. □

There is another deep fact which shows that $\sigma_{\mathrm{q}}(F)$ is the appropriate "discrete part" of the FMV-spectrum. Recall that, as mentioned in Theorem 1.2 (c), every nonzero spectral point of a *compact* linear operator is an eigenvalue. In the following Theorem 6.12 we will prove a nontrivial analogue to this fact for the FMV-spectrum of a class of operators which are in a certain sense "close" to linear operators. Let us call a continuous operator $F\colon X \to Y$ *asymptotically odd* if there exists an odd operator $\tilde{F}\colon X \to Y$ such that $[F - \tilde{F}]_{\mathrm{Q}} = 0$. For example, every asymptotically linear operator is certainly asymptotically odd. If F is compact and asymptotically odd, it is not hard to see that also its "asymptotic odd approximation" $\tilde{F}$ may be chosen compact.

Theorem 6.12. *Let $F\colon X \to X$ be compact and asymptotically odd. Then every $\lambda \in \sigma_{\mathrm{FMV}}(F) \setminus \{0\}$ is an asymptotic eigenvalue of F.*

Proof. Fix $\lambda \in \sigma_{\mathrm{FMV}}(F)$, $\lambda \neq 0$. From $[\lambda I - F]_{\mathrm{a}} \geq |\lambda| - [F]_{\mathrm{A}} = |\lambda| > 0$ it follows that $\lambda \in \sigma_\delta(F)$ or $\lambda \in \sigma_{\mathrm{q}}(F)$. We only have to exclude the first possibility, i.e., we must show that $\lambda I - F$ is stably solvable.

So let $G\colon X \to X$ be compact with $[G]_{\mathrm{Q}} = 0$. We suppose first that F is odd and G has bounded support, i.e., $G(x) \equiv \theta$ for $\|x\| \geq R$ with suitable $R > 0$. Without loss of generality, we may assume that $\lambda x - F(x) \neq \theta$ for $\|x\| \geq R$, because otherwise we already get $[\lambda I - F]_{\mathrm{q}} = 0$. Now, the operator $H\colon B_R(X) \to X$ defined by $H(x) := (F(x) + G(x))/\lambda$ coincides with F/λ on $S_R(X)$ and is odd there. By Borsuk's theorem (see Section 3.5), the topological degree $\deg(I - H, B_R^o(X), \theta)$ is odd, and so nonzero. Consequently, we find some $\hat{x} \in B_R^o(X)$ such that $\hat{x} = H(\hat{x})$, i.e., $\lambda\hat{x} - F(\hat{x}) = G(\hat{x})$. We conclude that $\lambda I - F$ is stably solvable, and so $\lambda \notin \sigma_\delta(F)$.

If the support of G is not bounded, we replace $G(x)$ by $G_n(x) = d_n(\|x\|)G(x)$, with $d_n(t)$ as in (6.8), and proceed as in the proof of Proposition 6.3. So we have proved the assertion in case F is compact and odd.

Assume now that F is compact and asymptotically odd, and choose a compact odd operator $\tilde{F}$ such that $[F - \tilde{F}]_Q = 0$. Since $F - \tilde{F}$ is compact, we have $p_{AQ}(F - \tilde{F}) = 0$, and so $\sigma_{\mathrm{FMV}}(F) = \sigma_{\mathrm{FMV}}(\tilde{F})$ and $\sigma_{\mathrm{q}}(F) = \sigma_{\mathrm{q}}(\tilde{F})$, by Lemma 6.6. From what we have just proved we conclude that

$$\sigma_{\mathrm{FMV}}(F) \setminus \{0\} = \sigma_{\mathrm{FMV}}(\tilde{F}) \setminus \{0\} \subseteq \sigma_{\mathrm{q}}(\tilde{F}) = \sigma_{\mathrm{q}}(F),$$

and so we are done. □

In the following example we show that "asymptotic eigenvalue" cannot be replaced by "unbounded eigenvalue" in Theorem 6.12.

Example 6.11. Let X be an infinite dimensional real Banach space, $e \in S(X)$ fixed, and $\ell \in X^*$ a bounded linear functional on X such that $\ell(e) = 1$. Define $F\colon X \to X$ by

$$F(x) = \frac{\|x\|}{1 + \|x\|}\ell(x)e. \tag{6.40}$$

Obviously, F is compact and odd, and $[F]_Q \le \|\ell\|$, and thus $\sigma_{\mathrm{FMV}}(F) \subseteq [-\|\ell\|, \|\ell\|]$, by (6.12). Choosing $x_n = ne$ we get

$$\frac{\|x_n - F(x_n)\|}{\|x_n\|} = \frac{|n - \frac{n}{1+n}n|}{n} = \frac{1}{n+1} \to 0 \quad (n \to \infty),$$

and so $1 \in \sigma_{\mathrm{q}}(F) \subseteq \sigma_{\mathrm{FMV}}(F)$. On the other hand, we claim that $\sigma_{\mathrm{p}}^0(F) = \emptyset$.

To see this, consider the eigenvalue equation $F(x) = \lambda x$. Any solution of this equation must be of the form $x = \mu e$ with $\lambda = |\mu|/(1 + |\mu|)$; viceversa, $x = \mu e$ with $\mu = \pm\lambda/(1 - \lambda)$ solves this equation. So we see that $\sigma_{\mathrm{p}}(F) = (0, 1)$, i.e., every $\lambda \in (0, 1)$ is a classical eigenvalue. However, the fact that the equation $F(x) = \lambda x$ has at most two nontrivial solutions x shows that λ cannot be an unbounded eigenvalue, and so $\sigma_{\mathrm{p}}^0(F) = \emptyset$. ♡

6.7 Notes, remarks and references

As pointed out several times above, the FMV-spectrum is one of most useful nonlinear spectra from the viewpoint of applications. The class of stably solvable operators which is basic in the definition of this spectrum was introduced in [121], where also some results and applications (without proofs) are given. Michael's selection theorem which we used in the proof of Lemma 6.1 may be found in [191], Dugundji's extension theorem which we used in the proof of Lemma 6.2 in [97].

The more general definition of k-stably solvable and strictly stably solvable operators was given in the recent note [16], where also the measure of stable solvability (6.4) was defined. Proposition 6.2 is proved in [16] in the same way as Proposition 6.1 in [122]. In fact, [16] contains Proposition 6.2 in a slightly more general form which reads as follows. Let us call an operator $F\colon X \to Y$ (k, l)-*stably solvable* $(k, l \geq 0)$ if the coincidence equation $F(x) = G(x)$ has a solution in X for every operator $G\colon X \to Y$ satisfying $[G]_{\mathrm{A}} \leq k$ and $[G]_{\mathrm{Q}} \leq l$.

Proposition 6.9. *Let $F\colon X \to Y$ be (k, l)-stably solvable, and suppose that $H\colon X \times [0, 1] \to Y$ is continuous with $[H(\cdot, 0)]_{\mathrm{Q}} < 1$ and*

$$\alpha(H(M \times [0, 1]) \leq k\alpha(M) \quad (M \subset X \text{ bounded}).$$

Let S be defined as in (6.1)*, and assume that $F(S)$ is bounded in Y. Then the equation $F(x) = H(x, 1)$ has a solution $\hat{x} \in X$.*

Obviously, in case $k = l$ we get exactly Proposition 6.2, since (k, k)-stable solvability means simply k-stable solvability in the sense of Section 6.1.

The highly nontrivial Example 6.4 which solves a long-standing open problem was found quite recently by Furi [118]. Unfortunately, this does not allow us to deduce that the subspectrum $\sigma_\delta(F)$ defined in (6.15) is closed; this is another interesting problem which so far is open.

FMV-regular operators have been introduced in the survey paper [122]. All results in Section 6.2 are taken from that paper. The role of essential compact vector fields in both the theory and applications of topological nonlinear analysis was illustrated by Granas in [138]. We remark that Nirenberg [206] has given a characterization of essential compact vector fields via stable homotopy theory, as well as several striking applications to partial differential equations. A certain generalization of Lemma 6.3 and Lemma 6.4 to larger classes of operators may be found in [16].

Proposition 6.3 shows that, in spite of the apparently very technical definition, the notion of FMV-regularity is quite natural. This is also illustrated by the following interesting connection between FMV-regularity and topology of Euclidean space.

Recall that two continuous functions $F, G\colon S(\mathbb{R}^n) \to S(\mathbb{R}^m)$ are called *homotopic* if there exists a continuous *homotopy* $H\colon S(\mathbb{R}^n) \times [0, 1] \to S(\mathbb{R}^m)$ such that $H(x, 0) = F(x)$ and $H(x, 1) = G(x)$ for all $x \in S(\mathbb{R}^n)$. We denote the corresponding homotopy class of F by $\langle F \rangle$. Now, suppose that $F \in \mathfrak{C}(\mathbb{R}^n, \mathbb{R}^m)$ is such that the set $F^-(\{\theta\})$ of all zeros of F is bounded; a sufficient condition for this is obviously $[F]_{\mathrm{q}} > 0$. Choose $R > 0$ with $F^-(\{\theta\}) \subseteq B_R(\mathbb{R}^n)$. Then the function $F_R\colon S(\mathbb{R}^n) \to S(\mathbb{R}^m)$ with $F_R(x) := F(Rx)/\|F(Rx)\|$ is well defined, and we call $\langle F_R \rangle$ the *homotopy class* of F. Since $\langle F_R \rangle$ is independent of R for sufficiently large R, we simply write $\langle F \rangle$ instead of $\langle F_R \rangle$. The following Proposition 6.10 was proved in [122].

Proposition 6.10. *Let $F \in \mathfrak{C}(\mathbb{R}^n, \mathbb{R}^m)$ be given with $[F]_q > 0$. Then F is FMV-regular if and only if the homotopy class $\langle F \rangle$ of F is nontrivial, i.e., contains only nonconstant functions.*

The applications of FMV-regular operators go far beyond the material we have presented in Section 6.2. For example, the following result which is Theorem 10.1.7 in [122] is usually referred to as *Borsuk–Ulam theorem* (on antipodal points).

Theorem 6.13. *Let X_0 be a proper closed subspace of a Banach space X, and let $F: S(X) \to X_0$ be a compact vector field. Then there exists a point $\hat{x} \in S(X)$ such that $F(-\hat{x}) = F(\hat{x})$.*

In the next chapter we will prove an extension of Theorem 6.13 from compact to α-contractive vector fields which uses another spectrum.

The results and examples in Sections 6.3, 6.4 and 6.5 are all taken from [122]. Only the proof of our Theorem 6.2, which is taken from [13], seems to be simpler than the proof of the corresponding Theorem 8.1.2 of [122].

Example 6.6 shows that the FMV-spectrum is not compatible with the usual notion of point spectrum $\sigma_p(F)$ in the sense of (3.18), but very well with the different notion of asymptotic point spectrum $\sigma_q(F)$ in the sense of (2.29). This has motivated the Chinese mathematician Wenying Feng to introduce another spectrum which has similar properties as the FMV-spectrum, but in addition contains the point spectrum $\sigma_p(F)$. We will discuss this spectrum in detail in the next chapter.

The fact that the FMV-spectrum does not contain the eigenvalues in the sense of definition (3.18) was apparently observed first in [103]. Example 6.6 may be extended in some sense as follows [103]: *if X is a Banach space and $F: X \to X$ satisfies $F(\theta) \neq \theta$ and $p_{AQ}(F) = 0$, then $\sigma_{\text{FMV}}(F) = \{0\}$ but $\sigma_p(F) = \mathbb{K} \setminus \{0\}$.* This shows that the disjointness of the FMV-spectrum and the point spectrum is not a rare exception, but a "typical" phenomenon for compact nonlinear operators of strictly sublinear growth.

One might think that the reason for the disjointness of the FMV-spectrum and the point spectrum in the result above is the assumption $F(\theta) \neq \theta$. However, even if this assumption is dropped, this disjointness result is in a certain sense "typical". The following example from [103] clarifies this point.

Example 6.12. Let X be any real Banach space, $e \in S(X)$ and $R > 0$ be fixed, and $F_R: X \to X$ be defined by

$$F_R(x) = \min\{\|x\|^2, R^2\}\, e. \tag{6.41}$$

Clearly, F_R is compact and satisfies $[F_R]_Q = 0$. Since F_R is not onto, from (6.12) it follows that $\sigma_{\text{FMV}}(F_R) = \{0\}$.

On the other hand, a simple geometric reasoning shows that $\sigma_p(F_R) = [-R, 0) \cup (0, R]$. In particular, 0 is not an eigenvalue of F_R, since $F_R(x) = \theta$ implies $x = \theta$. So $\sigma_{\text{FMV}}(F_R) \cap \sigma_p(F_R) = \emptyset$ for each $R > 0$.

Observe that we can make the point spectrum $\sigma_{\mathrm{p}}(F_R)$ arbitrarily large by letting $R \to \infty$, while the FMV-spectrum $\sigma_{\mathrm{FMV}}(F_R)$ is always the singleton $\{0\}$ for any R. Loosely speaking, this means that, in contrast to the point spectrum, the FMV-spectrum does not "feel" the influence of the truncation parameter R in (6.41). ♡

One special class of operators which has been studied in detail from the viewpoint of the FMV-spectrum in [103] is that of asymptotically linear operators. This is not surprising, because the FMV-spectrum has "asymptotic" nature, and the existence of $F'(\infty)$ means that F is "close to linear" on large spheres. Thus, Theorem 6.9 is one of the main results in [103].

All definitions and results in Section 6.6 are taken from the paper [16]. In particular, Example 6.9 shows that the FMV-spectrum may be strictly larger than the AGV-spectrum, since the latter need not contain all scalars λ with $[\lambda I - F]_{\mathrm{a}} = 0$. The estimate (6.25) which was used in Example 6.9 is a simple generalization of the Arzelà–Ascoli compactness criterion.

We point out that the following more general result than Theorem 6.12 holds which is a precise analogue to the "linear" result in Theorem 1.3 (c).

Theorem 6.14. *Let $F \in \mathfrak{A}(X)$ be asymptotically odd. Then every $\lambda \in \sigma_{\mathrm{FMV}}(F)$ satisfying $|\lambda| > [F]_{\mathrm{A}}$ is an asymptotic eigenvalue of F.*

Proposition 6.8, which is taken from [232], shows that (6.33) is a "good" definition of eigenvalue in the framework of the FMV-spectrum. However, as pointed out above, it is also possible (and perhaps more natural) to adapt the definition of eigenvalue in another way. So, the conditions (6.33) and (6.35) are by no means the only reasonable definitions of eigenvalues for nonlinear operators; other definitions will be given in Chapter 10 below.

Chapter 7

The Feng Spectrum

In this chapter we discuss another spectrum which is similar to the Furi–Martelli–Vignoli spectrum, but has the advantage to contain the eigenvalues. To this end, we first recall the definition and some properties of epi and k-epi operators. Afterwards we consider a class of regular operators which play the same role in the definition of the Feng spectrum as the FMV-regular operators in the definition of the FMV-spectrum. In the last section we give a comparison of all the spectra introduced so far from the viewpoint of their analytical and topological properties.

7.1 Epi and k-epi operators

Let X and Y be Banach spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Throughout the following, we denote by $\mathfrak{OBC}(X)$ the family of all open, bounded, connected subsets Ω of X with $\theta \in \Omega$. Given $\Omega \in \mathfrak{OBC}(X)$, a continuous operator $F\colon \overline{\Omega} \to Y$ is called *epi* on $\overline{\Omega}$ if $F(x) \neq \theta$ on $\partial\Omega$ and, for any compact operator $G\colon \overline{\Omega} \to Y$ satisfying $G(x) \equiv \theta$ on $\partial\Omega$, the equation

$$F(x) = G(x) \tag{7.1}$$

has a solution $x \in \Omega$. More generally, we call F a *k-epi* operator ($k \geq 0$) on $\overline{\Omega}$ if the above property holds for all operators with $[G]_{\mathrm{A}} \leq k$, not just compact operators. Clearly, a k-epi operator is also k'-epi for any $k' \leq k$; in particular, every k-epi operator is epi. Roughly speaking, one could say that, the larger one may choose k, the higher is the "degree of solvability" of equation (7.1).

This motivates the following definition. Given an operator $F\colon X \to Y$ and $\Omega \in \mathfrak{OBC}(X)$, we put

$$\nu_\Omega(F) := \inf\{k : k \geq 0,\ F \text{ is not } k\text{-epi on } \overline{\Omega}\} \tag{7.2}$$

and

$$\nu(F) := \inf_{\Omega \in \mathfrak{OBC}(X)} \nu_\Omega(F). \tag{7.3}$$

We call the number (7.3) the *measure of solvability of F*. (Here we put again $\inf \emptyset := \infty$.) Observe that $\nu(F) > 0$ if and only if there exists some $k > 0$ such that $F\colon \overline{\Omega} \to Y$ is k-epi for *every* $\Omega \in \mathfrak{OBC}(X)$. In particular, we have then $F(x) \neq \theta$ for all $x \neq \theta$.

In this chapter we will sometimes consider operators F which are defined only on the closure of some set $\Omega \in \mathfrak{OBC}(X)$. In this case the characteristics $[F|_{\overline{\Omega}}]_{\mathrm{A}}$ and

$[F|_{\overline{\Omega}}]_a$ refer to subsets of $\overline{\Omega}$, i.e., (2.12) and (2.14) are meant merely for $M \subseteq \overline{\Omega}$. These characteristics have the following obvious monotonicity property: if $\Omega' \subseteq \Omega$ then

$$[F|_{\overline{\Omega}}]_a \leq [F|_{\overline{\Omega'}}]_a \leq [F|_{\overline{\Omega'}}]_A \leq [F|_{\overline{\Omega}}]_A.$$

Observe that every continuous operator $G\colon \overline{\Omega} \to Y$ with $G|_{\partial\Omega} = \Theta$ admits a continuous extension

$$\hat{G}(x) = \begin{cases} G(x) & \text{if } x \in \overline{\Omega}, \\ \theta & \text{if } x \in X \setminus \overline{\Omega} \end{cases} \tag{7.4}$$

which satisfies $[\hat{G}]_A = [G|_{\overline{\Omega}}]_A$ and $[\hat{G}]_Q = 0$. This fact, which will be used several times, follows from the estimates

$$\alpha(\hat{G}(M)) = \alpha(\hat{G}(M \cap \overline{\Omega})) = \alpha(G(M \cap \overline{\Omega})) \leq [G]_A \alpha(M \cap \overline{\Omega}) \leq [G]_A \alpha(M)$$

for $M \subset X$ bounded. In what follows, we refer to (7.4) as the *trivial extension* of G. Although the above definitions are rather technical, the numbers (7.2) and (7.3) may be in fact calculated in some cases. We start with the most elementary case $X = Y = \mathbb{R}$; here we have of course $\mathfrak{OBC}(\mathbb{R}) = \{(a, b) : -\infty < a < 0 < b < \infty\}$.

Example 7.1. Let $F\colon [a, b] \to \mathbb{R}$ be continuous with $F(a)F(b) \neq 0$. From the intermediate value theorem it follows that F is k-epi (for every $k \geq 0$) if and only if $F(a)F(b) < 0$. In this case we have $\nu_{(a,b)}(F) = \infty$, in case $F(a)F(b) > 0$ we have $\nu_{(a,b)}(F) = 0$. ♡

Example 7.2. Let X and Y be Banach spaces and $L \in \mathfrak{L}(X, Y)$ a linear isomorphism. We claim that

$$\nu(L) \geq \lfloor L \rfloor,$$

where $\lfloor L \rfloor$ is the inner norm (1.79) of L. Indeed, suppose that $G\colon \overline{\Omega} \to X$ is continuous with $[G|_{\overline{\Omega}}]_A < \lfloor L \rfloor$ and $G|_{\partial\Omega} = \Theta$ for some $\Omega \in \mathfrak{OBC}(X)$. Then the operator $L^{-1}\hat{G}\colon X \to X$, with $\hat{G}$ as in (7.4), satisfies $[L^{-1}\hat{G}]_A \leq \|L^{-1}\| \, [G|_{\overline{\Omega}}]_A < 1$ and $[L^{-1}\hat{G}]_Q = 0$. From Theorem 2.2 it follows that $L^{-1}\hat{G}$ has a fixed point in X (actually, in Ω). This shows that $\nu_\Omega(L) \geq \lfloor L \rfloor$, hence also $\nu(L) \geq \lfloor L \rfloor$. ♡

The measure of solvability $\nu(I)$ is of particular interest. Of course, $\nu(I) = \infty$ in finite dimensions. Indeed, if $\Omega \in \mathfrak{OBC}(\mathbb{R}^n)$ and $G\colon \overline{\Omega} \to \mathbb{R}^n$ is continuous with $G(x) \equiv \theta$ on $\partial\Omega$, the extension (7.4) is compact with $[\hat{G}]_Q = 0$. Again from Theorem 2.2 it follows that G has a fixed point, and thus the set on the right-hand side of (7.2) is empty.

In Section 6.1 we have seen that $\mu(I) = 1$ in every infinite dimensional Banach space, with $\mu(F)$ being the measure of stable solvability (6.4). Later (Proposition 7.3) we will show that $\nu(I) = 1$ as well in every infinite dimensional Banach space. The next lemma provides a comparison between the measure of stable solvability (6.4) and the measure of solvability (7.3) for general operators F. Afterwards we give two examples where we estimate $\nu(I)$ from above in special spaces, just to illustrate its calculation.

Lemma 7.1. *For any $F \in \mathfrak{C}(X, Y)$ we have the estimate*

$$\mu(F) \leq \nu(F). \tag{7.5}$$

Proof. If $\nu(F) = \infty$ there is nothing to prove. Given $k > \nu(F)$, choose $\Omega \in \mathfrak{OBC}(X)$ and $G\colon \overline{\Omega} \to Y$ such that $[G|_{\overline{\Omega}}]_{\mathrm{A}} \leq k$, $G|_{\partial\Omega} = \Theta$, and $F(x) \neq G(x)$ for $x \in \overline{\Omega}$. The trivial extension (7.4) satisfies then $[\hat{G}]_{\mathrm{A}} \leq k$, $[\hat{G}]_{\mathrm{Q}} = 0$, and $F(x) \neq \hat{G}(x)$ for all $x \in X$, since $F(x) \neq \theta$ for $x \neq \theta$. So we see that $\mu(F) \leq k$ as claimed. □

Example 7.3. Let $X = l_2(\mathbb{Z})$ be the Hilbert space of all real sequences $x = (x_k)_k$ ($k \in \mathbb{Z}$) for which the norm

$$\|x\| = \Big(\sum_{k=-\infty}^{\infty} |x_k|^2 \Big)^{1/2}$$

is finite. Denote by $\{e_k : k \in \mathbb{Z}\}$ the canonical basis in X, i.e., $e_k = (\delta_{k,n})_n$, and define $L\colon X \to X$ by

$$L\Big(\sum_{k=-\infty}^{\infty} x_k e_k \Big) = \sum_{k=-\infty}^{\infty} (x_k - x_{k-1}) e_k.$$

Moreover, define $G\colon B(X) \to X$ by

$$G(x) = (1 - \|x\|)(Lx + e_0).$$

We claim that G has no fixed points in $B(X)$. Suppose that $G(\hat{x}) = \hat{x}$ for some $\hat{x} \in X$; it is clear that then $0 < \|\hat{x}\| < 1$, since $G(\theta) = e_0 \neq \theta$, and $G(S(X)) = \{\theta\}$. Writing the equality $G(\hat{x}) = \hat{x}$ in coordinates we obtain

$$(1 - \|\hat{x}\|)(\hat{x}_k - \hat{x}_{k-1}) = \hat{x}_k \quad (k \neq 0),$$

hence

$$\hat{x}_k \|\hat{x}\| = -(1 - \|\hat{x}\|)\hat{x}_{k-1} \quad (k \neq 0).$$

But for $0 < \|\hat{x}\| \leq \frac{1}{2}$ this implies that $|\hat{x}_k| \not\to 0$ as $k \to \infty$, while for $\frac{1}{2} \leq \|\hat{x}\| < 1$ it implies that $|\hat{x}_k| \not\to 0$ as $k \to -\infty$. In both cases $\hat{x}$ cannot belong to $l_2(\mathbb{Z})$.

We further claim that $[G|_{B(X)}]_{\mathrm{A}} \leq 2$. In fact, for $M \subseteq B(X)$ we have

$$G(M) \subseteq \overline{\mathrm{co}}(\{\theta\} \cup [L(M) + e_0])$$

and so $\alpha(G(M)) \leq \alpha(L(M)) \leq 2\alpha(M)$, since $\|L\| = 2$. This shows that

$$1 \leq \nu(I) \leq \nu_{B^o(X)}(I) \leq 2$$

in the space $X = l_2(\mathbb{Z})$. ♡

Example 7.4. Let c_0 be the Banach space of all real sequences $x = (x_n)_n$ $(n \in \mathbb{N})$ converging to zero, equipped with the maximum norm. It is easy to see that the map $G\colon B(X) \to S(X)$ defined by

$$G(x_1, x_2, x_3, \dots) = (1, x_1, x_2, \dots)$$

is an isometry (i.e., $\|G(x) - G(y)\| = \|x - y\|$), but has no fixed points in $B(X)$. Define $\tilde{G}\colon X \to X$ by $\tilde{G}(x) = d(\|x\|)G(x)$, where

$$d(t) := \begin{cases} 1 & \text{for } 0 \le t \le 1, \\ 2 - t & \text{for } 1 \le t \le 2, \\ 0 & \text{for } t \ge 2. \end{cases}$$

Again, $\tilde{G}$ has no fixed points in X. In fact, suppose that $\tilde{G}(\hat{x}) = \hat{x}$ for some $\hat{x} \in X$; it is clear that then $1 < \|\hat{x}\| < 2$. This means that $\hat{x} = (2 - \|\hat{x}\|)G(\hat{x})$ and

$$\|\hat{x}\| = \sup\left\{(2 - \|\hat{x}\|)^n : n = 1, 2, 3, \dots\right\} = 2 - \|\hat{x}\|,$$

hence $\|\hat{x}\| = 1$, a contradiction.

We claim that $[\tilde{G}]_{\mathrm{A}} = 1$. In fact, for $M \subseteq B_2(X)$ we have

$$\tilde{G}(M) \subseteq \overline{\mathrm{co}}(\{\theta\} \cup G(M))$$

and so $\alpha(\tilde{G}(M)) \le \alpha(G(M)) \le \alpha(M)$. This shows that

$$1 \le \nu(I) \le \nu_{B_2^o(X)}(I) \le 1,$$

i.e., $\nu(I) = 1$, in the space $X = c_0$. ♡

Finally, in the next example we present an operator which is epi, but not k-epi on $\overline{\Omega} = B(X)$ for any $k > 0$.

Example 7.5. Let X and F be defined as in Example 6.4. We claim that F is epi on $B(X)$. To see this, let $G\colon B(X) \to X$ be compact with $G(x) \equiv \theta$ on $S(X)$, and denote by $\hat{G}\colon X \to X$ its trivial extension (7.4). Since F is stably solvable, we find $\hat{x} \in X$ such that $F(\hat{x}) = \hat{G}(\hat{x})$.

Now, the assumption $\|\hat{x}\| > 1$ leads to $1 < \|\hat{x}\|^2 = \|F(\hat{x})\| = \|\hat{G}(\hat{x})\| = 0$, a contradiction. Consequently, we have $\|\hat{x}\| \le 1$, hence $\hat{G}(\hat{x}) = G(\hat{x})$. This shows that F is in fact epi on $B(X)$.

Suppose now that F is k-epi for some $k > 0$. Choosing $G = F - F_n$ for $kn \ge 3$ as in Example 6.4, and considering again the trivial extension $\hat{G}$ of this G, we see that the equation $F(x) = G(x)$, i.e., $F_n(x) = \theta$, has no solution in $B(X)$, and so we are done. ♡

Given $k \geq 0$, we define an operator $F\colon X \to Y$ to be *k-proper* if

$$\alpha(F^{-}(N)) \leq k\alpha(N) \quad (N \subset Y \text{ bounded}). \tag{7.6}$$

Evidently, every operator which is k-proper for some $k > 0$ is proper in the sense of the usual definition (see Section 3.1), and every k-proper operator is also k'-proper for $k' \geq k$. So the characteristic

$$[F]_{\pi} = \inf\{k : k \geq 0,\ F \text{ is } k\text{-proper}\} \tag{7.7}$$

is of interest; the following lemma shows that we actually know already this characteristic, at least for coercive operators F.

Lemma 7.2. *Suppose that $F\colon X \to Y$ is coercive. Then the equality*

$$[F]_{\pi} = \frac{1}{[F]_{\mathrm{a}}}$$

is true, where $[F]_{\mathrm{a}}$ is defined as in (2.15).

Proof. Without loss of generality we may assume that $[F]_{\pi} < \infty$ and $[F]_{\mathrm{a}} > 0$. Then for fixed $k > [F]_{\pi}$ we have

$$\alpha(M) \leq \alpha(F^{-}(F(M))) \leq k\alpha(F(M)) \quad (M \subset X)$$

hence

$$\frac{\alpha(F(M))}{\alpha(M)} \geq \frac{1}{k},$$

which shows that $[F]_{\mathrm{a}} \geq 1/k$. Since $k > [F]_{\pi}$ is arbitrary we get $[F]_{\mathrm{a}} \geq 1/[F]_{\pi}$.

On the other hand, for fixed $k < [F]_{\mathrm{a}}$ and $F^{-}(N)$ bounded we have

$$\alpha(N) \geq \alpha(F(F^{-}(N))) \geq k\alpha(F^{-}(N)) \quad (N \subset Y)$$

hence

$$\frac{\alpha(F^{-}(N))}{\alpha(N)} \leq \frac{1}{k},$$

which shows that $[F]_{\pi} \leq 1/k$. Since $k < [F]_{\mathrm{a}}$ is arbitrary we get $[F]_{\pi} \leq 1/[F]_{\mathrm{a}}$. □

The next lemma connects k-proper and k-epi operators. More precisely, it provides an upper estimate for $\nu(F)$ in terms of $[F]_{\pi}$, and vice versa.

Lemma 7.3. *Suppose that $\Omega \in \mathfrak{OBC}(X)$, $F : \overline{\Omega} \to Y$ is injective and k-proper, and $F(\Omega)$ is open. Then F is k'-epi on $\overline{\Omega}$ for every $k' \in [0, 1/k)$ if and only if $\theta \in F(\Omega)$.*

Proof. Since F is injective and proper, the inverse operator $F^{-1}\colon F(\overline{\Omega}) \to \overline{\Omega}$ is continuous. Moreover, $\partial F(\Omega) = F(\partial\Omega)$, since $\overline{F(\Omega)} = F(\overline{\Omega})$ and $F(\Omega)$ is open.

Now, if F is epi, the trivial choice $G(x) \equiv \theta$ shows that $\theta \in F(\Omega)$.

Conversely, suppose that $\theta \in F(\Omega)$, and let $G\colon \overline{\Omega} \to Y$ be continuous with $[G|_{\overline{\Omega}}]_{\mathrm{A}} \leq k' < 1/k$ and $G(x) \equiv \theta$ on $\partial\Omega$. Define $H\colon Y \to Y$ by

$$H(y) = \begin{cases} G(F^{-1}(y)) & \text{if } y \in \overline{F(\Omega)}, \\ \theta & \text{if } y \in Y \setminus \overline{F(\Omega)}. \end{cases}$$

For $y \in \partial F(\Omega) = F(\partial\Omega)$, i.e., $y = F(x)$ with $x \in \partial\Omega$, we have $H(y) = G(x) = \theta$; this shows that H is continuous. Furthermore, for bounded $M \subset Y$ we have

$$\alpha(H(M)) = \alpha(G(F^{-}[M \cap \overline{F(\Omega)}])) \leq k'\alpha(F^{-}[M \cap \overline{F(\Omega)}]) \leq k'k\alpha(M)$$

which shows that $[H]_{\mathrm{A}} \leq k'k < 1$. Finally, $H(Y)$ is bounded, since

$$H(Y) = H(\overline{F(\Omega)}) \cup \{\theta\} = G(\overline{\Omega}) \cup \{\theta\},$$

and so $[H]_{\mathrm{Q}} = 0$. From Theorem 2.2 it follows that there exists $\hat{y} \in Y$ such that $\hat{y} = H(\hat{y}) = G(F^{-1}(\hat{y}))$; here $\hat{y} \in F(\Omega)$ since $\theta \in F(\Omega)$. Consequently, $\hat{x} := F^{-1}(\hat{y}) \in \Omega$ solves the equation $F(x) = G(x)$, and so we are done. □

Using the characteristic (7.7) we may write the assertion of Lemma 7.3 in the more compact form

$$\nu_{\Omega}(F) \geq \frac{1}{[F|_{\overline{\Omega}}]_{\pi}}.$$

An essentially stronger version of this will be given in Theorem 7.1 below.

Observe that Lemma 7.3 gives again the statement of Example 7.2. In fact, if $L \in \mathfrak{L}(X, Y)$ is an isomorphism, all the hypotheses of Lemma 7.3 are satisfied for $\Omega = B^{o}(X)$ and $k = \|L^{-1}\|$.

Now we state five important properties of k-epi operators. In particular, Property 7.4 gives a certain continuation principle for epi and k-epi operators which is parallel to Proposition 6.4 and will be used many times in the sequel. It is instructive to compare these properties with those of the topological degree which we enumerated in Section 3.5.

Property 7.1 (Existence). *Suppose that $F\colon \overline{\Omega} \to Y$ is epi on $\overline{\Omega}$. Then the equation $F(x) = \theta$ has a solution in Ω.*

Proof. The assertion follows by the trivial choice $G(x) \equiv \theta$. □

Property 7.2 (Normalization). *Suppose that $F\colon \overline{\Omega} \to X$ is compact with $F(x) \equiv \theta$ on $\partial\Omega$. Then $I - F$ is k-epi on $\overline{\Omega}$ for every $k < 1$.*

Proof. Let $H : \overline{\Omega} \to X$ be continuous with $H|_{\partial\Omega} = \Theta$ and $[H|_{\overline{\Omega}}]_A \le k$, where $k < 1$ is fixed. Then the operator $G = F + H$ satisfies $[G|_{\overline{\Omega}}]_A \le [F|_{\overline{\Omega}}]_A + [H|_{\overline{\Omega}}]_A \le k < 1$, i.e., is α-contractive, and so is its trivial extension (7.4). Moreover, $G(x) \equiv \theta$ on $\partial\Omega$. Theorem 2.2 implies that there exists $\hat{x} \in \Omega$ with $G(\hat{x}) = \hat{x}$, i.e., $(I - F)(\hat{x}) = H(\hat{x})$. □

Property 7.3 (Localization). *Let $F : \overline{\Omega} \to Y$ be k-epi on $\overline{\Omega}$ and $F^{-}(\{\theta\}) \subseteq \Omega'$, where $\Omega' \in \mathfrak{OBC}(X)$ and $\Omega' \subseteq \Omega$. Then F is also k-epi on $\overline{\Omega}'$.*

Proof. Let $G : \overline{\Omega}' \to Y$ be continuous with $[G|_{\overline{\Omega}'}]_A \le k$ and $G|_{\partial\Omega'} = \Theta$. Denote by $\hat{G}$ the trivial extension (7.4) of G outside Ω'. The inclusion $\partial\Omega \subset X \setminus \Omega'$ implies that $\hat{G}(x) \equiv \theta$ on $\partial\Omega$. Thus the equation $F(x) = \hat{G}(x)$ has a solution $\hat{x} \in \Omega$. But from the assumption $F^{-}(\{\theta\}) \subseteq \Omega'$ it follows that $\hat{x} \in \Omega'$, and so we are done. □

Property 7.4 (Homotopy). *Suppose that $F_0 : \overline{\Omega} \to Y$ is k_0-epi on $\overline{\Omega}$ for some $k_0 \ge 0$, and $H : \overline{\Omega} \times [0, 1] \to Y$ is continuous with $H(x, 0) \equiv \theta$ and*

$$\alpha(H(M \times [0, 1])) \le k\alpha(M) \quad (M \subseteq \overline{\Omega})$$

for some $k \le k_0$. Let

$$S = \{x \in \overline{\Omega} : F_0(x) + H(x, t) = \theta \text{ for some } t \in [0, 1]\}. \tag{7.8}$$

If $S \cap \partial\Omega = \emptyset$ then the operator $F_1 := F_0 + H(\,\cdot\,, 1)$ is k_1-epi on $\overline{\Omega}$ for $k_1 \le k_0 - k$.

Proof. Let $G : \overline{\Omega} \to Y$ be continuous with $[G|_{\overline{\Omega}}]_A \le k_1$ and $G|_{\partial\Omega} = \Theta$. We have to show that the equation $F_1(x) = F_0(x) + H(x, 1) = G(x)$ has a solution $\hat{x} \in \Omega$.

Since F_0 is k_0-epi and $k_1 \le k_0$, we find $\tilde{x} \in \Omega$ such that $F_0(\tilde{x}) + H(\tilde{x}, 0) = F_0(\tilde{x}) = G(\tilde{x})$. This shows that the set

$$S_G := \{x \in \overline{\Omega} : F_0(x) + H(x, t) = G(x) \text{ for some } t \in [0, 1]\}$$

is nonempty. Moreover, $S_G \cap \partial\Omega = \emptyset$. We claim that S_G is also closed. Indeed, fix $x_0 \in X \setminus S_G$. By the compactness of the interval $[0, 1]$ we may find finitely many sets $I_1, I_2, \ldots, I_m \subseteq [0, 1]$ covering $[0, 1]$, and corresponding neighbourhoods $U_1, U_2, \ldots, U_m \subseteq X$ of x_0 such that

$$F_0(x) + H(x, t) \ne G(x) \quad (x \in U_1 \cup U_2 \cup \cdots \cup U_m,\ 0 \le t \le 1).$$

Then $U := U_1 \cap U_2 \cap \cdots \cap U_m$ is a neighbourhood of x_0 which is disjoint from S_G, and so $x_0 \notin \overline{S_G}$.

Now define $\pi : \overline{\Omega} \to [0, 1]$ by

$$\pi(x) = \frac{\operatorname{dist}(x, \partial\Omega)}{\operatorname{dist}(x, \partial\Omega) + \operatorname{dist}(x, S_G)},$$

and $\hat{G}\colon \overline{\Omega} \to Y$ by

$$\hat{G}(x) = G(x) - H(x, \pi(x)).$$

Then $[\hat{G}|_{\overline{\Omega}}]_{\mathrm{A}} \le k_0$ and $\hat{G}(x) = G(x) - H(x, 0) \equiv \theta$ for $x \in \partial\Omega$. Consequently, there exists $\hat{x} \in \Omega$ such that

$$F_0(\hat{x}) = \hat{G}(\hat{x}) = G(\hat{x}) - H(\hat{x}, \pi(\hat{x})).$$

Since $0 \le \pi(\hat{x}) \le 1$, this implies that $\hat{x} \in S_G$, hence actually $\pi(\hat{x}) = 1$. Consequently, $F_0(\hat{x}) = G(\hat{x}) - H(\hat{x}, 1)$, and so $F_1(\hat{x}) = G(\hat{x})$ as claimed. □

Property 7.5 (Degree). *Suppose that $F\colon \overline{\Omega} \to X$ is α-contractive with $F(x) \ne x$ on $\partial\Omega$, and assume that* $\deg(I - F, \Omega, \theta) \ne 0$. *Then $I - F$ is k-epi on $\overline{\Omega}$ for every $k \in [0, 1 - [F|_{\overline{\Omega}}]_{\mathrm{A}})$.*

Proof. Let $H\colon \overline{\Omega} \to Y$ be continuous with $H|_{\partial\Omega} = \Theta$ and $[H|_{\overline{\Omega}}]_{\mathrm{A}} \le k$, where $k < 1 - [F|\overline{\Omega}]_{\mathrm{A}}$. Then the operator $G = F + H$ satisfies $[G|_{\overline{\Omega}}]_{\mathrm{A}} \le [F|_{\overline{\Omega}}]_{\mathrm{A}} + [H|_{\overline{\Omega}}]_{\mathrm{A}} < 1$, i.e., is α-contractive. Moreover, $G(x) \equiv F(x)$ on $\partial\Omega$. From the boundary dependence of the topological degree (Property 3.7) we deduce that

$$\deg(I - G, \Omega, \theta) = \deg(I - F, \Omega, \theta) \ne 0,$$

and so there exists $\hat{x} \in \Omega$ with $\hat{x} = G(\hat{x}) = F(\hat{x}) + H(\hat{x})$, hence $\hat{x} - F(\hat{x}) = H(\hat{x})$ as claimed. □

7.2 Feng-regular operators

In this section we need again the lower characteristics $[F]_{\mathrm{b}}$ in (2.7) and $[F]_{\mathrm{a}}$ in (2.15). Let us call a continuous operator $F\colon X \to X$ *Feng-regular* (or *F-regular*, for short) if the numbers $[F]_{\mathrm{a}}$ and $[F]_{\mathrm{b}}$ are strictly positive, and F is epi on $\overline{\Omega}$ for all $\Omega \in \mathfrak{O}(X)$. The following remarkable theorem shows that this already implies that F is k-epi for some positive k; we will use this result several times in this and the following chapter.

Theorem 7.1. *Let $F\colon \overline{\Omega} \to Y$ be epi on $\overline{\Omega}$ and $[F|_{\overline{\Omega}}]_{\mathrm{a}} > 0$. Then F is even k-epi on $\overline{\Omega}$ for sufficiently small $k > 0$. More precisely, the lower estimate*

$$\nu_{\Omega}(F) \ge [F|_{\overline{\Omega}}]_{\mathrm{a}} \tag{7.9}$$

for the characteristic (7.2) *holds true.*

Proof. Assume that we are given an operator $G\colon \overline{\Omega} \to Y$ with $[G|_{\overline{\Omega}}]_{\mathrm{A}} < [F|_{\overline{\Omega}}]_{\mathrm{a}}$ and $G(x) \equiv \theta$ on $\partial\Omega$. We have to prove that the equation $F(x) = G(x)$ has a solution in Ω.

To this end, let $\mathfrak{U}$ denote the set of all subsets $U \subseteq X$ with

$$F^{-}(\overline{\mathrm{co}}[G(U) \cup \{\theta\}]) \subseteq U. \tag{7.10}$$

(Recall that $F^{-}(M) = \{x \in \overline{\Omega} : F(x) \in M\}$ is defined even if M is not a subset of $F(\overline{\Omega})$.) In particular, $\mathfrak{U}$ is nonempty, since $\overline{\Omega} \in \mathfrak{U}$. The idea of the proof is to show that $\mathfrak{U}$ contains some set U for which the set

$$W := \overline{\text{co}}(G(U) \cup \{\theta\}) \tag{7.11}$$

is compact. If such a set exists, the statement can be proved as follows. Since W is convex, compact, and nonempty (because $\theta \in W$), there exists a retraction $\rho\colon Y \to W$ onto W, i.e., ρ is continuous with $\rho(y) = y$ on W. Define now an operator $G_0\colon \overline{\Omega} \to Y$ by $G_0(x) = \rho(G(x))$. Then G_0 attains its values in the compact set W, and, for $x \in \partial\Omega$, we have $G_0(x) = \rho(\theta) = \theta$, because $\theta \in W$. Since F is epi on $\overline{\Omega}$, we thus find a solution $\hat{x} \in \Omega$ of the equation $F(x) = G_0(x)$. Since $G_0(\hat{x}) \in W$, we have $\hat{x} \in F^{-}(W) \subseteq U$, and so $G(\hat{x}) \in W$ which implies

$$G(\hat{x}) = \rho(G(\hat{x})) = G_0(\hat{x}) = F(\hat{x}).$$

To prove that $\mathfrak{U}$ contains a set U such that (7.11) is compact, we show first that $\mathfrak{U}$ contains a set U_0 which satisfies the relation

$$F^{-}(\overline{\text{co}}[G(U_0) \cup \{\theta\}]) = U_0. \tag{7.12}$$

Indeed, we already observed that $\mathfrak{U} \neq \emptyset$. Consequently, the set

$$U_0 := \bigcap_{U \in \mathfrak{U}} U$$

is well-defined. This set belongs to $\mathfrak{U}$, because for any $U \in \mathfrak{U}$ the set

$$U_1 := F^{-}(\overline{\text{co}}[G(U_0) \cup \{\theta\}])$$

is contained in $F^{-}(\overline{\text{co}}[G(U) \cup \{\theta\}]) \subseteq U$, and so $U_1 \subseteq U_0$. Moreover, this inclusion also implies that

$$F^{-}(\overline{\text{co}}[G(U_1) \cup \{\theta\}]) \subseteq F^{-}(\overline{\text{co}}[G(U_0) \cup \{\theta\}]) = U_1,$$

and so we have $U_1 \in \mathfrak{U}$ as well. The definition of U_0 thus implies $U_0 \subseteq U_1$, and so we actually have $U_0 = U_1$, i.e., equality (7.12) holds.

We prove now that for any set $U = U_0$ which satisfies (7.12), the set (7.11) is compact which in view of the above observations implies the statement. The compactness of U already follows from the inclusion

$$F^{-}(\overline{\text{co}}[G(U) \cup \{\theta\}]) \supseteq U. \tag{7.13}$$

Observe that (7.13) is dual to the relation (7.10) which defines the elements of $\mathfrak{U}$ so that we actually need both inclusion of the set equality (7.12) for our proof. In fact, the inclusion (7.13) is equivalent to the inclusion

$$F(U) \subseteq \overline{\text{co}}(G(U) \cup \{\theta\})$$

which implies

$$[F|_{\overline{\Omega}}]_{\mathrm{a}}\alpha(U) \le \alpha(F(U)) \le \alpha(\overline{\mathrm{co}}[G(U) \cup \{\theta\}]) = \alpha(G(U)) \le [G|_{\overline{\Omega}}]_{\mathrm{A}}\alpha(U).$$

Since $[G|_{\overline{\Omega}}]_{\mathrm{A}} < [F|_{\overline{\Omega}}]_{\mathrm{a}}$, this is only possible if $\alpha(U) = 0$; so $\overline{U}$ is compact. Since G is continuous, the set $G(\overline{U})$ is compact, too. In particular, $G(U)$ is precompact which implies, in view of Mazur's lemma, that the set (7.11) is compact as claimed. □

The following result establishes a connection between the F-regularity of an operator and its FMV-regularity in the sense of Section 6.2.

Theorem 7.2. *Every F-regular operator F is FMV-regular.*

Proof. Let F be F-regular, i.e., $[F]_{\mathrm{b}} > 0$, $[F]_{\mathrm{a}} > 0$, and $\nu(F) > 0$. The inequality $[F]_{\mathrm{b}} > 0$ implies, in particular, that $F(\theta) = \theta$. From (2.8) we get then $[F]_{\mathrm{q}} > 0$; so it remains to show that F is stably solvable. Thus, let $G\colon X \to Y$ be compact with $[G]_{\mathrm{Q}} = 0$. If G has bounded support, i.e., $\|G(x)\| = 0$ for $\|x\| \ge R$ with some $R > 0$, we may use the fact that F is epi on $B_R(X)$ and conclude that the equation $F(x) = G(x)$ has a solution in $B_R(X)$. If the support of G is not compact, we replace $G(x)$ by $G_n(x) = d_n(\|x\|)G(x)$ with d_n as in (6.8) and follow the same reasoning as in the proof of Proposition 6.3. □

In spite of its technical character, the following theorem shows that the definition of Feng-regularity is quite natural; compare this with Lemma 6.1 and Theorem 6.1.

Theorem 7.3. *Every F-regular operator $F\colon X \to Y$ is surjective. Furthermore, $L \in \mathfrak{L}(X, Y)$ is F-regular if and only if L is an isomorphism.*

Proof. The first statement is a trivial consequence of Theorem 7.2 and the fact that FMV-regular operators are stably solvable, hence surjective.

Consider now an operator $L \in \mathfrak{L}(X, Y)$. If L is F-regular, then L is surjective, by what we just proved, and also injective with $\|L^{-1}y\| \le [L]_{\mathrm{b}}^{-1}\|y\|$, since $[L]_{\mathrm{b}} > 0$.

Conversely, suppose that L is a linear isomorphism. Then $L^{-1} \in \mathfrak{L}(Y, X)$ and $\|Lx\| \ge \lfloor L \rfloor\, \|x\|$, with $\lfloor L \rfloor$ as in (1.79), hence $[L]_{\mathrm{b}} \ge \lfloor L \rfloor$ and $[L]_{\mathrm{a}} \ge \lfloor L \rfloor$. It remains to show that $\nu(L) > 0$. But being an isomorphism, L satisfies $\nu(L) \ge \lfloor L \rfloor > 0$, as we have seen in Example 7.2. □

The following example shows that the converse of Theorem 7.2 or Theorem 7.3 is not true.

Example 7.6. Let $X = \mathbb{R}$ and $F(x) = x - x^3$. Here we have $F(0) = 0$ and

$$[F]_{\mathrm{q}} = [F]_{\mathrm{a}} = \infty, \quad [F]_{\mathrm{b}} = 0,$$

and so F is not F-regular. However, F is clearly stably solvable, hence FMV-regular and also surjective. ♡

We introduce another notion for further reference. Given $\Omega \in \mathfrak{OBC}(X)$, we call an operator $F\colon \overline{\Omega} \to Y$ *strictly epi* on $\overline{\Omega}$ if

$$\operatorname{dist}(\theta, F(\partial\Omega)) = \inf_{x\in\partial\Omega} \|F(x)\| > 0, \tag{7.14}$$

i.e., $F(\partial\Omega)$ is bounded away from zero, and $\nu_\Omega(F) > 0$, i.e., F is k-epi on $\overline{\Omega}$ for some $k > 0$. This definition modifies that of epi operators in rather the same way as the definition of strictly stably solvable operators modifies that of stably solvable operators, see Section 6.1.

The following perturbation result of Rouché type is parallel to Lemma 6.4 and shows that, if F is strictly epi, then also $F + G$ is strictly epi if $[G]_{\mathrm{A}}$ and $\|G(x)\|$ are sufficiently small on $\partial\Omega$. More precisely, the following is true.

Lemma 7.4. *Suppose that $F\colon \overline{\Omega} \to Y$ is strictly epi on $\overline{\Omega}$, and $G\colon \overline{\Omega} \to Y$ satisfies*

$$\sup_{x\in\partial\Omega} \|G(x)\| < \operatorname{dist}(\theta, F(\partial\Omega)) \tag{7.15}$$

and

$$[G|_{\overline{\Omega}}]_{\mathrm{A}} < \nu_\Omega(F). \tag{7.16}$$

Then $F + G$ is strictly epi on $\overline{\Omega}$.

Proof. The triangle inequality

$$\|F(x) + G(x)\| \geq \|F(x)\| - \|G(x)\| \quad (x \in \partial\Omega)$$

implies that

$$\operatorname{dist}(\theta, (F+G)(\partial\Omega)) \geq \operatorname{dist}(\theta, F(\partial\Omega)) - \sup_{x\in\partial\Omega} \|G(x)\| > 0,$$

i.e., (7.14) holds for $F + G$. To prove that $F + G$ is k-epi for some $k > 0$ we use Property 7.4 of k-epi operators for $F_0 := F$. Fix k with $[G|_{\overline{\Omega}}]_{\mathrm{A}} \leq k < \nu_\Omega(F)$, and define $H\colon \overline{\Omega} \times [0, 1] \to Y$ by $H(x, t) = tG(x)$. Then

$$\alpha(H(M \times [0, 1])) \leq \alpha(G(M)) \leq [G|_{\overline{\Omega}}]_{\mathrm{A}}\alpha(M) \quad (M \subseteq \overline{\Omega}),$$

and the set (7.8) is here

$$S = \{x \in \overline{\Omega} : F(x) + tG(x) = \theta \text{ for some } t \in [0, 1]\}.$$

We claim that $S \cap \partial\Omega = \emptyset$. In fact, suppose that $F(\hat{x}) + tG(\hat{x}) = \theta$ for some $\hat{x} \in \partial\Omega$ and $t \in [0, 1]$. Then

$$\inf_{x\in\partial\Omega} \|F(x)\| \leq \|F(\hat{x})\| = t\|G(\hat{x})\| \leq \sup_{x\in\partial\Omega} \|G(x)\|,$$

contradicting (7.15). From Property 7.4 we conclude that the operator $F + H(\cdot, 1) = F + G$ is k_1-epi on $\overline{\Omega}$ for $0 \leq k_1 \leq k - [G|_{\overline{\Omega}}]_{\mathrm{A}}$. □

The proof of Lemma 7.4 shows that the Rouché type inequality

$$\nu(F+G) \geq \nu(F) - [G]_{\mathrm{A}} \tag{7.17}$$

holds true for the characteristic (7.3) which is parallel to (6.6).

7.3 The Feng spectrum

The notion of F-regularity may be used to define a new spectrum in rather the same way as we defined the FMV-spectrum by means of FMV-regularity, or the AGV-spectrum by means of AGV-regularity. Given $F \in \mathfrak{C}(X)$, we call the set

$$\rho_{\mathrm{F}}(F) := \{\lambda \in \mathbb{K} : \lambda I - F \text{ is F-regular}\} \tag{7.18}$$

the *Feng resolvent set* and its complement

$$\sigma_{\mathrm{F}}(F) := \mathbb{K} \setminus \rho_{\mathrm{F}}(F) \tag{7.19}$$

the *Feng spectrum* of F. Theorem 7.3 shows that this gives the usual definition for linear operators. Moreover, from Theorem 7.2 it follows immediately that $\sigma_{\mathrm{FMV}}(F) \subseteq \sigma_{\mathrm{F}}(F)$.

As before, we have the decomposition

$$\sigma_{\mathrm{F}}(F) = \sigma_{\nu}(F) \cup \sigma_{\mathrm{a}}(F) \cup \sigma_{\mathrm{b}}(F), \tag{7.20}$$

where we have put

$$\sigma_{\nu}(F) := \{\lambda \in \mathbb{K} : \ \nu(\lambda I - F) = 0\}, \tag{7.21}$$

$\sigma_{\mathrm{a}}(F)$ is defined by (2.31), and $\sigma_{\mathrm{b}}(F)$ by (2.30). As all the other spectra considered so far, the Feng spectrum is empty for the operator F in Example 3.18. On the other hand, we have the following three results which are parallel to Theorems 6.2, 6.3, and 6.4.

Theorem 7.4. *The spectrum $\sigma_{\mathrm{F}}(F)$ is closed.*

Proof. Suppose that $\lambda \in \rho_{\mathrm{F}}(F)$, i.e., $[\lambda I - F]_{\mathrm{a}} > 0$, $[\lambda I - F]_{\mathrm{b}} > 0$, and $\lambda I - F$ is k-epi on $\overline{\Omega}$ for some $k > 0$ and every $\Omega \in \mathfrak{O}(X)$. Take

$$\delta < \frac{1}{2} \min\{k, [\lambda I - F]_{\mathrm{a}}, [\lambda I - F]_{\mathrm{b}}\},$$

and fix $\mu \in \mathbb{K}$ with $|\lambda - \mu| < \delta$. First of all, from Propositions 2.3 (c) and 2.4 (d) it follows that

$$[\mu I - F]_{\mathrm{a}} \geq \frac{1}{2}[\lambda I - F]_{\mathrm{a}} > 0, \quad [\mu I - F]_{\mathrm{b}} \geq \frac{1}{2}[\lambda I - F]_{\mathrm{b}} > 0.$$

We show that $\nu_\Omega(\mu I - F) > 0$ for every $\Omega \in \mathfrak{OBC}(X)$; to this end, we use Property 7.4 with $F_0 := \lambda I - F$ and $H(x,t) := t(\mu - \lambda)x$. It is clear that $H(x,0) \equiv \theta$ and

$$\alpha(H(M \times [0,1])) \leq |\lambda - \mu|\alpha(M) \quad (M \subseteq \overline{\Omega}).$$

For the set (7.8) we get

$$S = \{x \in \overline{\Omega} : \lambda x - F(x) + t(\mu - \lambda)x = \theta \text{ for some } t \in [0,1]\}.$$

We claim that $S = \{\theta\}$, and so $S \cap \partial\Omega = \emptyset$. In fact, for every $x \in S$ we have

$$t|\mu - \lambda|\,\|x\| = \|\lambda x - F(x)\| \geq [\lambda I - F]_{\mathrm{b}}\|x\|,$$

and so $\|x\| = 0$, since $|\mu - \lambda| < \delta < [\lambda I - F]_{\mathrm{b}}$. Thus, from Proposition 7.4 we conclude that $F_1 = F_0 + H(\cdot, 1) = \lambda I - F + (\mu - \lambda)I = \mu I - F$ is k-epi on $\overline{\Omega}$ for $0 \leq k < \delta - |\mu - \lambda|$. So we have proved that $\mu \in \rho_{\mathrm{F}}(F)$. □

Theorem 7.5. *Suppose that $F \in \mathfrak{C}(X)$ satisfies $[F]_{\mathrm{A}} < \infty$ and $[F]_{\mathrm{B}} < \infty$. Then the spectrum $\sigma_{\mathrm{F}}(F)$ is bounded, hence compact.*

Proof. We show that $\lambda \in \rho_{\mathrm{F}}(F)$ if $|\lambda| > \max\{[F]_{\mathrm{A}}, [F]_{\mathrm{B}}\}$. First of all, from (2.34) and (2.35) we deduce that $\lambda \notin \sigma_{\mathrm{a}}(F) \cup \sigma_{\mathrm{b}}(F)$ for such λ.

We claim that $\lambda I - F$ is k-epi on $\overline{\Omega}$ for $\Omega \in \mathfrak{OBC}(X)$ and $0 \leq k < |\lambda| - [F]_{\mathrm{A}}$. Since $\nu(\lambda I) \geq \lVert \lambda I \rVert = |\lambda|$, by Example 7.2, from (7.17) we conclude that

$$\nu(\lambda I - F) \geq \nu(\lambda I) - [F]_{\mathrm{A}} \geq |\lambda| - [F]_{\mathrm{A}} > k$$

as claimed. □

If we define the *Feng spectral radius* of $F \in \mathfrak{C}(X)$ by

$$r_{\mathrm{F}}(F) = \sup\{|\lambda| : \lambda \in \sigma_{\mathrm{F}}(F)\}, \tag{7.22}$$

the proof of Theorem 7.5 shows that

$$r_{\mathrm{F}}(F) \leq \max\{[F]_{\mathrm{A}}, [F]_{\mathrm{B}}\}. \tag{7.23}$$

This is of course analogous to the estimate (6.12) for the FMV-spectral radius.

The next theorem shows that also the Feng spectrum is upper semicontinuous in a suitable topology. Theorem 7.5 suggests that the most appropriate class is the intersection $\mathfrak{A}(X) \cap \mathfrak{B}(X)$, since the map $F \mapsto \sigma_{\mathrm{F}}(F)$ is compact-valued on this class. In contrast to the rather complicated FMV-topology which we considered in Section 6.3, the class $\mathfrak{A}(X) \cap \mathfrak{B}(X)$ is even a *Banach space* with norm

$$\|F\|_{AB} = \max\{[F]_{\mathrm{A}}, [F]_{\mathrm{B}}\}. \tag{7.24}$$

The following continuity result for the Feng spectrum is parallel to Theorem 6.4.

Theorem 7.6. *The multivalued map* $\sigma_F\colon \mathfrak{A}(X) \cap \mathfrak{B}(X) \to 2^{\mathbb{K}}$ *which associates to each* F *its Feng spectrum is upper semicontinuous in the norm* (7.24).

Proof. We use again Lemma 5.4 for $p(F) = \|F\|_{AB}$. Fix $\lambda \in \rho_F(F)$, hence $[\lambda I - F]_a > 0$, $[\lambda I - F]_b > 0$, and $\nu(\lambda I - F) > 0$. Thus, we find some $k > 0$ such that $\lambda I - F$ is k-epi on $\overline{\Omega}$ for every $\Omega \in \mathfrak{OBC}(X)$. Since $\rho_F(F)$ is open, by Theorem 7.4, there exists $\delta = \delta(\lambda) > 0$ such that

$$[\mu I - F]_a > \frac{1}{2}[\lambda I - F]_a, \quad [\mu I - F]_b > \frac{1}{2}[\lambda I - F]_b,$$

and $\mu I - F$ is $\frac{k}{2}$-epi on $\overline{\Omega}$, provided that $|\mu - \lambda| < \delta(\lambda)$. Here we take

$$\delta(\lambda) < \frac{1}{2}\min\{[\lambda I - F]_a, [\lambda I - F]_b, k\}.$$

Choose $G \in \mathfrak{A}(X) \cap \mathfrak{B}(X)$ with $\|G - F\|_{AB} < \delta(\lambda)$. From Proposition 2.4 (d) and our choice of $\delta(\lambda)$ we have then

$$[\mu I - G]_a \geq [\mu I - F]_a - [F - G]_A > 0$$

and

$$\|\mu x - G(x)\| \geq \|\mu x - F(x)\| - \|F(x) - G(x)\| > (\tfrac{1}{2}[\lambda I - F]_b - \delta(\lambda))\|x\|,$$

hence $[\mu I - G]_b > 0$. Moreover, from (7.17) we obtain

$$\begin{aligned}\nu(\mu I - G) &\geq \nu(\lambda I - F) - [(\lambda - \mu)I + (F - G)]_A \\ &\geq k - |\lambda - \mu| - \|F - G\|_{AB} \\ &> k - 2\delta(\lambda) > 0.\end{aligned}$$

We have proved that the complement of the graph of the map σ_F is open in $[\mathfrak{A}(X) \cap \mathfrak{B}(X)] \times \mathbb{K}$. Moreover, the estimate (7.23) shows that (5.39) holds true, and so the assertion is proved. □

Our discussion shows that the Feng spectrum is very similar to the FMV-spectrum. However, since the characteristics $[F]_B$ and $[F]_b$ play the same role for $\sigma_F(F)$ as the characteristics $[F]_Q$ and $[F]_q$ for $\sigma_{FMV}(F)$, one sees that the Feng spectrum is "global", while the FMV-spectrum is "asymptotic". This has a pleasant consequence: *in contrast to the Furi–Martelli–Vignoli spectrum, the Feng spectrum contains the eigenvalues*. Indeed, from $F(x) = \lambda x$ it follows that $[\lambda I - F]_b = 0$, and so $\sigma_p(F) \subseteq \sigma_b(F) \subseteq \sigma_F(F)$, by definition.

Example 7.7. To illustrate this, consider again the scalar function $F(x) = \sqrt{|x|}$ from Example 3.16. As we have seen in Example 6.6, every $\lambda \in \mathbb{R} \setminus \{0\}$ is an eigenvalue of F, and $\sigma_{FMV}(F) = \{0\}$. On the other hand, since the Feng spectrum contains the eigenvalues and is closed, we see that $\sigma_F(F) = \mathbb{R}$ in this example. Of course, this may also be checked directly. ♡

In Proposition 6.5 we have shown that the subspectrum $\sigma_\pi(F) = \sigma_a(F) \cup \sigma_q(F)$ contains the boundary of the FMV-spectrum. We prove now a parallel result for the Feng spectrum. In analogy to the subspectrum (6.16) we put

$$\sigma_\varphi(F) = \sigma_a(F) \cup \sigma_b(F). \tag{7.25}$$

Proposition 2.5 (b) implies that $\sigma_\pi(F) \subseteq \sigma_\varphi(F)$.

Proposition 7.1. *The subspectrum* (7.25) *is closed, and the inclusion*

$$\partial\sigma_F(F) \subseteq \sigma_\varphi(F) \tag{7.26}$$

holds.

Proof. The closedness of $\sigma_\varphi(F)$ follows from Theorem 2.4. Put

$$U := \sigma_F(F) \setminus \sigma_\varphi(F); \tag{7.27}$$

we claim that U is an open subset of $\mathbb{K}$. Indeed, suppose that there exists $\lambda \in U$ which is not an interior point of U. Then we can find a sequence $(\lambda_n)_n$ in $\mathbb{K} \setminus U$ with $\lambda_n \to \lambda$. Since $\lambda \notin \sigma_\varphi(F)$ and $\sigma_\varphi(F)$ is closed, we may assume that $\lambda_n \notin \sigma_\varphi(F)$ for all n, and thus $\lambda_n \notin \sigma_F(F)$. In particular, the operator $\lambda_n I - F$ is epi on $\overline{\Omega}$ for every $\Omega \in \mathfrak{OBC}(X)$. So, for any compact operator $G\colon \overline{\Omega} \to X$ with $G|_{\partial\Omega} = \Theta$, there exists some $x_n \in \Omega$ with

$$\lambda_n x_n - F(x_n) = G(x_n). \tag{7.28}$$

Since Ω is bounded, we have

$$(\lambda I - F)(x_n) - G(x_n) = (\lambda - \lambda_n)x_n \to \theta \quad (n \to \infty),$$

and so the set $\{(\lambda I - F)(x_n) : n \in \mathbb{N}\}$ is precompact. The fact that $[\lambda I - F]_a > 0$ implies that the set $\{x_n : n \in \mathbb{N}\}$ is precompact as well, and so we find a convergent subsequence $(x_{n_k})_k$ of $(x_n)_n$, say $x_{n_k} \to \hat{x}$. Passing to the limit in (7.28), we find that $\lambda\hat{x} - F(\hat{x}) = G(\hat{x})$ which shows that $\lambda I - F$ is epi on $\overline{\Omega}$. Since $\lambda \notin \sigma_\varphi(F)$, this implies $\lambda \in \rho_F(F)$, contradicting our definition (7.27) of U.

Suppose now that $\lambda \in \partial\sigma_F(F)$. Then λ is not an interior point of $U = \sigma_F(F) \setminus \sigma_\varphi(F)$, and thus $\lambda \notin U$. Since $\lambda \in \sigma_F(F)$, by the closedness of the Feng spectrum, we must have $\lambda \in \sigma_\varphi(F)$, and so we are done. □

7.4 Special classes of operators

In this section we consider the special class of homogeneous operators. Recall that $F\colon X \to Y$ is called *τ-homogeneous* ($\tau > 0$) if

$$F(tx) = t^\tau F(x) \quad (x \in X,\ t > 0). \tag{7.29}$$

In case $\tau = 1$ we have considered this condition in (3.7). We will use the following obvious property of the zeros of a τ-homogeneous operator F: if $F(x_0) = \theta$ for some $x_0 \in X$, then also $F(x) = \theta$ for every x on the "ray" $\{tx_0 : 0 \leq t < \infty\}$. The following proposition shows how "local" and "global" properties of homogeneous operators are actually equivalent.

Proposition 7.2. *Let $F: X \to Y$ be τ-homogeneous for some $\tau > 0$. Then the following is true.*

(a) *F is strictly epi on some $\Omega \in \mathfrak{OBC}(X)$ if and only if F is strictly epi on every $\Omega \in \mathfrak{OBC}(X)$.*

(b) *F is epi on some $\Omega \in \mathfrak{OBC}(X)$ if and only if F is epi on every $\Omega \in \mathfrak{OBC}(X)$.*

(c) *If $\tau = 1$, then $[F|_{\overline{\Omega}}]_{\mathrm{A}} = [F]_{\mathrm{A}}$ for every $\Omega \in \mathfrak{OBC}(X)$.*

(d) *If $\tau = 1$, then $[F|_{\overline{\Omega}}]_{\mathrm{a}} = [F]_{\mathrm{a}}$ for every $\Omega \in \mathfrak{OBC}(X)$.*

(e) *If $\tau = 1$, then $[F|_{\overline{\Omega}}]_{\mathrm{B}} = [F]_{\mathrm{B}}$ for every $\Omega \in \mathfrak{OBC}(X)$.*

(f) *If $\tau = 1$, then $[F|_{\overline{\Omega}}]_{\mathrm{b}} = [F]_{\mathrm{b}}$ for every $\Omega \in \mathfrak{OBC}(X)$.*

Proof. To prove (a), assume that F is k-epi ($k > 0$) on $\overline{\Omega}$ for some $\Omega \in \mathfrak{OBC}(X)$. This means that $F(x) \neq \theta$ on $\partial\Omega$ and, whenever $G: \overline{\Omega} \to Y$ satisfies $G|_{\partial\Omega} = \Theta$ and $[G|_{\overline{\Omega}}]_{\mathrm{A}} \leq k$, the equation $F(x) = G(x)$ is solvable in Ω. Let $\Omega' \in \mathfrak{OBC}(X)$ be fixed. The τ-homogeneity of F guarantees that $F(x) \neq \theta$ for $x \neq \theta$. Moreover, if $\mathrm{dist}(\theta, F(\partial\Omega)) > 0$, then $\mathrm{dist}(\theta, F(\partial\Omega')) > 0$ as well, again by homogeneity.

Choose $r > 0$ such that $B_r(X) \subseteq \Omega$. Then F is also k-epi on $B_r(X)$, by Property 7.3 of k-epi operators. Now take $R > 0$ so large that $\Omega' \subseteq B_R^o(X)$, and let $H: B_R(X) \to Y$ be an operator satisfying $H|_{S_R(X)} = \Theta$ and $[H|_{B_R(X)}]_{\mathrm{A}} \leq kR^{\tau-1}r^{1-\tau}$. Then the operator $G: B_r(X) \to Y$ defined by

$$G(x) := \frac{r^\tau}{R^\tau} H\left(\frac{R}{r}x\right)$$

satisfies $G|_{S_r(X)} = \Theta$. Moreover, the inequalities

$$\alpha(G(M)) = \frac{r^\tau}{R^\tau}\alpha\left(H\left(\frac{R}{r}M\right)\right) \leq \frac{r^\tau}{R^\tau}[H|_{B_R(X)}]_{\mathrm{A}}\alpha\left(\frac{R}{r}M\right) = \frac{r^\tau}{R^\tau}k\frac{R^{\tau-1}}{r^{\tau-1}}\frac{R}{r}\alpha(M)$$

show that $[G|_{B_r(X)}]_{\mathrm{A}} \leq k$. By assumption, we find $\hat{x} \in B_r^o(X)$ such that $F(\hat{x}) = G(\hat{x})$, and so $\tilde{x} := R\hat{x}/r \in B_R^o(X)$ solves the equation $F(x) = H(x)$. We conclude that F is k-epi on $B_R(X)$, and so also on Ω', again by Property 7.3.

This proves the statement (a); the statement (b) is proved similarly.

Suppose now that F is 1-homogeneous and $[F|_{\overline{\Omega}}]_{\mathrm{A}} < \infty$ for some $\Omega \in \mathfrak{OBC}(X)$. For any bounded subset M of X we may find $t > 0$ such that $tM \subseteq \Omega$. Consequently,

$$t\alpha(F(M)) = \alpha(F(tM)) \leq [F|_{\overline{\Omega}}]_{\mathrm{A}}\alpha(tM) = t\,[F|_{\overline{\Omega}}]_{\mathrm{A}}\alpha(M),$$

which shows that $[F]_A = [F|_{\overline{\Omega}}]_A$ is actually independent of Ω. This proves (c); the proof of (d) is analogous.

To see that (e) is true, suppose that F is 1-homogeneous and $[F|_{\overline{\Omega}}]_B < \infty$ for some $\Omega \in \mathfrak{OBC}(X)$. For any $x \in X$ we may find $t > 0$ such that $tx \in \Omega$, hence

$$\frac{\|F(x)\|}{\|x\|} = \frac{\|F(tx)\|}{\|tx\|} \leq [F|_{\overline{\Omega}}]_B < \infty,$$

which shows that $[F]_B = [F|_{\overline{\Omega}}]_B$ is also independent of Ω. This proves (e); the proof of (f) is again analogous. □

Specializing the characteristics (7.2) and (7.3) to balls, let us put

$$\tilde{\nu}_r(F) := \nu_{B_r^o(X)}(F) \quad (r > 0) \tag{7.30}$$

and

$$\tilde{\nu}(F) = \inf_{r>0} \tilde{\nu}_r(F). \tag{7.31}$$

The proof of Proposition 7.2 shows that $\tilde{\nu}_r(F)$ is actually independent of $r > 0$ if F is 1-homogeneous, and so it suffices to consider F on the unit ball $B(X)$. One could expect that this provides a simpler description of the Feng spectrum and similar spectra; this is in fact true.

As we have seen in Example 6.5, there is no analogue for the Gel'fand formula (1.9) for the FMV-spectrum. The same is true for the Feng spectrum, as Example 7.8 below shows which is very similar to Example 6.5. However, we have an upper estimate for 1-homogeneous operators which we will give in the subsequent Theorem 7.7.

Example 7.8. Let $F: \mathbb{R} \to \mathbb{R}$ be defined by

$$F(x) = \begin{cases} 0 & \text{if } x \leq 1, \\ x - 1 & \text{if } 1 \leq x \leq 2, \\ 1 & \text{if } x \geq 2. \end{cases}$$

It is not hard to see that $\sigma_F(F) = [0, 1/2]$, hence $r_F(F) = 1/2$. On the other hand, since $F^n(x) \equiv \theta$ for $n \geq 2$, we conclude that $\|F^n\|_{AB} = 0$ for $n \geq 2$, and so the formula (1.9) with $\|\cdot\|$ replaced by $\|\cdot\|_{AB}$ cannot be true. This example also shows that the spectral mapping theorem fails for the Feng spectrum. ♡

Theorem 7.7. *Let $F: X \to X$ be 1-homogeneous with $[F]_A < \infty$. Then the Feng spectral radius* (7.22) *satisfies the estimate*

$$r_F(F) \leq \max\{[F]_A, \liminf_{n\to\infty} [F^n]_B^{1/n}\}. \tag{7.32}$$

In particular,

$$r_F(F) \leq \liminf_{n\to\infty} [F^n]_B^{1/n} \tag{7.33}$$

if F is compact.

Proof. Suppose that both $|\lambda| > [F]_A$ and $|\lambda| > \liminf_{n\to\infty}[F^n]_B^{1/n}$. We use Property 7.4 for $\Omega = B^o(X)$, $F_0 := \lambda I$ and $H(x,t) := -tF(x)$. So consider the set

$$S = \{x \in B(X) : \lambda x - tF(x) = \theta \text{ for some } t \in [0,1]\}.$$

We claim that $S = \{\theta\}$. For $t = 0$ this is trivial. Assume that there exist $\hat{x} \in B(X)\setminus\{\theta\}$ and $\hat{t} \in (0,1]$ such that $\lambda\hat{x} = \hat{t}F(\hat{x})$. Then we get, by homogeneity,

$$[F]_B = \sup_{\|x\|=1} \|F(x)\| \geq \left\| F\left(\frac{\hat{x}}{\|\hat{x}\|}\right)\right\| \geq |\lambda|.$$

Moreover,

$$\left\| F^2\left(\frac{\hat{x}}{\|\hat{x}\|}\right)\right\| = \left\| F\left(\frac{\lambda\hat{x}}{\hat{t}\|\hat{x}\|}\right)\right\| \geq \frac{|\lambda|^2}{\hat{t}^2}.$$

So we have $[F^2]_B \geq |\lambda|^2$. By induction, we obtain $[F^n]_B \geq |\lambda|^n$, contradicting our assumption $|\lambda| > \liminf_{n\to\infty}[F^n]_B^{1/n}$. Since $[\lambda I]_A \leq |\lambda|$ and $[H]_A \leq [F]_A < |\lambda|$, from Property 7.4 it follows that the operator $F_0 + H(\cdot, 1) = \lambda I - F$ is $(|\lambda| - [F]_A)$-epi on $B(X)$, and so $\nu(\lambda I - F) > 0$, i.e., $\lambda \in \rho_F(F)$. □

The next example shows that the estimate (7.33) is sharp.

Example 7.9. Let X and F be as in Example 2.46, so F is continuous, compact and 1-homogeneous. Trivially, $[F^n]_B = 1$ for all $n \in \mathbb{N}$, and so (7.33) shows that $r_F(F) \leq 1$. But from $F(e) = e$ it follows that $\lambda = 1$ is an eigenvalue of F, and so $r_F(F) = 1$. ♡

In Theorem 1.2 (c) we have shown that, for a compact linear operator L, every point $\lambda \in \sigma(L) \setminus \{0\}$ is an eigenvalue of L. The same holds, with obvious modifications, for the FMV-spectrum of an asymptotically odd compact nonlinear operator (see Theorem 6.12). Interestingly, an analogous result holds for the Feng spectrum of an operator which is "very close" to being linear. We state this result for the larger class of α-contractive operators, since the proof is not very different from that for compact operators. In this way, we get the following analogue to Theorem 6.12.

Theorem 7.8. *Let $F \in \mathfrak{A}(X)$ be 1-homogeneous and odd. Then every $\lambda \in \sigma_F(F)$ satisfying $|\lambda| > [F]_A$ is an eigenvalue of F.*

Proof. We fix $\lambda \in \sigma_F(F)$ with $|\lambda| > [F]_A$ and show that λ belongs to $\sigma_b(F)$. In fact, suppose that $[\lambda I - F]_b > 0$. Since F is odd, the topological degree $\deg(I - F/\lambda, \Omega, \theta)$ of the α-contractive vector field $I - F/\lambda$ is odd, and so nonzero, for every $\Omega \in \mathfrak{OBC}(X)$. Consequently, for fixed $k \in ([F]_A/|\lambda|, 1)$, the operator $I - F/\lambda$ is $(k - [F]_A)/|\lambda|$-epi on $\overline{\Omega}$, by Property 7.5 of k-epi operators. Consequently, $\lambda I - F$ is $(|\lambda|k - [F]_A)$-epi on $\overline{\Omega}$. Moreover, we certainly have $[\lambda I - F]_a > 0$, by (2.35). Altogether this means that $\lambda \in \rho_F(F)$, a contradiction.

So we conclude that $[\lambda I - F]_{\mathrm{b}} = 0$, which means that there exists a sequence $(x_n)_n$ in X such that

$$\|\lambda x_n - F(x_n)\| \leq \frac{1}{n}\|x_n\|.$$

Putting $e_n := x_n/\|x_n\|$ we thus have $\|\lambda e_n - F(e_n)\| \to 0$ as $n \to \infty$. In addition, the set $M := \{e_1, e_2, e_3, \dots\}$ satisfies

$$[\lambda I - F]_{\mathrm{a}}\alpha(M) \leq \alpha((\lambda I - F)(M)) = 0,$$

which shows that $(e_n)_n$ admits a convergent subsequence $(e_{n_k})_k$, say $e_{n_k} \to e$. By continuity, we have then $e \in S(X)$ and $F(e) = \lambda e$, i.e., $\lambda \in \sigma_{\mathrm{p}}(F)$ as claimed. □

We will see later (see Example 12.1) that Theorem 7.8 is false without the oddness assumption on F.

As a consequence of Theorem 7.8 we can prove the following Borsuk–Ulam type theorem for α-contractive operators which generalizes Theorem 6.13.

Theorem 7.9. *Let X_0 be a proper closed subspace of a Banach space X, and let $F\colon S(X) \to X_0$ be an α-contractive vector field, i.e., $F = I - F_0$ with $[F_0]_{\mathrm{A}} < 1$. Then there exists a point $\hat{x} \in S(X)$ such that $F(-\hat{x}) = F(\hat{x})$.*

Proof. Suppose first in addition that $F_0 = I - F$ is odd. Let F_1 denote the homogeneous extension (6.7) of F, i.e.,

$$F_1(x) = \begin{cases} x - \|x\| F_0\left(\frac{x}{\|x\|}\right) & \text{if } x \neq \theta, \\ \theta & \text{if } x = \theta. \end{cases}$$

Then F_1 maps the space X into the subspace X_0. Indeed, otherwise we could find $x_0 \in X$ such that $F_1(x_0) \in X \setminus X_0$, hence $F(x_0/\|x_0\|) = F_1(x_0)/\|x_0\| \in X \setminus X_0$, a contradiction. So the operator F_1 is not surjective which implies that $1 \in \sigma_{\mathrm{F}}(I - F_1)$, by Theorem 7.3. Also, $[I - F_1]_{\mathrm{A}} = [I - F]_{\mathrm{A}} = [F_0]_{\mathrm{A}} < 1$, by Proposition 3.3 and our assumption. Since $I - F_1$ is clearly 1-homogeneous and odd, from Theorem 7.8 we conclude that $1 \in \sigma_{\mathrm{p}}(I - F_1)$. This means that there exists $\tilde{x} \in X \setminus \{\theta\}$ such that $\tilde{x} - F_1(\tilde{x}) = \tilde{x}$, and so $\tilde{x} = \|\tilde{x}\| F_0(\tilde{x}/\|\tilde{x}\|)$. In other words, the element $\hat{x} := \tilde{x}/\|\tilde{x}\| \in S(X)$ is a fixed point of F_0. Since we supposed F_0 to be odd, this implies that

$$F(\hat{x}) - F(-\hat{x}) = \hat{x} - F_0(\hat{x}) + \hat{x} + F_0(-\hat{x}) = 2\hat{x} - 2F_0(\hat{x}) = 2\hat{x} - 2\hat{x} = \theta.$$

Now we drop the additional assumption on $F_0 = I - F$ to be odd. Then the operator G_0 defined by

$$G_0(x) := \frac{1}{2}(F_0(x) - F_0(-x))$$

is certainly odd and satisfies $[G_0]_A = [F_0]_A < 1$ and $G_0(X) \subseteq X_0$. By what we have just proved, there exists $\hat{x} \in S(X)$ such that $G(-\hat{x}) = G(\hat{x})$, where $G := I - G_0$. But then

$$F(\hat{x}) - F(-\hat{x}) = 2G(\hat{x}) = G(\hat{x}) - G(-\hat{x}) = \theta,$$

since G is odd. So the proof is complete. □

7.5 A comparison of different spectra

In this and the preceding three chapters we have discussed spectra for various classes of continuous nonlinear operators. The purpose of this section is to compare these spectra from the viewpoint of their analytical properties.

First of all, we have seen that only the Neuberger spectrum is always nonempty in complex Banach spaces (Theorem 4.1 and Example 3.18). On the other hand, the Neuberger spectrum may be unbounded (Example 4.1) or not closed (Example 4.7). In contrast, the Kachurovskij spectrum is always compact (Theorem 5.1). Both the Rhodius and Dörfner spectra need not be bounded (Examples 4.1 and 5.7) or closed (Examples 4.1 and 5.8). Finally, the FMV-spectrum, the AGV-spectrum, and the Feng spectrum are always closed (Theorems 6.2, 6.11, and 7.4), but may be unbounded (Examples 6.10 and 7.7).

If we impose additional conditions, the spectra get more structure. Thus, all spectra are nonempty for compact operators in infinite dimensional spaces (Theorems 4.3, 5.2 and 6.6). Moreover, the FMV-spectrum and AGV-spectrum are compact in case $[F]_A < \infty$ and $[F]_Q < \infty$ (Theorems 6.3 and 6.11), while the Feng spectrum is compact in case $[F]_A < \infty$ and $[F]_B < \infty$ (Theorem 7.5). Finally, all the spectra discussed in this and the previous chapter are upper semicontinuous in a suitable topology. So we may summarize these properties in the following table which is a refinement of Table 5.1.

Table 7.1

spectrum	$\neq \emptyset$	closed	bounded	compact	$\supseteq \sigma_p(F)$	u.s.c.
$\sigma_N(F)$	yes	no	no	no	yes	——
$\sigma_K(F)$	no*	yes	yes	yes	yes	yes
$\sigma_R(F)$	no*	no	no	no	yes	——
$\sigma_D(F)$	no*	no	no	no	yes	no
$\sigma_{FMV}(F)$	no*	yes	no**	no**	no	yes
$\sigma_{AGV}(F)$	no*	yes	no**	no**	no	yes
$\sigma_F(F)$	no*	yes	no***	no***	yes	yes

*: *yes* if $\dim X = \infty$ and $[F]_A = 0$
**: *yes* if $[F]_A < \infty$ and $[F]_Q < \infty$
***: *yes* if $[F]_A < \infty$ and $[F]_B < \infty$

We also give a scheme, for the reader's ease, of all possible inclusions between these spectra in the following Table 7.2.

Table 7.2

$$\begin{array}{ccccccccc}
\sigma_{\mathrm{AGV}}(F) & \subseteq & \sigma_{\mathrm{FMV}}(F) & \subseteq & \sigma_{\mathrm{F}}(F) & \subseteq & \sigma_{\mathrm{D}}(F) & \subseteq & \sigma_{\mathrm{K}}(F) \\
 & & & & \cup\!| & & \cup\!| & & \cup\!| \\
 & & & & \sigma_{\mathrm{p}}(F) & \subseteq & \sigma_{\mathrm{R}}(F) & \subseteq & \sigma_{\mathrm{N}}(F)
\end{array}$$

Table 7.2 shows again that $\sigma_{\mathrm{AGV}}(F)$ and $\sigma_{\mathrm{FMV}}(F)$ are the only spectra which may not contain the eigenvalues. Moreover, $\sigma_{\mathrm{K}}(F) = \emptyset$ implies that all other spectra are also empty, whenever they are defined. (Recall that the operator in Example 3.18 is not differentiable at any point.)

Let us also make some remarks on possible strict inclusions in this scheme. Examples for $\sigma_{\mathrm{AGV}}(F) \subset \sigma_{\mathrm{FMV}}(F)$, $\sigma_{\mathrm{FMV}}(F) \subset \sigma_{\mathrm{F}}(F)$, and $\sigma_{\mathrm{F}}(F) \subset \sigma_{\mathrm{D}}(F)$ have already been given. Example 4.2 shows that the strict inclusions $\sigma_{\mathrm{p}}(F) \subset \sigma_{\mathrm{F}}(F)$, $\sigma_{\mathrm{p}}(F) \subset \sigma_{\mathrm{R}}(F)$, $\sigma_{\mathrm{R}}(F) \subset \sigma_{\mathrm{D}}(F)$, and $\sigma_{\mathrm{R}}(F) \subset \sigma_{\mathrm{N}}(F)$ may actually occur. It remains to find an example for the strict inequality $\sigma_{\mathrm{D}}(F) \subset \sigma_{\mathrm{K}}(F)$.

Example 7.10. Let $X = \mathbb{R}$, $F(x) = (x-1)^2 + 1$ for $1 \le x \le 2$, and $F(x) = x$ otherwise. Obviously, $F \in \mathfrak{Lip}(X)$ with $[F]_{\mathrm{Lip}} = 2$, but $[F]_{\mathrm{B}} = 1$. Moreover, F is a bijection with $[F]_{\mathrm{b}} = 2(\sqrt{2}-1)$, and hence $0 \in \rho_{\mathrm{D}}(F)$. On the other hand, $0 \in \sigma_{\mathrm{K}}(F)$ since F^{-1} is not Lipschitz continuous. More precisely, by means of a straightforward computation one may show that $\sigma_{\mathrm{D}}(F) = [2(\sqrt{2}-1), 1]$ and $\sigma_{\mathrm{K}}(F) = [0, 2]$. ♡

Of course, usually one need not calculate all spectra in Table 7.2 to get a complete picture. Sometimes it suffices to calculate the smallest spectrum $\sigma_{\mathrm{AGV}}(F)$ and to give some upper estimate for the "size" of the largest spectrum $\sigma_{\mathrm{K}}(F)$. We illustrate this by means of an operator which we already considered before.

Example 7.11. Let $X = l_2$ and F be defined by

$$F(x_1, x_2, x_3, \dots) = (\|x\|, x_1, x_2, \dots). \tag{7.34}$$

We have seen in Example 3.15 that $[F]_{\mathrm{Lip}} = \sqrt{2}$, and so

$$\sigma_{\mathrm{K}}(F) \subseteq \overline{\mathbb{D}}_{\sqrt{2}}, \tag{7.35}$$

by (5.11), where $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$. We show that $\mu(\lambda I - F) = 0$ for $|\lambda| \le \sqrt{2}$, and so

$$\sigma_{\mathrm{AGV}}(F) \supseteq \overline{\mathbb{D}}_{\sqrt{2}}, \tag{7.36}$$

i.e., we have equality in the upper row in Table 7.2. Indeed, the operator $G(x_1, x_2, x_3, \dots) := (1, 0, 0, \dots)$ is compact with $[G]_{\mathrm{Q}} = 0$. Fix λ with $0 < |\lambda| \le \sqrt{2}$, and suppose that the equation $F(x) = G(x)$ has a solution $\hat{x} = (\hat{x}_1, \hat{x}_2, \hat{x}_3, \dots) \in X$. Then

$$\hat{x}_n = \frac{1 + \|\hat{x}\|}{\lambda^n} \quad (n = 1, 2, 3, \dots).$$

For $|\lambda| \le 1$, the element $\hat{x}$ cannot belong to X. On the other hand, for $|\lambda| > 1$ we obtain

$$\left(\frac{\|\hat{x}\|}{1 + \|\hat{x}\|}\right)^2 = \frac{1}{|\lambda|^2 - 1},$$

and this implies $|\lambda|^2 - 1 > 1$, hence $|\lambda| > \sqrt{2}$. This contradiction shows that $\lambda I - F$ is not stably solvable for $|\lambda| \le \sqrt{2}$, and so (7.36) is proved. The same reasoning shows that $\lambda I - F$ is not a homeomorphism for $|\lambda| \le \sqrt{2}$, which means that

$$\sigma_{\mathrm{R}}(F) = \overline{\mathbb{D}}_{\sqrt{2}}.$$

It remains to calculate the point spectrum $\sigma_{\mathrm{p}}(F)$. It is easy to see that $\lambda = 0$ is not an eigenvalue of F. For $\lambda \ne 0$, the eigenvalue equation $F(x) = \lambda x$ leads to the relation

$$\frac{x}{\|x\|} = \left(\frac{1}{\lambda}, \frac{1}{\lambda^2}, \frac{1}{\lambda^3}, \dots\right),$$

hence

$$\frac{1}{|\lambda|^2 - 1} = \frac{|\lambda|^{-2}}{1 - |\lambda|^{-2}} = \sum_{n=1}^{\infty} |\lambda|^{-2n} = 1,$$

which implies that $|\lambda|^2 = 2$. Conversely, for any $\lambda \in \mathbb{K}$ with $|\lambda| = \sqrt{2}$, the element

$$x_\lambda := (\lambda^{-1}, \lambda^{-2}, \lambda^{-3}, \dots)$$

satisfies $x_\lambda \in X$ and $F(x_\lambda) = \lambda x_\lambda$. So we have calculated all spectra for F, and Table 7.2 becomes for this operator as shown in Table 7.3 below. ♡

Table 7.3

$$\begin{array}{ccccccccc}
\overline{\mathbb{D}}_{\sqrt{2}} & = & \overline{\mathbb{D}}_{\sqrt{2}} & = & \overline{\mathbb{D}}_{\sqrt{2}} & = & \overline{\mathbb{D}}_{\sqrt{2}} & = & \overline{\mathbb{D}}_{\sqrt{2}} \\
 & & & & \cup & & \| & & \| \\
 & & & & \mathbb{S}_{\sqrt{2}} & \subset & \overline{\mathbb{D}}_{\sqrt{2}} & = & \overline{\mathbb{D}}_{\sqrt{2}}
\end{array}$$

An extension of Table 7.2, containing more spectra and point spectra, will be discussed in Section 8.5 in the next chapter.

7.6 Notes, remarks and references

There are many papers on epi and k-epi operators, a rather detailed list of references may be found in the recent book [148]. The notion of epi operators is due to Furi, Martelli and Vignoli [123]; they call them *zero-epi maps*. In [123] the authors prove the "existence", "boundary dependence", "normalization", "localization" and "homotopy" properties which we have given in Section 7.1 and which are similar to those of topological degree theory. For example, Property 7.4 may be viewed as some kind of homotopy invariance of k-epi operators. Some kind of axiomatic approach in a very general setting may be found in the recent paper [18].

We point out, however, that the theory of epi mappings has some advantages compared with topological degree theory. First, it requires only elementary tools such as the Schauder fixed point theorem and the Tietze–Uryson lemma. Second, epi operators F may act between different spaces, in contrast to the vector fields $I - F$ considered in classical degree theory. Thus the theory of epi operators applies directly, say, to differential or functional differential equations, while to apply degree theory it is necessary to reformulate the problems for nonlinear operators acting in one and the same space.

There is another advantage of epi operators: they may be useful even if the topological degree is zero, and so degree theory does not apply. However, this advantage is in a certain sense only apparent. For example, the function $F\colon \Omega \to \mathbb{R}$ defined by $F(x) = x^2 - 4$ on $\Omega = (-3, -1) \cup (1, 3)$ is epi, but has (Brouwer) degree 0 on Ω. Of course, in this example the degree of F is -1 on $(-3, -1)$ and $+1$ on $(1, 3)$. So, the natural question arises whether or not there exist epi operators with vanishing degree on connected sets.

An example of an epi operator F with $[F]_{\mathrm{A}} = 1/4$ and vanishing (Nussbaum–Sadovskij) degree in an infinite dimensional space has been given recently by Ding [78]. However, as was observed by Väth [260], the calculation in [78] leading to this example is false. Moreover, Väth proves in the same paper that every operator of the form $F = I - C$ with $[C]_{\mathrm{A}} < 1/2$ which is epi on a Jordan domain Ω has non-zero degree. Interestingly, in [131] it was shown that this is true even for $[C] < 1$ and arbitrary $\Omega \in \mathfrak{OBC}(X)$. If Ω is only an open subset of X (not necessarily connected), then there exists a connected component $\Omega' \subset \Omega$ such that F is epi on $\overline{\Omega'}$ if and only if F has non-vanishing degree on Ω'.

The class of k-epi operators has been introduced by Martelli [183] and, independently, by Tarafdar and Thompson [248]. The characteristic (7.31) is called *measure of non-solvability* in [248]. Feng [105] calls the numbers (7.30) and (7.31) *measures of solvability*, as we do for the more general characteristics (7.2) and (7.3), which seems more reasonable. The following example which is due to Väth shows that an operator may be epi on some set $\Omega \in \mathfrak{OBC}(X)$, but not epi on any ball $B_r(X)$.

Example 7.12. Let $X = \mathbb{R}^2$ and $\Omega \in \mathfrak{OBC}(X)$ be the ellipse $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + 4y^2 < 16\}$. By the classical Tietze–Uryson lemma we may find a smooth function

$\varphi\colon \mathbb{R}^2 \to [0, 1]$ such that $\varphi(x, y) \equiv 1$ on $\overline{\Omega}$ and $\varphi(x, y) \equiv 0$ for $\mathrm{dist}((x, y), \overline{\Omega}) \geq 1$, say. Clearly, the operator $F\colon X \to X$ defined by $F(x, y) = (\varphi(x, y)x, \varphi(x, y)y)$ is then k-epi on $\overline{\Omega}$ for any $k > 0$, and so $\nu_\Omega(F) = \infty$. On the other hand, F is *not* epi on any disc $B_r(X)$, since F has zeros on $S_r(X)$ for each $r > 0$. ♡

The definition of *k-proper operators* and Lemma 7.3 are taken from [248]. Of course, this notion is closely related to our lower measure of noncompactness (2.15), as Lemma 7.2 shows.

In [248] the authors claim to have an example of an epi operator which is not k-epi for any $k > 0$. However, this claim is not true. To the best of our knowledge, the only known example of such an operator is due to Furi [118]; this is our Example 7.5. By Theorem 7.1, every operator of this type *must* satisfy $[F]_\mathrm{a} = 0$.

The (trivial) estimate (7.5) is useful for obtaining upper bounds for the characteristic $\mu(F)$, or lower bounds for the characteristic $\nu(F)$. Let us call the number

$$V(X) := \nu(I) = \inf\{k : k > 0, \text{there exists a fixed point free operator } F\colon B(X) \to B(X) \text{ with } [F]_\mathrm{A} \leq k\} \tag{7.37}$$

the *Väth constant* of the Banach space X. From Darbo's fixed point principle (Theorem 2.1) it follows that $V(X) \geq 1$ in every Banach space. Example 7.3 which is taken from [248] shows that $V(l_2) \leq 2$, while Example 7.4 shows that $V(c_0) = 1$. Quite recently, however, it was shown by Caponetti and Trombetta [60] that $V(X) = 1$ in every infinite dimensional Banach space X. In fact, since the work of Lin and Sternfeld [172] one knows that in an infinite dimensional Banach space one may always find a Lipschitz continuous fixed point free map $F\colon B(X) \to B(X)$. If $[F]_\mathrm{Lip} > 1$ for such a map, for $\varepsilon > 0$ we may pass from F to the operator $F_\varepsilon\colon B(X) \to B(X)$ defined by

$$F_\varepsilon(x) := x + \frac{\varepsilon(F(x) - x)}{[F]_\mathrm{Lip} - 1}. \tag{7.38}$$

Clearly, $[F_\varepsilon]_\mathrm{A} \leq [F_\varepsilon]_\mathrm{Lip} \leq 1 + \varepsilon$, and the operator (7.38) is fixed point free, because every fixed point of F_ε is also a fixed point of F. This shows that $V(X) \leq 1 + \varepsilon$, and the assertion follows.

We point out that Väth's paper [259] is concerned with the problem for what kind of spaces X the value $V(X) = [F]_\mathrm{A} = 1$ is actually *attained* by some fixed point free operator F. This problem is by far more difficult than just proving the equality $V(X) = 1$. For instance, it is shown in [259] that the infimum $V(X) = 1$ is attained if X is Hilbert, or a closed subspace of l_p $(1 \leq p \leq \infty)$, or c_0.

Observe that Väth's constant $V(X)$ and Wośko's constant $W(X)$ introduced in (2.36) are related by the estimate $V(X) \leq W(X)$. In fact, if $k > W(X)$ and $\rho\colon B(X) \to S(X)$ is a retraction with $[\rho]_\mathrm{A} \leq k$, then $F = -\rho\colon B(X) \to B(X)$ is a fixed point free operator with $[F]_\mathrm{A} \leq k$, and so $V(X) \leq k$.

Throughout Sections 7.2–7.4 we followed Feng's beautiful survey paper [105]. That is of course the reason why we call (7.19) the *Feng spectrum*. The main motivation for introducing this spectrum has been the flaw of the Furi–Martelli–Vignoli spectrum of not containing the eigenvalues; compare again Example 6.6 with Example 7.7.

We point out that Feng defined in [105] the class of operators which we call F-regular by means of the three conditions $[F]_{\mathrm{a}} > 0$, $[F]_{\mathrm{b}} > 0$, and $\tilde{\nu}(F) > 0$ (see (7.31)), i.e., F is required to be k-epi on each ball for some $k > 0$. As Theorem 7.1 shows, the latter condition is unnecessarily strong, because the positivity of $\nu(F)$ already follows from the condition $[F]_{\mathrm{a}} > 0$. Of course, Theorem 7.1, which is due to Väth [262], was still unknown when Feng defined her spectrum.

All results contained in Section 7.4 are also taken from Feng's paper [105]. In particular, our Theorem 7.8 is [105, Theorem 4.6]. Of course, this is an exact analogue to Theorems 1.3 (d) and 6.14. We will use results of this type in Chapter 12 below. The Borsuk–Ulam Theorem 7.9 may be found in the recent paper [106], even for condensing instead of α-contractive operators; it generalizes the analogous theorem for compact operators from [122] which we stated as Theorem 6.13.

Observe that a direct analogue to Lemma 6.6 for the Feng spectrum would be completely trivial, because $\|F - \tilde{F}\|_{AB} = 0$ (see (7.24)) implies that $F = \tilde{F}$. However, it was shown in [280] that $\sigma_{\mathrm{F}}(F) = \sigma_{\mathrm{F}}(\tilde{F})$ if there is some linear homeomorphism $L \in \mathfrak{L}(X)$ such that $\tilde{F} = L^{-1}FL$. In fact, this is a direct consequence of the estimates

$$[\tilde{F}]_{\mathrm{a}} \geq [L^{-1}]_{\mathrm{a}}\,[F]_{\mathrm{a}}\,[L]_{\mathrm{a}}, \; [\tilde{F}]_{\mathrm{b}} \geq \|L\|\; [F]_{\mathrm{b}}\,\|L^{-1}\|, \; \nu(\tilde{F}) \geq [L^{-1}]_{\mathrm{A}}^{-1}\,\nu(F)\,[L]_{\mathrm{A}}^{-1}.$$

Finally, let us remark that the comparison of the Neuberger, Kachurovskij, Rhodius, Dörfner, Furi–Martelli–Vignoli, and Feng spectrum in Section 7.5 is taken from [13], where one may also find some further examples and comments. An essential completion of this may be found in Section 8.5 below.

Chapter 8
The Väth Phantom

In this chapter we discuss two spectra which have been recently introduced by Väth under the name "phantoms". Roughly speaking, the definition of phantoms is based on a notion of "stably" epi operators, in rather the same way as the FMV-spectrum and the AGV-spectrum have been constructed in Chapter 6 by means of stably solvable operators. In contrast to the "asymptotic" FMV-spectrum and the "global" Feng spectrum, the Väth phantom is "local", and so reflects quite well the local character of nonlinear problems.

We also associate to the Väth phantom a new notion of point spectrum (eigenvalues) called point phantom. This new definition of eigenvalues has surprisingly many properties in common with the linear case; for example, the union of 0 and the point phantom of a compact operator is always compact. It also turns out that the Väth phantom is the smallest "reasonable" spectrum which contains the point phantom.

8.1 Strictly epi operators

Given a Banach space X, we denote as in the last chapter by $\mathfrak{OBC}(X)$ the class of all open, bounded, connected subsets $\Omega \subset X$ containing θ. Recall that a continuous operator $F\colon \overline{\Omega} \to Y$ is called epi on $\overline{\Omega}$ if $F(x) \neq \theta$ on $\partial\Omega$ and the equation $F(x) = G(x)$ has a solution in Ω for any compact operator $G\colon \overline{\Omega} \to Y$ which vanishes on $\partial\Omega$. As we have seen in Furi's Example 7.5, this notion is not "stable enough" to be invariant under small perturbations, and hence does not provide a closed spectrum.

The reason for this unpleasant phenomenon is two-fold. First, the boundary condition $\theta \notin F(\partial\Omega)$ need not remain true if we perturb F slightly; second, the solvability of the equation $F(x) = G(x)$ may get lost if G is no longer compact, even if $[G]_{\mathrm{A}}$ is "very small". This is the reason why we have introduced the notion of *strictly epi operators* at the end of Section 7.2. Recall that $F\colon \overline{\Omega} \to Y$ is strictly epi on $\overline{\Omega}$ if

$$\operatorname{dist}(\theta, F(\partial\Omega)) = \inf_{x\in\partial\Omega} \|F(x)\| > 0, \tag{8.1}$$

i.e., $F(\partial\Omega)$ is bounded away from zero, and $\nu_\Omega(F) > 0$, i.e., F is k-epi on $\overline{\Omega}$ for some $k > 0$. This definition modifies that of epi operators in rather the same way as the definition of strictly stably solvable operators modifies that of stably solvable operators. In case $\Omega = B_r^o(X)$, being strictly epi on $\overline{\Omega} = B_r(X)$ simply means that

$$\inf_{\|x\|=r} \|F(x)\| > 0, \quad \tilde{\nu}_r(F) > 0, \tag{8.2}$$

with $\tilde{\nu}_r(F)$ as in (7.30).

The following theorem rephrases Väth's fundamental Theorem 7.1 in terms of strictly epi operators.

Theorem 8.1. *Every operator $F\colon \overline{\Omega} \to Y$ which is epi on $\overline{\Omega}$ and satisfies $[F|_{\overline{\Omega}}]_{\mathrm{a}} > 0$ is even strictly epi on $\overline{\Omega}$.*

Proof. Since F is epi, we know that $F(x) \neq \theta$ on $\partial\Omega$. The hypothesis $[F|_{\overline{\Omega}}]_{\mathrm{a}} > 0$ implies that F is proper on $\overline{\Omega}$, by Proposition 2.4 (b), and thus F maps closed sets into closed sets, by Theorem 3.1 (b). In particular, $F(\partial\Omega)$ is closed in Y, and so (8.1) is true.

The fact that the condition $[F|_{\overline{\Omega}}]_{\mathrm{a}} > 0$ implies the positivity of $\nu_\Omega(F)$ is a consequence of (7.9). □

Theorem 8.1 admits a simple corollary which is similar to Lemma 7.4.

Lemma 8.1. *Suppose that $F\colon \overline{\Omega} \to Y$ is epi on $\overline{\Omega}$ with $[F|_{\overline{\Omega}}]_{\mathrm{a}} > 0$, and $G\colon \overline{\Omega} \to Y$ satisfies*

$$\sup_{x\in\partial\Omega} \|G(x)\| < \mathrm{dist}(\theta, F(\partial\Omega)) \tag{8.3}$$

and $[G|_{\overline{\Omega}}]_{\mathrm{A}} < [F|_{\overline{\Omega}}]_{\mathrm{a}}$. Then $F + G$ is strictly epi on $\overline{\Omega}$ with $[(F+G)|_{\overline{\Omega}}]_{\mathrm{a}} > 0$.

Proof. From Theorem 8.1 it follows that F is strictly epi on $\overline{\Omega}$ with $\nu_\Omega(F) \geq [F|_{\overline{\Omega}}]_{\mathrm{a}}$. Consequently,

$$[G|_{\overline{\Omega}}]_{\mathrm{A}} < \nu_\Omega(F),$$

and so $F + G$ is (strictly) epi on $\overline{\Omega}$, by Lemma 7.4. Moreover, we have

$$[(F+G)|_{\overline{\Omega}}]_{\mathrm{a}} \geq [F|_{\overline{\Omega}}]_{\mathrm{a}} - [G|_{\overline{\Omega}}]_{\mathrm{A}} > 0,$$

by Proposition 2.4 (d), and the proof is complete. □

In view of Theorem 8.1 and Lemma 8.1 we introduce a new notion. Given $\Omega \in \mathfrak{OBC}(X)$, let us call $F\colon \overline{\Omega} \to Y$ *properly epi* on $\overline{\Omega}$ if F is epi on $\overline{\Omega}$ and $[F|_{\overline{\Omega}}]_{\mathrm{a}} > 0$. Thus, being properly epi is stronger than being strictly epi, by Theorem 8.1. Moreover, in this terminology Lemma 8.1 is a precise analogue to Lemma 7.4, with "strictly epi" replaced by "properly epi".

In applications one has often to deal with α-contractive vector fields, i.e., operators of the form $F = I - C$, where $[C]_{\mathrm{A}} < 1$. For such operators the following test is convenient.

Proposition 8.1. *Suppose that $F = I - C\colon \overline{\Omega} \to Y$ with $[C|_{\overline{\Omega}}]_{\mathrm{A}} < 1$. Then the following three statements are equivalent*:

(a) *F is epi on $\overline{\Omega}$.*

(b) *F is strictly epi on $\overline{\Omega}$.*

(c) *F is properly epi on $\overline{\Omega}$.*

Proof. The assertion follows easily from the trivial estimate $[F|_{\overline{\Omega}}]_a = [I - C|_{\overline{\Omega}}]_a \geq 1 - [C|_{\overline{\Omega}}]_A > 0$ and from Theorem 8.1. □

Proposition 8.1 shows that the difference between epi operators and strictly epi operators appears rather small from the viewpoint of applications. Nevertheless, the above three properties are not always equivalent. Indeed, the operator F in Example 7.5 satisfies (a), but neither (b) nor (c). Of course, this operator is not an α-contractive vector field, since $(I - F)(S_r(X)) = S_{r^2-r}(X)$ for all $r > 1$.

8.2 The phantom and the large phantom

The definitions and results of the preceding section have obvious similarities to the corresponding definitions and results in Chapter 6. Roughly speaking, when passing from "asymptotic" conditions (i.e., for $\|x\| \to \infty$) to "local" conditions (i.e., for $x \in \Omega$ with $\Omega \in \mathfrak{OBC}(X)$), one has to replace FMV-regularity or AGV-regularity by some regularity involving epi or strictly epi operators on some set $\Omega \in \mathfrak{OBC}(X)$. However, this would depend on our choice of Ω which is not very natural. To get rid of this dependence, let us call $F\colon X \to Y$ *v-regular* if there exists some $\Omega \in \mathfrak{OBC}(X)$ such that F is strictly epi on $\overline{\Omega}$, and *V-regular* if there exists some $\Omega \in \mathfrak{OBC}(X)$ such that F is properly epi on $\overline{\Omega}$.

The following proposition provides a comparison between these and the other types of regularity studied in Sections 6.2 and 6.6.

Proposition 8.2. *Every AGV-regular operator is v-regular, every FMV-regular operator is V-regular, and every V-regular operator is v-regular.*

Proof. AGV-regularity of $F\colon X \to Y$ means that $[F]_q > 0$ and $\mu(F) > 0$, i.e., F is k-stably solvable for some $k > 0$. From Lemma 7.1 it follows then that also $\nu(F) > 0$, i.e., F is k-epi on some set $\Omega \in \mathfrak{OBC}(X)$, and so F is v-regular.

FMV-regularity of F in turn means that $[F]_q > 0$, $[F]_a > 0$, and F is stably solvable; we have to show that this implies that F is epi on $\overline{\Omega}$ for some $\Omega \in \mathfrak{OBC}(X)$.

From $[F]_q > 0$ it follows that there exists $r > 0$ such that $F(x) \neq \theta$ for $\|x\| \geq r$. We claim that F is epi on $\overline{\Omega} = B_r(X)$. So let $G\colon B_r(X) \to Y$ be compact with $G(x) \equiv \theta$ on $S_r(X)$, and let $\hat{G}$ be the trivial extension (7.4) of G. Since F is stably solvable, we find $\hat{x} \in X$ such that $F(\hat{x}) = \hat{G}(\hat{x})$. Since $F(x) \neq \theta = \hat{G}(x)$ for $\|x\| \geq r$, we conclude that $\hat{x} \in \Omega$, and so we are done.

Finally, the fact that V-regularity implies v-regularity is just a reformulation of Theorem 8.1. □

We may summarize the four regularity properties contained in Proposition 8.2 in the following table.

Table 8.1

$F(x) = G(x)$ solvable	asymptotic	local
for $[G]_A > 0$ and $[F]_a \geq 0$	F AGV-regular	F v-regular
for $[G]_A = 0$ and $[F]_a > 0$	F FMV-regular	F V-regular

Proposition 8.2 shows that passing from asymptotic to local conditions one gets a more general notion of regularity or, in other words, "sharper" solvability results. In Example 6.9 we have seen that there exist FMV-regular operators which are not AGV-regular. The following two examples show that also the implications in Proposition 8.2 cannot be inverted.

Example 8.1. In $X = \mathbb{R}$, consider the "sawtooth function" $F\colon X \to X$ be defined by

$$F(x) = \begin{cases} x & \text{if } |x| \leq 1, \\ -2-x & \text{if } -2 \leq x \leq -1, \\ 2-x & \text{if } 1 \leq x \leq 2, \\ 0 & \text{if } |x| \geq 2. \end{cases} \tag{8.4}$$

Since F is k-epi on $\overline{\Omega} = [-1, 1]$ for any $k > 0$, F is certainly both v-regular and V-regular. On the other hand, since F is not onto, F is neither AGV-regular nor FMV-regular.

To see that v-regularity is weaker than V-regularity, we can use the same example which we already used to show that AGV-regularity is weaker than FMV-regularity.

Example 8.2. Let F be the operator from Example 6.9. We already know that F is AGV-regular. Consequently, F is v-regular, by Proposition 8.2. On the other hand, consider the set M_c from Example 6.9. For any $\Omega \in \mathfrak{OBC}(X)$, the intersection $M_c \cap \Omega$ is not precompact, because Ω is a neighborhood of θ in an infinite dimensional space. But the image $F(M_c \cap \Omega) = \{\theta\}$ is compact, and so we must have $[F|_{\overline{\Omega}}]_a = 0$, i.e., F cannot be V-regular.

Given $F\colon X \to X$, we call the set

$$\phi(F) = \{\lambda \in \mathbb{K} : \lambda I - F \text{ is not v-regular}\} \tag{8.5}$$

the *Väth phantom* of F and the set

$$\Phi(F) = \{\lambda \in \mathbb{K} : \lambda I - F \text{ is not V-regular}\} \tag{8.6}$$

the *large Väth phantom* of F. Thus, a scalar λ belongs to $\phi(F)$ if and only if $\lambda I - F$ fails to be strictly epi on *any* set $\Omega \in \mathfrak{OBC}(X)$. Likewise, we have $\lambda \in \Phi(F)$ if and only if, for any set $\Omega \in \mathfrak{OBC}(X)$, either $[(\lambda I - F)|_{\overline{\Omega}}]_a = 0$ or $\lambda I - F$ is not epi on $\overline{\Omega}$.

Proposition 8.2 trivially implies that

$$\phi(F) \subseteq \sigma_{AGV}(F), \tag{8.7}$$

$$\Phi(F) \subseteq \sigma_{FMV}(F), \tag{8.8}$$

and

$$\phi(F) \subseteq \Phi(F). \tag{8.9}$$

The following Table 8.2 corresponds to Table 8.1 and describes the four sets occurring in (8.7)–(8.9).

Table 8.2

$\lambda x - F(x) = G(x)$ not solvable	asymptotic	local
for $[G]_A > 0$ and $[F]_a \geq 0$	$\lambda \in \sigma_{AGV}(F)$	$\lambda \in \phi(F)$
for $[G]_A = 0$ and $[F]_a > 0$	$\lambda \in \sigma_{FMV}(F)$	$\lambda \in \Phi(F)$

Let us make some comments on Table 8.2. Example 8.2 shows that the condition $[F]_a > 0$, which is crucial for FMV-regularity or V-regularity, may fail even if F has very good "surjectivity properties". This means that the AGV-spectrum and the Väth phantom are more appropriate than the FMV-spectrum and the large Väth phantom if we are interested in *existence of solutions* to nonlinear problems. Nevertheless, there are two reasons why the condition $[F]_a > 0$ may be of interest anyway. First, it might be easier to verify than solvability conditions for "more general right hand sides". Second, the condition $[F]_a = 0$ means, loosely speaking, that the operator F is "extremely non-injective", and one might want a spectrum to reflect this drawback. Indeed, injectivity is the crucial property if one is interested in *uniqueness of solutions* to nonlinear problems. For instance, for the scalar operator F in Example 8.1 we have

$$\phi(F) = \Phi(F) = \emptyset, \quad \sigma_{AGV}(F) = \sigma_{FMV}(F) = \{0\}.$$

It is also illuminating to illustrate the above discussion by means of the following *linear* example.

Example 8.3. Let $X = l_p$ $(1 \leq p \leq \infty)$, and let $L \in \mathfrak{L}(X)$ be defined by

$$L(x_1, x_2, x_3, x_4, \ldots) = (x_2, x_4, x_6, x_8, \ldots).$$

Then $\|L\| = 1$, $[L]_a = 0$, and L has the infinite dimensional nullspace

$$N(L) = \{(x_1, 0, x_3, 0, \dots) : x_1, x_3, \dots \in \mathbb{K}\};$$

so L is far from being injective. Nevertheless, L is onto with bounded right inverse

$$K(y_1, y_2, y_3, y_4, \dots) = (0, y_1, 0, y_2, \dots).$$

This implies that L is k-epi on every ball $B_r(X)$ for $k < 1/\|K\| = 1$. In fact, if $G\colon B_r(X) \to X$ is continuous with $[G]_A < 1$ and $G(x) \equiv \theta$ on $S_r(X)$, for the trivial extension (7.4) of G we have $[\hat{G}K]_A \le [G]_A\|K\| < 1$ and $[\hat{G}K]_Q \le [\hat{G}]_Q\|K\| = 0$. Theorem 2.2 implies that there exists $\hat{y} \in X$ with $\hat{G}(K(\hat{y})) = \hat{y}$, and so the element $\hat{x} := K\hat{y}$ solves the equation $Lx = G(x)$. ♡

Let us now discuss some properties of the two phantoms (8.5) and (8.6). The following is parallel to Theorems 6.2, 6.11 and 7.4.

Theorem 8.2. *Both the phantom $\phi(F)$ and the large phantom $\Phi(F)$ are closed.*

Proof. We prove that the complement of $\phi(F)$ is open. Fix $\lambda \in \mathbb{K} \setminus \phi(F)$. This means that the operator $\lambda I - F$ is v-regular, i.e., we find some $\Omega \in \mathfrak{O}\mathfrak{B}\mathfrak{C}(X)$ such that $\lambda I - F$ is strictly epi on $\overline{\Omega}$. In particular,

$$\delta := \min\left\{\frac{\operatorname{dist}(\theta, (\lambda I - F)(\partial\Omega))}{\sup_{x\in\partial\Omega}\|x\|}, \nu_\Omega(\lambda I - F)\right\} > 0.$$

For any $\mu \in \mathbb{K}$ with $|\mu - \lambda| < \delta$, the operator $G = (\mu - \lambda)I$ satisfies

$$\sup_{x\in\partial\Omega}\|G(x)\| < \delta \sup_{x\in\partial\Omega}\|x\| \le \operatorname{dist}(\theta, (\lambda I - F)(\partial\Omega))$$

and

$$[G|_{\overline{\Omega}}]_A = |\mu - \lambda| < \delta \le \nu_\Omega(F).$$

Lemma 7.4 thus implies that $(\lambda I - F) + G = \mu I - F$ is strictly epi on $\overline{\Omega}$, i.e., $\mu \in \phi(F)$.

The proof for $\Phi(F)$ is analogous, with the only difference that we put

$$\delta := \min\left\{\frac{\operatorname{dist}(\theta, (\lambda I - F)(\partial\Omega))}{\sup_{x\in\partial\Omega}\|x\|}, [(\lambda I - F)|_{\overline{\Omega}}]_a\right\}$$

and use Lemma 8.1 instead of Lemma 7.4. □

In the next theorem we give a sufficient boundedness condition for the Väth phantoms. We recall that the FMV-spectrum is bounded for operators in $\mathfrak{A}(X) \cap \mathfrak{Q}(X)$ (Theorem 6.3), while the Feng spectrum is bounded for operators in $\mathfrak{A}(X) \cap \mathfrak{B}(X)$

(Theorem 7.5).For the phantom, the sets $\mathfrak{Q}(X, Y)$ and $\mathfrak{B}(X, Y)$ do not play any role, of course, and the set $\mathfrak{A}(X, Y)$ can be replaced by

$$\mathfrak{V}(X, Y) := \{F \in \mathfrak{C}(X, Y) : [F|_{\overline{\Omega}}]_{\mathrm{A}} < \infty \text{ for some } \Omega \in \mathfrak{OBC}(X)\}. \tag{8.10}$$

As usual, we write $\mathfrak{V}(X, X) =: \mathfrak{V}(X)$. Evidently, $\mathfrak{V}(X) \supseteq \mathfrak{A}(X)$, where the inclusion may be strict.

Theorem 8.3. *For $G \in \mathfrak{V}(X)$, both the phantom $\phi(G)$ and the large phantom $\Phi(G)$ are bounded, hence compact.*

Proof. By assumption, we find some $\Omega \in \mathfrak{OBC}(X)$ with $[G|_{\overline{\Omega}}]_{\mathrm{A}} < \infty$ and

$$c := \frac{\sup_{x \in \partial\Omega} \|G(x)\|}{\inf_{x \in \partial\Omega} \|x\|} < \infty.$$

Fix $\lambda \in \mathbb{K}$ with $|\lambda| > \max\{c, [G|_{\overline{\Omega}}]_{\mathrm{A}}\}$; we apply Lemma 8.1 to the linear operator $F = -\lambda I$ on Ω. It is clear that F is epi on $\overline{\Omega}$ and $[F|_{\overline{\Omega}}]_{\mathrm{a}} \geq |\lambda| > 0$. Moreover, by our choice of λ we have

$$\sup_{x \in \partial\Omega} \|G(x)\| = c\,\mathrm{dist}(\theta, \partial\Omega) < |\lambda|\,\mathrm{dist}(\theta, \partial\Omega) = \mathrm{dist}(\theta, F(\partial\Omega)),$$

so (8.3) holds true. Finally, $[G|_{\overline{\Omega}}]_{\mathrm{A}} < |\lambda| \leq [F|_{\overline{\Omega}}]_{\mathrm{a}}$. So from Lemma 8.1 it follows that $F + G = G - \lambda I$ is both v-regular and V-regular, and so we are done. □

The proof of Theorem 8.3 shows that the *phantom radius*

$$r_\Phi(F) := \sup\{|\lambda| : \lambda \in \Phi(F)\} \tag{8.11}$$

of an operator $F \in \mathfrak{A}(X)$ satisfies the upper estimate

$$r_\Phi(F) \leq \inf_{\Omega \in \mathfrak{OBC}(X)} \max\left\{[F|_{\overline{\Omega}}]_{\mathrm{A}}, \frac{\sup_{x \in \partial\Omega} \|F(x)\|}{\inf_{x \in \partial\Omega} \|x\|}\right\}. \tag{8.12}$$

In Chapter 6 we have proved a semicontinuity result for the FMV-spectrum and AGV-spectrum with respect to the FMV-topology. Recall that the FMV-topology is generated by the seminorm

$$p_{AQ}(F) = \max\{[F]_{\mathrm{A}}, [F]_{\mathrm{Q}}\}.$$

This seminorm reflects both the compactness condition $[F]_{\mathrm{A}} < \infty$ and the asymptotic condition $[F]_{\mathrm{Q}} < \infty$ which guarantee the compactness of the spectra $\sigma_{\mathrm{FMV}}(F)$ and $\sigma_{\mathrm{AGV}}(F)$. Since the phantoms (8.5) and (8.6) are "local", not "asymptotic", it is evident that we have to choose another topology for them.

The estimate (8.12) suggests to choose, for $m = 1, 2, 3, \ldots$, the family of pseudo-norms

$$q_m(F) := \max\left\{[F|_{B_m(X)}]_{\mathrm{A}}, \sup_{\|x\| \leq m} \|F(x)\|\right\}. \tag{8.13}$$

This family generates a topology on $\mathfrak{C}(X, Y)$ which we will call the *Väth topology* in what follows. The Väth topology is neither finer nor coarser than the FMV-topology. Observe that multiplication by scalars is not continuous in the Väth topology, and so $(\mathfrak{C}(X, Y), q_m)$ is not a topological vector space. However, since the Väth topology is Hausdorff and generated by a countable number of pseudonorms, it is metrizable by the metric

$$d_{\mathfrak{V}}(F, G) = \sum_{m=1}^{\infty} \frac{1}{2^m} \frac{q_m(F - G)}{1 + q_m(F - G)}, \tag{8.14}$$

which we call the *Väth metric* on $\mathfrak{V}(X, Y)$. On the subspace of all $F \in \mathfrak{C}(X, Y)$ for which $q_m(F) < \infty$ for all $m \in \mathbb{N}$, the pseudonorms q_m are seminorms, and so this subspace becomes a locally convex vector space with the Väth topology. However, the following important result holds even on the essentially larger space $\mathfrak{V}(X, Y)$.

Theorem 8.4. *The multivalued maps $\phi\colon \mathfrak{V}(X) \to 2^{\mathbb{K}}$ and $\Phi\colon \mathfrak{V}(X) \to 2^{\mathbb{K}}$ which associate to each F its phantom or large phantom are upper semicontinuous in the Väth topology.*

Proof. By Lemma 5.4, it suffices to show that the multivalued maps ϕ and Φ are locally compact and have closed graph. Concerning the latter, we show first that they even have closed graph on $\mathfrak{C}(X)$.

The proof is analogous to the proof of Theorem 8.2 (actually, Theorem 8.2 is a special case). We prove that the complement is open. Thus, fix some (F_0, λ_0) in the complement of the graph of ϕ, which means that $\lambda_0 \notin \phi(F_0)$. Then the operator $\lambda_0 I - F_0$ is strictly epi on $\overline{\Omega}$ for some $\Omega \in \mathfrak{O}\mathfrak{B}\mathfrak{C}(X)$, and so

$$\delta := \min\{\operatorname{dist}(\theta, (\lambda_0 I - F_0)(\partial\Omega)), \nu_\Omega(\lambda_0 I - F_0)\} > 0.$$

Choose $m \in \mathbb{N}$ with $\overline{\Omega} \subseteq B_m(X)$. We claim that, for any $F \in \mathfrak{C}(X)$ and $\lambda \in \mathbb{K}$ with

$$q_m(F - F_0) < \frac{\delta}{2}, \quad |\lambda - \lambda_0| < \frac{\delta}{2m},$$

the pair (F, λ) belongs to the complement of the graph of ϕ. Indeed, the operator $G\colon \overline{\Omega} \to X$ defined by $G = (\lambda I - F) - (\lambda_0 I - F_0)$ satisfies

$$\begin{aligned} q_m(G) &= q_m((\lambda - \lambda_0)I - (F - F_0)) \\ &\leq |\lambda - \lambda_0| q_m(I) + q_m(F - F_0) \\ &= m|\lambda - \lambda_0| + q_m(F - F_0) \\ &< m\frac{\delta}{2m} + \frac{\delta}{2} = \delta, \end{aligned}$$

and so in particular

$$\max\left\{ \sup_{x \in \partial\Omega} \|G(x)\|, [G|_{\overline{\Omega}}]_{\mathrm{A}} \right\} < \delta.$$

By Lemma 7.4, this implies that $(\lambda_0 I - F_0) + G = \lambda I - F$ is strictly epi on $\overline{\Omega}$, and so $\lambda \notin \phi(F)$, as claimed.

The proof of the closedness of the graph of Φ is analogous, with the only difference that we put

$$\delta := \min\{\mathrm{dist}(\theta, (\lambda_0 I - F_0)(\partial\Omega)), [(\lambda_0 I - F_0)|_{\overline{\Omega}}]_{\mathrm{a}}\}$$

and use Lemma 8.1 instead of Lemma 7.4.

It remains to show that ϕ and Φ are locally compact. Since the maps $F \mapsto \phi(F)$ and $F \mapsto \Phi(F)$ attain values in the (finite-dimensional!) space $\mathbb{K}$, we have to show that they are uniformly bounded in a neighborhood of any given $F \in \mathfrak{V}(X)$. But this is an immediate consequence of (8.12). □

In the proof of Theorem 8.4 we have also shown that $\mathfrak{V}(X, Y)$ is an open subset of $\mathfrak{C}(X, Y)$ in the Väth topology.

Let us now see how the phantom of a *linear* operator looks like. Fortunately, we get precisely what we expect.

Theorem 8.5. *For $L \in \mathfrak{L}(X)$ we have*

$$\phi(L) = \Phi(L) = \sigma(L), \tag{8.15}$$

where $\sigma(L)$ is the usual spectrum (1.5) *of L.*

Proof. If $\lambda I - L$ is v-regular, then $\mathrm{dist}(\theta, (\lambda I - L)(\partial\Omega)) > 0$ and $\nu_\Omega(\lambda I - L) > 0$ for some $\Omega \in \mathfrak{OBC}(X)$. The first condition implies that $\lambda I - L$ is injective, the second one that it is surjective.

To see the latter, assume that the range of $\lambda I - L$ misses some point $y \in X$. Then $y \neq \theta$, and the range of $\lambda I - L$ misses all points μy with $\mu \neq 0$. In particular, the equation $(\lambda I - L)x = G(x)$ has no solution for the compact operator $G(x) = \mathrm{dist}(x, \partial\Omega)y$ which satisfies $G(x) \equiv \theta$ on $\partial\Omega$. Consequently, $\lambda I - L$ is not epi on $\overline{\Omega}$ which contradicts $\nu_\Omega(\lambda I - L) > 0$.

In this way we have proved that $\sigma(L) \subseteq \phi(L)$. On the other hand, Theorem 6.1 and the inclusion (8.8) show that $\Phi(L) \subseteq \sigma(L)$. □

8.3 The point phantom

In Section 6.6 we have seen that the appropriate notion of "eigenvalue" in connection with the FMV-spectrum and AGV-spectrum is provided by the subspectrum $\sigma_{\mathrm{q}}(F)$. Recall that $\lambda \in \sigma_{\mathrm{q}}(F)$ if there exists an unbounded sequence $(x_n)_n$ such that

$$\frac{\|\lambda x_n - F(x_n)\|}{\|x_n\|} \to 0 \quad (n \to \infty). \tag{8.16}$$

In this section we propose an "eigenvalue theory" which takes into account the structure of the Väth phantom; the corresponding part of the phantom will be called "point phantom".

Let us say that $\lambda \in \mathbb{K}$ is a *connected eigenvalue* of $F\colon X \to X$ if the nullset

$$N(\lambda I - F) = \{x \in X : F(x) = \lambda x\} \tag{8.17}$$

of $\lambda I - F$ contains an unbounded connected set containing θ. The set

$$\phi_{\mathrm{p}}(F) := \{\lambda \in \mathbb{K} : \lambda \text{ connected eigenvalue for } F\} \tag{8.18}$$

will be called *point phantom* of F in the sequel. It is clear that, in case of a bounded linear operator L this gives the familiar definition of eigenvalue, i.e.,

$$\phi_{\mathrm{p}}(L) = \sigma_{\mathrm{p}}(L) \tag{8.19}$$

with $\sigma_{\mathrm{p}}(L)$ as in (1.21). For nonlinear F, however, the sets $\phi_{\mathrm{p}}(F)$ and $\sigma_{\mathrm{p}}(F)$ may be quite different.

For instance, for the "seagull" F from Example 6.6 we have

$$\sigma_{\mathrm{p}}(F) = \mathbb{R} \setminus \{0\}, \quad \sigma_{\mathrm{q}}(F) = \{0\}, \quad \phi_{\mathrm{p}}(F) = \emptyset. \tag{8.20}$$

Similarly, for the "sawtooth" F from Example 8.1 we have

$$\sigma_{\mathrm{p}}(F) = [0, 1], \quad \sigma_{\mathrm{q}}(F) = \{0\}, \quad \phi_{\mathrm{p}}(F) = \emptyset. \tag{8.21}$$

These examples illustrate again the fact that the point spectrum $\sigma_{\mathrm{p}}(F)$ takes into account the global behaviour of F, the asymptotic point spectrum $\sigma_{\mathrm{q}}(F)$ the asymptotic behaviour of F, and the point phantom $\phi_{\mathrm{p}}(F)$ the local behaviour of F (near zero). In general, in the real scalar case $X = \mathbb{R}$, the point phantom contains at most 2 elements. But already in the complex scalar case $X = \mathbb{C}$ the point phantom may be a continuum, as the following example shows.

Example 8.4. Let $X = \mathbb{C}$ and $F\colon X \to X$ be defined by

$$F(z) = |\sin(\arg z)|z. \tag{8.22}$$

The nullset of $\lambda I - F$ is then

$$N(\lambda I - F) = \{z \in \mathbb{C} : |\sin(\arg z)| = \lambda\} \cup \{0\},$$

and this set contains a one-dimensional subspace for each $\lambda \in [0, 1]$. On the other hand, for $\lambda \in \mathbb{C} \setminus [0, 1]$ the equation $F(z) = \lambda z$ has no solution at all, and so $\phi_{\mathrm{p}}(F) = [0, 1]$ in this example. ♡

In contrast to the "naive" definition of the point spectrum $\sigma_{\mathrm{p}}(F)$, the point phantom has several properties in common with the usual point spectrum of a linear operator. For example, we will see in Chapter 10 (Theorem 10.8) that $\phi_{\mathrm{p}}(F) \cup \{0\}$ is *compact* for compact operators F. Moreover, the Väth phantom $\phi(F)$ is the "smallest reasonable spectrum" which contains the point phantom, as we shall show now.

To this end, let us consider some analogue to what we have done in (6.39) for the asymptotic point spectrum. Given $F: X \to X$, we call the set

$$\begin{aligned}\phi_{\mathrm{q}}(F) := \{\lambda \in \mathbb{K} :\ &\text{there exist sequences } (F_n)_n \text{ and } (\lambda_n)_n \text{ with} \\ &d_{\mathfrak{V}}(F_n, F) \to 0,\ \lambda_n \to \lambda,\ \text{and } \lambda_n \in \phi_{\mathrm{p}}(F_n) \text{ for all } n \in \mathbb{N}\}\end{aligned} \tag{8.23}$$

the *approximate point phantom* of F, where $d_{\mathfrak{V}}$ denotes the Väth metric (8.14). We remark that the set (8.23) may be defined equivalently in the form

$$\begin{aligned}\phi_{\mathrm{q}}(F) = \{\lambda \in \mathbb{K} :\ &\text{there exists sequence } (F_n)_n \text{ with} \\ &d_{\mathfrak{V}}(F_n, F) \to 0 \text{ and } \lambda \in \phi_{\mathrm{p}}(F_n) \text{ for all } n \in \mathbb{N}\},\end{aligned} \tag{8.24}$$

i.e., the sequence $(\lambda_n)_n$ may be chosen *constant*. This follows from the fact that $G_n := F_n + (\lambda - \lambda_n)I \to F$ and $\lambda \in \phi_{\mathrm{p}}(G_n)$.

The definition (8.23) of the approximate point phantom appears rather technical. However, for linear operators this is nothing new, as the following shows.

Theorem 8.6. *For $L \in \mathfrak{L}(X)$, the equality*

$$\phi_{\mathrm{q}}(L) = \sigma_{\mathrm{q}}(L) \tag{8.25}$$

holds true, where $\sigma_{\mathrm{q}}(L)$ is the approximate point spectrum (1.50) *of L.*

Proof. Given $\lambda \in \phi_{\mathrm{q}}(L)$, let $(F_n)_n$ be a sequence of (not necessarily linear) operators converging to L in the Väth topology and satisfying $\lambda \in \phi_{\mathrm{p}}(F_n)$ for all n. The latter condition implies that we can find a sequence $(e_n)_n$ in $S(X)$ such that $F_n(e_n) = \lambda e_n$ for each n. Given $\varepsilon > 0$, choose $n \in \mathbb{N}$ such that $q_1(F_n - L) \leq \varepsilon$, where

$$q_1(F) = \max\{[F|_{B(X)}]_{\mathrm{A}}, \sup\{\|F(x)\| : x \in B(X)\}\},$$

see (8.13). We get then

$$\|\lambda e_n - Le_n\| \leq \|\lambda e_n - F_n(e_n)\| + \|F_n(e_n) - Le_n\| \leq \varepsilon,$$

which shows that

$$\inf_{\|x\|=1} \|\lambda x - Lx\| = 0,$$

i.e., $\lambda \in \sigma_{\mathrm{q}}(L)$. Conversely, fix $\lambda \in \sigma_{\mathrm{q}}(L)$ and let $(e_n)_n$ be a sequence in $S(X)$ such that $\|\lambda e_n - Le_n\| \to 0$ as $n \to \infty$. Define $F_n: X \to X$ by

$$F_n(x) := Lx + \|x\|(\lambda e_n - Le_n).$$

For fixed $m \in \mathbb{N}$ we have then, on the one hand,

$$\sup_{\|x\| \le m} \|F_n(x) - Lx\| \le m\|\lambda e_n - Le_n\| \to 0 \quad (n \to \infty).$$

But we also have $[(F_n - L)|_{B_m(X)}]_{\mathrm{A}} = 0$, on the other, since $(F_n - L)(B_m(X))$ is a bounded subset of a one-dimensional space, hence precompact. This shows that $q_m(F_n - L) \to 0$, as $n \to \infty$, for each $m \in \mathbb{N}$, and so $d_{\mathfrak{V}}(F_n, L) \to 0$ as $n \to \infty$.

It remains to show that λ belongs to each of the point phantoms $\phi_{\mathrm{p}}(F_n)$. But for each n the ray $T_n := \{\mu e_n : 0 \le \mu < \infty\}$ belongs to the nullset $N(\lambda I - F_n)$, because

$$F_n(\mu e_n) = \mu L e_n + \|e_n\|\mu\lambda e_n - \|e_n\|\mu L e_n = \lambda\mu e_n,$$

and so $\lambda \in \phi_{\mathrm{p}}(F_n)$, hence $\lambda \in \phi_{\mathrm{q}}(L)$, as claimed. □

As we shall see later, the inclusion $\phi_{\mathrm{q}}(F) \subseteq \sigma_{\mathrm{q}}(F)$ holds for nonlinear F, where the inclusion may be strict. For now we are interested in the connection between $\phi_{\mathrm{q}}(F)$ and the point phantom $\phi_{\mathrm{p}}(F)$, on the one hand, and between $\phi_{\mathrm{q}}(F)$ and the whole phantom $\phi(F)$, on the other.

Theorem 8.7. *The inclusions*

$$\phi_{\mathrm{p}}(F) \subseteq \phi_{\mathrm{q}}(F) \subseteq \phi(F) \subseteq \Phi(F) \tag{8.26}$$

are true.

Proof. The first inclusion is trivial, and the last inclusion has already been proved in (8.10). It is not hard to see that $\phi_{\mathrm{p}}(F) \subseteq \phi(F)$. Indeed, if λ is a connected eigenvalue for F, we find an unbounded connected subset C of $N(\lambda I - F)$ which contains θ. For any $\Omega \in \mathfrak{OBC}(X)$, we have $\theta \notin (\lambda I - F)(\partial\Omega)$, since otherwise C could be divided into the disjoint open (in C) sets $\Omega \cap C$ and $C \setminus \overline{\Omega}$. Consequently, $\lambda \in \phi(F)$.

So we have proved the inclusion $\phi_{\mathrm{p}}(F) \subseteq \phi(F)$. Now, since the graph of ϕ is closed (Theorem 8.4), it follows that the graph of ϕ_{q} is contained in the graph of ϕ, and so the second inclusion in (8.26) holds true as well. □

We will give a much stronger result than (8.26) for compact odd operators in Theorem 8.9 below. In case of a linear operator L, the corresponding inclusions

$$\sigma_{\mathrm{p}}(L) \subseteq \sigma_{\mathrm{q}}(L) \subseteq \sigma(L) \tag{8.27}$$

are of course trivial. The analogy between σ_{q} and ϕ_{q} is actually much larger. In fact, the points $\lambda \in \sigma_{\mathrm{q}}(F)$ are characterized by the "asymptotic boundary condition" $[\lambda I - F]_{\mathrm{q}} = 0$. As the next theorem shows, the approximate point phantom $\phi_{\mathrm{q}}(F)$ may be characterized similarly by a "local boundary condition".

Proposition 8.3. *A point λ belongs to $\phi_{\mathrm{q}}(F)$ if and only if*

$$\operatorname{dist}(\theta, (\lambda I - F)(\partial\Omega)) = 0 \tag{8.28}$$

for all $\Omega \in \mathfrak{OBC}(X)$.

Proof. The necessity of condition (8.28) is evident (recall the proof of Theorem 8.7). Indeed, if we can find a set $\Omega \in \mathfrak{OBC}(X)$ such that $\delta := \operatorname{dist}(\theta, (\lambda I - F)(\partial\Omega)) > 0$, then we choose $m \in \mathbb{N}$ with $\partial\Omega \subseteq B_m(X)$. For any $G \in \mathfrak{C}(X)$ and any $\mu \in \mathbb{K}$ with

$$q_m(F - G) < \frac{\delta}{2}, \quad |\mu - \lambda| < \frac{\delta}{2m}$$

we have then $\theta \notin (\mu I - G)(\partial\Omega)$, i.e., the nullset $N(\mu I - G)$ does not intersect $\partial\Omega$. Consequently, any subset $C \subseteq N(\mu I - G)$ can be divided into the disjoint open (in C) sets $\Omega \cap C$ and $C \setminus \overline{\Omega}$, and so any connected set $C \subseteq N(\mu I - G)$ containing θ is contained in the bounded set Ω which implies $\mu \notin \phi_{\mathrm{p}}(G)$. So we have shown that $\lambda \notin \phi_{\mathrm{q}}(F)$ if (8.28) *fails* for some $\Omega \in \mathfrak{OBC}(X)$.

To prove sufficiency let us assume that $\lambda \notin \phi_{\mathrm{q}}(F)$. If $F(\theta) \neq \theta$, the statement is trivial, because then the continuity of F implies that

$$\operatorname{dist}(\theta, F(S_r(X))) > \frac{\|F(\theta)\|}{2}$$

for sufficiently small $r > 0$. So assume that $F(\theta) = \theta$. Putting $A := \lambda I - F$, we see then that, for any given $\varepsilon > 0$, the closed set $N_\varepsilon = \{x \in X : \|A(x)\| \leq \varepsilon\}$ contains θ.

We define now a subset $M_\varepsilon \subseteq N_\varepsilon$ with $\theta \in M_\varepsilon$ as follows. For $x \in N_\varepsilon$, put

$$\delta(x) := \sup\{\delta \in (0, 1) : \text{any } y \in X \text{ with } \|y - x\| \leq \delta \text{ satisfies } \|A(y)\| \leq 2\varepsilon\}. \tag{8.29}$$

Note that $\delta(x) > 0$ for any $x \in N_\varepsilon$, because A is continuous. Given $x \in N_\varepsilon$, let us call a finite sequence $x_1, \ldots, x_k \in N_\varepsilon$ an *ε-connection for x* if $x_1 = \theta$, $x_k = x$, and

$$\|x_j - x_{j+1}\| \leq \frac{1}{2}\delta(x_j) \quad (j = 1, \ldots, k - 1). \tag{8.30}$$

Denote by M_ε the set of all $x \in N_\varepsilon$ for which there exists an ε-connection. Then $\theta \in M_\varepsilon$, by definition. Moreover, M_ε is open in N_ε. To see this, fix $x \in M_\varepsilon$, and let $x_1, \ldots, x_k$ be an ε-connection for x. Then every $y \in N_\varepsilon$ with $\|y - x\| \leq \delta(x)/2$ has the ε-connection $x_1, \ldots, x_k, x_{k+1} := y \in N_\varepsilon$, and so $y \in M_\varepsilon$.

We prove now that also the complement $M_\varepsilon^c = N_\varepsilon \setminus M_\varepsilon$ is open in N_ε. More precisely, we show that, for $y \in M_\varepsilon^c$, every point $x \in N_\varepsilon$ with $\|x - y\| \leq \delta(y)/3$ belongs to M_ε^c as well. To see that $x \in M_\varepsilon^c$, observe first that, for any $z \in X$ with $\|z - x\| < 2\delta(y)/3$, we have $\|z - y\| < \delta(y)$ and thus $\|A(z)\| \leq 2\varepsilon$. From (8.29) we thus deduce the lower estimate $\delta(x) \geq 2\delta(y)/3$. It follows that $\|y - x\| \leq \delta(x)/2$. So, the assumption $x \in M_\varepsilon$ would imply $y \in M_\varepsilon$, as we have seen before, a contradiction. Consequently, we must have $x \in M_\varepsilon^c$ as claimed.

We show now that M_ε is bounded for sufficiently small $\varepsilon > 0$. To this end, we prove that the inclusion

$$M_\varepsilon \subseteq B_{1/\varepsilon}(X) \tag{8.31}$$

holds for ε sufficiently small. To see this, suppose that $\varepsilon > 0$ is such that (8.31) fails. We construct an operator $F_\varepsilon \in \mathfrak{C}(X)$ with $\lambda \in \phi_{\mathrm{p}}(F_\varepsilon)$ which satisfies $q_m(F_\varepsilon - F) \le 2\varepsilon$ for any $m < 1/\varepsilon$.

Since (8.31) fails, there is some $x_0 \in M_\varepsilon$ such that $\|x_0\| > 1/\varepsilon$. Let $x_1, \dots, x_k \in N_\varepsilon$ be a corresponding ε-connection for x_0. Put $Z_\varepsilon = [x_1, x_2] \cup [x_2, x_3] \cup \dots \cup [x_{k-1}, x_k]$, where $[x_j, x_{j+1}] = \mathrm{co}\{x_j, x_{j+1}\}$ denotes the closed line segment connecting x_j with x_{j+1}. Note that, by (8.30), for any $x \in [x_j, x_{j+1}]$ we have $\|x - x_j\| \le \delta(x_j)/2$. So (8.29) implies

$$\|A(x)\| \le 2\varepsilon \quad (x \in Z_\varepsilon). \tag{8.32}$$

By construction, Z_ε contains a path connecting $x_1 = \theta$ with $x_k = x_0$ which intersects at most in a finite number of points. Successively eliminating loops, if necessary, we find a path $P_\varepsilon \subseteq Z_\varepsilon$ connecting θ with x_0 which is homeomorphic to an interval. By the Tietze–Uryson lemma, we may thus extend the identity on P_ε to a continuous function $\rho_\varepsilon \colon X \to P_\varepsilon$.

Let $\psi_\varepsilon \colon [0, \infty) \to [0, 1]$ be continuous with $\psi_\varepsilon(t) \equiv 1$ for $0 \le t \le 1/\varepsilon$ and $\psi_\varepsilon(t) \equiv 0$ for $t \ge \|x_0\|$ $(> 1/\varepsilon)$. Now we define $F_\varepsilon \colon X \to X$ by

$$F_\varepsilon(x) = \lambda x - \psi_\varepsilon(\|x\|)[A(x) - A(\rho_\varepsilon(x))].$$

Then the nullset of $A_\varepsilon := \lambda I - F_\varepsilon$ contains the connected unbounded set $P_\varepsilon \cup \{x \in X : \|x\| \ge \|x_0\|\}$. In fact, for $x \in P_\varepsilon$ we have $A(x) - A(\rho_\varepsilon(x)) = \theta$, while for $\|x\| \ge \|x_0\|$ we have $\psi_\varepsilon(\|x\|) = 0$. This implies that $\lambda \in \phi_{\mathrm{p}}(F_\varepsilon)$. Moreover, for any $x \in B_{1/\varepsilon}(X)$ we have

$$F_\varepsilon(x) - F(x) = -\psi_\varepsilon(\|x\|)[A(x) - A(\rho_\varepsilon(x))] + A(x) = A(\rho_\varepsilon(x)) \in A(P_\varepsilon).$$

As A maps the compact set P_ε into a compact set, it follows that $[(F_\varepsilon - F)|_{B_{1/\varepsilon}(X)}]_{\mathrm{A}} = 0$. On the other hand, in view of (8.32) we get

$$\|F_\varepsilon(x) - F(x)\| \le 2\varepsilon \quad (x \in B_{1/\varepsilon}(X)).$$

Consequently, F_ε has the required properties.

Now we can prove that (8.31) holds for all sufficiently small $\varepsilon > 0$. Otherwise, we find a sequence $(\varepsilon_n)_n$ of positive numbers with $\varepsilon_n \to 0$ such that (8.31) fails for any $\varepsilon = \varepsilon_n$. For the corresponding sequence of operators F_{ε_n} constructed above, we have then $\lambda \in \phi_{\mathrm{p}}(F_{\varepsilon_n})$ and $F_{\varepsilon_n} \to F$ in the Väth topology, as $n \to \infty$. But this contradicts our assumption $\lambda \notin \phi_{\mathrm{q}}(F)$.

Summarizing, we thus have proved that we find some $\varepsilon > 0$ such that the corresponding set N_ε has the following three properties: N_ε contains some bounded subset M_ε with $\theta \in M_\varepsilon$, M_ε is open in N_ε, and the complement $M_\varepsilon^c = N_\varepsilon \setminus M_\varepsilon$ is also open in N_ε. It follows that M_ε and M_ε^c are also closed in N_ε, and thus closed in X, because N_ε is closed. We thus find disjoint open sets $\Omega_1 \subset X$ and $\Omega_2 \subset X$ such that $M_\varepsilon \subseteq \Omega_1$ and $M_\varepsilon^c \subseteq \Omega_2$. Replacing Ω_1 by $\Omega_1 \cap B_r^o(X)$, if necessary, where $r > 0$ is so large that $M_\varepsilon \subseteq B_r^o(X)$, it is no loss of generality to assume that Ω_1 is bounded.

Let Ω be the connected component of θ in Ω_1. Since Ω_1 is open, its path-components are open and are precisely its components, and so $\Omega \in \mathfrak{OBC}(X)$. Moreover, since components are closed in Ω_1, we have $\overline{\Omega} \cap \Omega_1 = \Omega$, and so, since Ω is open,

$$\partial\Omega = \overline{\Omega} \setminus \Omega = [(\overline{\Omega} \cap \Omega_1) \cup (\overline{\Omega} \setminus \Omega_1)] \setminus \Omega \subseteq \overline{\Omega} \setminus \Omega_1 \subseteq \overline{\Omega}_1 \setminus \Omega_1.$$

In view of $\overline{\Omega}_1 \cap M_\varepsilon^c = \emptyset$ and $M_\varepsilon \subseteq \Omega_1$, it follows that $\partial\Omega$ is disjoint from $M_\varepsilon^c \cup M_\varepsilon = N_\varepsilon$. But our definition of N_ε shows that

$$\inf_{x\in\partial\Omega} \|A(x)\| = \operatorname{dist}(\theta, (\lambda I - F)(\partial\Omega)) \geq \varepsilon.$$

So we have shown that (8.28) is also sufficient for λ to belong to $\phi_q(F)$, and the proof is complete. □

We remark that the use of "ε-connections" in the previous proof is motivated by the proof of Proposition 8.4 in the following Section 8.4.

In Theorem 8.6 we have proved the equality $\phi_q(L) = \sigma_q(L)$ for $L \in \mathfrak{L}(X)$. If F is nonlinear, one may obtain the inclusion

$$\phi_q(F) \subseteq \sigma_q(F) \tag{8.33}$$

as a simple consequence of Proposition 8.3. In fact, $\lambda \in \phi_q(F)$ implies that

$$\inf_{\|x\|=n} \|\lambda x - F(x)\| = 0 \quad (n \in \mathbb{N}),$$

by Proposition 8.3. In particular, for each $n \in \mathbb{N}$ we find an element $x_n \in S_n(X)$ such that $\|\lambda x_n - F(x_n)\| \leq \frac{1}{n}$, and so $\lambda \in \sigma_q(F)$.

Note that Proposition 8.3 also implies the inclusion $\phi_q(F) \subseteq \phi(F)$ stated in Theorem 8.7. Actually, Proposition 8.3 is the result which justifies the definition of the phantom which at first glance may look somewhat artificial: if one *wants* that the phantom $\phi(F)$ contains the point phantom $\phi_p(F)$ and has a closed graph, then it *must* contain the approximate point phantom $\phi_q(F)$. Consequently, the complement of the phantom $\phi(F)$ may only contain points λ with the property that $\operatorname{dist}(\theta, (\lambda I - F)(\partial\Omega)) > 0$ for some $\Omega \in \mathfrak{OBC}(X)$; the requirement that $\lambda I - F$ be strictly epi on $\overline{\Omega}$ for some $\Omega \in \mathfrak{OBC}(X)$ now just adds a certain "surjectivity condition". But this property describes precisely the complement of $\phi(F)$.

8.4 Special classes of operators

We have several times encountered the phenomenon that spectral theory gets richer if we restrict ourselves to compact operators. The same is true for phantoms. We start with a result which is analogous to Proposition 8.3 but whose proof is surprisingly difficult.

Proposition 8.4. *For $F \in \mathfrak{K}(X)$, the following is true*:

(a) *A point $\lambda \neq 0$ belongs to $\phi_{\mathrm{p}}(F)$ if and only if*

$$\theta \in (\lambda I - F)(\partial\Omega) \tag{8.34}$$

for all $\Omega \in \mathfrak{OBC}(X)$.

(b) *If X is finite dimensional, then (a) holds also for $\lambda = 0$.*

Proof. The fact that (8.34) holds for $\lambda \in \phi_{\mathrm{p}}(F)$ is trivial and holds even without the assumptions $F \in \mathfrak{K}(X)$ and $\lambda \neq 0$.

For the converse implication, assume that $\lambda \notin \phi_{\mathrm{p}}(F)$; we have to show that (8.34) is false. If $\theta \notin N(\lambda I - F)$, then we find some $r > 0$ with $B_r(X) \cap N(\lambda I - F) = \emptyset$, and so we have $\theta \notin (\lambda I - F)(\partial\Omega)$ for $\Omega = B_r^o(\theta)$. Consequently, we may assume that $\theta \in N(\lambda I - F)$.

Given $\varepsilon > 0$ and $x \in N(\lambda I - F)$, let us call a finite sequence $x_1, \ldots, x_k \in N(\lambda I - F)$ an *ε-chain for x* if $x_1 = \theta$, $x_k = x$, and $\|x_j - x_{j+1}\| \leq \varepsilon$ for $j = 1, \ldots, k-1$. Similarly as in the proof of Proposition 8.3, we denote by M_ε the set of all $x \in N(\lambda I - F)$ for which there exists an ε-chain. As before, one may show that both M_ε and $M_\varepsilon^c := N(\lambda I - F) \setminus M_\varepsilon$ are open in $N(\lambda I - F)$. Moreover, M_ε is bounded for all sufficiently small $\varepsilon > 0$. To see this, we assume by contradiction that there is a sequence $(\varepsilon_n)_n$ of positive numbers with $\varepsilon_n \to 0$ such that M_{ε_n} is unbounded.

Given $R > 0$, the set $N_R := N(\lambda I - F) \cap B_R(X)$ is compact. Indeed, if $x_n \in N_R$, then $(x_n)_n$ is a bounded sequence in $N(\lambda I - F)$, and so $(x_n)_n = (\lambda^{-1} F x_n)_n$ has a convergent subsequence. Since N_R is closed, the limit of this subsequence belongs to N_R.

Let $C_R \subseteq N_R$ be the set of all $x \in N_R$ with the property that there is an ε-chain for x *in* N_R for any $\varepsilon > 0$. Note that in general the inclusion

$$C_R \subset \bigcap_{\varepsilon > 0} M_\varepsilon \cap B_R(X)$$

is strict, because the ε-chains for $x \in M_\varepsilon$ may leave the ball $B_R(X)$. Nevertheless, we will prove that C_R is connected and contains an element x_R with $\|x_R\| = R$.

Let us first show that C_R contains all accumulation points of any sequence $(x_n)_n$ in N_R, where x_n has an ε_n-chain in N_R; in particular, C_R contains all accumulation points of C_R. Indeed, if such an accumulation point x (hence $x \in N_R$) and $\varepsilon > 0$

are given, we find some n with $\|x_n - x\| < \varepsilon$ and $\varepsilon_n < \varepsilon$, and x_n has some ε_n-chain $y_1, \ldots, y_k \in N_R$. Then $y_1, \ldots, y_k, y_{k+1} := x \in N_R$ is an ε-chain for x.

We prove now that C_R is connected. Assume by contradiction that C_R is not connected. Then C_R can be divided into two nonempty disjoint closed (in C_R) subsets $A, B \subseteq C_R$, where without loss of generality $\theta \in A$. We have proved above that C_R is closed in the compact set N_R. Consequently, since the sets A and B are closed in C_R, they are compact, and thus their distance $d := \operatorname{dist}(A, B)$ is positive. On the other hand, for any n we find some ε_n-chain $x_{n,1}, \ldots, x_{n,k_n} \in N_R$ with $x_{n,k_n} \in B$. Since $\operatorname{dist}(x_{n,k_n}, A) \geq d$, for any n with $\varepsilon_n < d/2$ we may choose the smallest index j_n with the property that $\operatorname{dist}(x_{n,j_n}, A) \geq d/4$. Then the estimate

$$\operatorname{dist}(x_{n,j_n}, A) < \varepsilon_n + \frac{d}{4} \leq \frac{3d}{4}$$

implies that also $\operatorname{dist}(x_{n,j_n}, B) \geq d/4$. By the compactness of N_R, the sequence $(x_{n,j_n})_n$ has an accumulation point $x \in N_R$. As we have observed above, this implies $x \in C_R$. But on the other hand, $\operatorname{dist}(x, A \cup B) \geq d/4$ implies $x \notin C_R$, a contradiction.

Now we show that C_R contains an element x_R of norm $\|x_R\| = R$. Since M_{ε_n} is unbounded, by assumption, we find an ε_n-chain $x_{n,1}, \ldots, x_{n,k_n+1} \in N_R$ with $\|x_{n,k_n+1}\| > R$. By decreasing k_n, if necessary, it is no loss of generality to assume that all except for the last element of this chain belong to $B_R(X)$, i.e., $x_{n,1}, \ldots, x_{n,k_n} \in N_R$. Note that the relation $\|x_{n,k_n+1} - x_{n,k_n}\| < \varepsilon_n$ implies, in particular, that $\|x_{n,k_n}\| > R - \varepsilon_n$. Since N_R is compact, the sequence $(x_{n,k_n})_n$ has some accumulation point x_R. Then $\|x_R\| = R$, and $x_R \in C_R$, by what we have proved above.

So we have found a connected set $C_R \subseteq N_R \subseteq N(\lambda I - F)$ which contains θ and some point x_R with $\|x_R\| = R$, i.e., the component of θ in $N(\lambda I - F)$ contains a point of norm R. Since $R > 0$ was arbitrary, we may conclude that the component of θ in $N(\lambda I - F)$ contains points of arbitrarily large norm, and thus is unbounded. But this means precisely that $\lambda \in \phi_{\mathrm{p}}(F)$, contradicting our assumption.

Summarizing, this contradiction shows that the set $M_\varepsilon \subseteq N(\lambda I - F)$ is bounded for all sufficiently small $\varepsilon > 0$. Since both M_ε and its complement $N(\lambda I - F) \setminus M_\varepsilon$ are open in $N(\lambda I - F)$, the same argument as at the end of the proof of Proposition 8.3 implies that there is some $\Omega \in \mathfrak{OBC}(X)$ with $\partial\Omega \cap N(\lambda I - F) = \emptyset$. But the latter means $\theta \notin (\lambda I - F)(\partial\Omega)$ which completes the proof of (a).

The proof of (b) is almost trivial. In fact, if X has finite dimension, the case $\lambda = 0$ may be reduced to the case $\lambda = 1$ by replacing F by the (compact!) operator $I - F$. □

An inspection of the proof shows that Proposition 8.4 holds without the two assumptions $\lambda \neq 0$ and $F \in \mathfrak{K}(X)$ if we require instead that the set $N(\lambda I - F) \cap B_R(X)$ be compact for any $R > 0$; this observation provides also another proof for part (b).

We have mentioned before that it is somewhat strange that Proposition 8.4 is so hard to prove. Indeed, the relation $\lambda \notin \phi_{\mathrm{p}}(F)$ means that the component C of θ in the nullset $N(\lambda I - F)$ is *bounded*. Since components of $N(\lambda I - F)$ are closed in X, one might expect that one can "separate" C from the unbounded components of $N(\lambda I - F)$ by open sets. This would imply, in particular, that $C \subseteq \Omega$ and $N(\lambda I - F) \cap \partial\Omega = \emptyset$ for some $\Omega \in \mathfrak{OBC}(X)$.

Surprisingly, this separation is not always possible! The reason is that the union of all unbounded components of $N(\lambda I - F)$ may contain a point of C as an accumulation point. If $N(\lambda I - F)$ were compact, one could try to apply some separation theorem in compact spaces from elementary topology (those proofs are usually based on ε-chains, and our proof of Proposition 8.4 is modelled after this). However, the main difficulty in the above proof is that $N(\lambda I - F)$ is *not compact*, and so one somehow has to restrict to the compact set $N_R = N(\lambda I - F) \cap B_R(X)$ which is not trivial.

The following example shows that, in general, $N(\lambda I - F)$ might fail to have the mentioned separation property. At the same time, this example shows that the assumption $\lambda \neq 0$ may not be dropped in Proposition 8.4 if X is infinite dimensional; this implies that the assumption $F \in \mathfrak{K}(X)$ may not be dropped either.

Example 8.5. Let X be an arbitrary infinite dimensional Banach space, and let $(e_n)_n$ be a sequence in $S(X)$ satisfying $\|e_m - e_n\| \geq 1/2$ for $m \neq n$. Consider the rays

$$R_n := \{te_n : t \geq 1/n\} \quad (n = 1, 2, 3, \dots).$$

For each $n \in \mathbb{N}$ we can find some $c_n \in (0, 1/2n)$ such that $\operatorname{dist}(R_m, R_n) \geq 3c_n$ for $m \neq n$. Put

$$U_n := \{x \in X : \operatorname{dist}(x, R_n) < c_n\} \quad (n = 1, 2, 3, \dots),$$

and define a function $\varphi\colon X \to [0, 1]$ by

$$\varphi(x) := \begin{cases} \dfrac{1}{c_n} \operatorname{dist}(x, R_n) & \text{if } x \in U_n, \\ 1 & \text{if } x \in X \setminus (U_1 \cup U_2 \cup \dots). \end{cases}$$

Then φ is continuous on $X \setminus \{\theta\}$ and bounded near θ. So the operator $F\colon X \to X$ defined by

$$F(x) = \varphi(x)\|x\|e, \tag{8.35}$$

where $e \in S(X)$ is some fixed element, is continuous on X. Moreover, since its range $R(F)$ is one-dimensional, F is certainly compact.

We claim that $0 \notin \phi_{\mathrm{p}}(F)$, i.e., 0 is not a connected eigenvalue of F. Indeed, the connected component C of θ in the nullset $N(F)$ of F is just the singleton $\{\theta\}$. To see this, suppose the contrary. Then the set $C \cap U_k$ would be nonempty for some $k \in \mathbb{N}$. But $C \cap U_k$ is both open and closed in C, because $\partial U_k \cap C \subseteq \partial U_k \cap N(F) = \emptyset$. Since C is connected, we conclude that $C \cap U_k = C$, contradicting the fact that $\theta \notin U_k$. So we have shown that $0 \notin \phi_{\mathrm{p}}(F)$.

On the other hand, (8.34) is true for $\lambda = 0$ and any $\Omega \in \mathfrak{O}(X)$. In fact, given $\Omega \in \mathfrak{OBC}(X)$, we have $R_n \cap \Omega \neq \emptyset$ for some $n \in \mathbb{N}$. Since Ω is open and R_n is connected, we must have $\partial\Omega \cap R_n \neq \emptyset$, since otherwise the set $\Omega \cap R_n$ would be both open and closed in R_n although $\emptyset \neq \Omega \cap R_n \neq R_n$. From $R_n \subseteq N(F)$ we see that $\theta \in F(\partial\Omega)$, and so (8.34) holds true for $\lambda = 0$. ♡

The following Proposition 8.5 is an interesting corollary of Proposition 8.4.

Proposition 8.5. *Let F_n converge to F uniformly on bounded sets, and $\lambda_n \in \phi_{\mathrm{p}}(F_n)$. If $F \in \mathfrak{K}(X)$, then $\phi_{\mathrm{p}}(F)$ contains all nonzero accumulation points of the sequence $(\lambda_n)_n$. If X has finite dimension, $\phi_{\mathrm{p}}(F)$ contains all accumulation points of the sequence $(\lambda_n)_n$.*

Proof. Assume there is some accumulation point $\lambda \neq 0$ (or also $\lambda = 0$ if X has finite dimension) with $\lambda \notin \phi_{\mathrm{p}}(F)$. By Proposition 8.4, we find some $\Omega \in \mathfrak{OBC}(X)$ with $\theta \notin (\lambda I - F)(\partial\Omega)$. Since $\lambda I - F$ is proper on $\overline{\Omega}$, by Proposition 3.2, and thus $(\lambda I - F)(\partial\Omega)$ is closed, by Theorem 3.1 (b), this relation is actually equivalent to $\mathrm{dist}(\theta, (\lambda I - F)(\partial\Omega)) > 0$. For some n, we thus have

$$\mathrm{dist}(\theta, (\lambda_n I - F_n)(\partial\Omega)) > \frac{1}{2}\,\mathrm{dist}(\theta, (\lambda I - F)(\partial\Omega)),$$

and so $\theta \notin (\lambda_n I - F_n)(\partial\Omega)$. But this means that $\lambda_n \notin \phi_{\mathrm{p}}(F_n)$, a contradiction. □

Note that the assumptions $\lambda \neq 0$ and $F \in \mathfrak{K}(X)$ are actually needed in the proof only to ensure the compactness of $N(\lambda I - F) \cap B_R(X)$ for any $R > 0$, and the closedness of $(\lambda I - F)(\partial\Omega)$. In particular, one may therefore replace these two assumption by requiring that $\lambda I - F$ be proper.

In Proposition 8.5 we may alternatively require that $F_n \to F$ with respect to the Väth topology, because this implies that $F_n \to F$ uniformly on bounded sets. However, in the connection of Proposition 8.5, the topology of uniform convergence on bounded sets is more natural, because the definition of $\phi_{\mathrm{p}}(F)$ is not related to any compactness assumptions.

Proposition 8.5 in turn implies the following result which exhibits a striking analogy to the linear case.

Theorem 8.8. *For $F \in \mathfrak{K}(X)$, the following is true:*

(a) *The equality*

$$\phi_{\mathrm{q}}(F) \cup \{0\} = \phi_{\mathrm{p}}(F) \cup \{0\} \tag{8.36}$$

holds.

(b) *If X is finite dimensional, then even*

$$\phi_{\mathrm{q}}(F) = \phi_{\mathrm{p}}(F). \tag{8.37}$$

(c) *The multivalued map* $\phi_p \cup \{0\}: \mathfrak{C}(X) \to 2^{\mathbb{K}}$ *which associates to each* F *the set* $\phi_p(F) \cup \{0\}$ *is upper semicontinuous with respect to the topology of uniform convergence on bounded sets.*

(d) *If* X *is finite dimensional, then the map* $\phi_p: \mathfrak{C}(X) \to 2^{\mathbb{K}}$ *is upper semicontinuous everywhere.*

Proof. The assertions (a) and (b) are immediate consequences of Proposition 8.5. Concerning (c), fix $F \in \mathfrak{C}(X)$. We claim that $\phi_p(G)$ is uniformly bounded for all $G \in \mathfrak{C}(X)$ in a neighborhood of F. Indeed, we find a constant $c > 0$, a set $\Omega \in \mathfrak{OBC}(X)$, and a neighborhood U of F such that $\|G(x)\| \leq c$ on $\overline{\Omega}$ for all $G \in U$. For $|\lambda| > c$, we thus have $\theta \notin (\lambda I - G)(\partial\Omega)$ which in view of Proposition 8.4 implies that $\lambda \notin \phi_p(G)$. This shows that $\phi_p(G)$ is bounded by c for all $G \in U$.

If $\phi_p \cup \{0\}$ is not upper semicontinuous at F, we find an open neighborhood V of $\phi_p(F) \cup \{0\}$, and sequences $(F_n)_n$ and $(\lambda_n)_n$ with $\lambda_n \in \phi_p(F_n) \setminus V$, $\lambda_n \to \lambda$, and $F_n \to F$. By what we have proved above, we may assume that $(\lambda_n)_n$ is bounded and thus contains some accumulation point λ in the closed set $\mathbb{K} \setminus V$. But then Proposition 8.5 implies $\lambda \in \phi_p(F)$, a contradiction. The proof of (d) is analogous. □

As remarked before, parts (c) and (d) imply that $\phi_p \cup \{0\}$ and ϕ_p are also upper semicontinuous at all points of $\mathfrak{K}(X)$ with respect to the Väth topology.

If we suppose, apart from compactness, that F is odd, we get the following very strong result.

Theorem 8.9. *If* $F \in \mathfrak{K}(X)$ *is odd and* X *is finite dimensional, then all phantoms coincide, i.e.,*

$$\phi_p(F) = \phi_q(F) = \phi(F) = \Phi(F). \tag{8.38}$$

If X *is infinite dimensional, these sets can differ at most by the point* 0.

Proof. In view of (8.26), it suffices to show that any $\lambda \in \Phi(F)$ belongs to $\phi_p(F)$, if either $\lambda \neq 0$ or X is finite dimensional. Assume that $\lambda \notin \phi_p(F)$. Proposition 8.4 implies that there is some $\Omega_0 \in \mathfrak{OBC}(X)$ with $\theta \notin (\lambda I - F)(\partial\Omega_0)$. Since $\lambda I - F$ is odd, we also have $\theta \notin (\lambda I - F)(\partial(-\Omega_0))$. For the symmetric set $\Omega := \Omega_0 \cup (-\Omega_0) \in \mathfrak{OBC}(X)$, we thus have $\theta \notin (\lambda I - F)(\partial\Omega)$. By Borsuk's theorem (see Section 3.5), the degree $\deg(I - F/\lambda, \Omega, \theta)$ (respectively, $\deg(\lambda I - F, \Omega, \theta)$ in finite dimensions) is odd, and so $\lambda I - F$ is epi on $\overline{\Omega}$, by Property 7.5 of epi operators. In view of Proposition 8.1, the map $\lambda I - F$ thus is V-regular, i.e., $\lambda \notin \Phi(F)$. □

Observe that Theorem 8.9 may be considered as an analogue for phantoms to the "discreteness" conditions for spectra given in Theorems 6.12 and 7.8. For the sake of completeness, we reformulate this as follows.

Theorem 8.10. *Let* $F: X \to X$ *be compact and odd. Then every* $\lambda \in \phi(F) \setminus \{0\} = \Phi(F) \setminus \{0\}$ *is a connected eigenvalue of* F.

There is another class of operators for which we get remarkably precise results, namely τ-homogeneous operators (see Section 7.4). It turns out that many spectra and phantoms *coincide* for such operators. This is surprising only at first glance; in fact, the homogeneity implies that "local", "asymptotic", and "global" properties are rather the same.

Theorem 8.11. *Let $F\colon X \to X$ be* 1*-homogeneous. Then we have*

$$\sigma_{\mathrm{AGV}}(F) = \phi(F) \subseteq \Phi(F) = \sigma_{\mathrm{FMV}}(F) = \sigma_{\mathrm{F}}(F). \tag{8.39}$$

Proof. Let us first prove the last two equalities in (8.39). We already know from Theorem 7.2 and Proposition 8.2 that

$$\Phi(F) \subseteq \sigma_{\mathrm{FMV}}(F) \subseteq \sigma_{\mathrm{F}}(F),$$

so we only have to show that every 1-homogeneous V-regular operator is F-regular. Thus, suppose that there exists $\Omega \in \mathfrak{OBC}(X)$ such that F is epi on $\overline{\Omega}$ and $[F|_{\overline{\Omega}}]_{\mathrm{a}} > 0$. Since F is 1-homogeneous, Proposition 7.2 (b) and (d) imply that F is epi on *every* $\Omega \in \mathfrak{OBC}(X)$, and $[F]_{\mathrm{a}} > 0$ on the whole space.

We still have to prove that $[F]_{\mathrm{b}} > 0$. From Proposition 8.2 we know that F is v-regular, so in particular

$$\mathrm{dist}(\theta, F(\partial\Omega)) = \inf_{x\in\partial\Omega} \|F(x)\| > 0.$$

Putting $s := \sup\{\|x\| : x \in \partial\Omega\}$ we have, again by homogeneity,

$$\inf_{x\neq\theta} \frac{\|F(x)\|}{\|x\|} = \inf_{\|x\|=1} \|F(x)\| \geq \frac{1}{s} \inf_{x\in\partial\Omega} \|F(x)\| > 0$$

which shows that $[F]_{\mathrm{b}} > 0$.

Now we prove the first equality in (8.39). We already know from (8.7) that

$$\phi(F) \subseteq \sigma_{\mathrm{AGV}}(F),$$

so we only have to show that every 1-homogeneous v-regular operator is AGV-regular. Again, suppose that there exists $\Omega \in \mathfrak{OBC}(X)$ such that F is k-epi on $\overline{\Omega}$ for some $k > 0$. As we have seen above, we have then $[F]_{\mathrm{q}} \geq [F]_{\mathrm{b}} > 0$.

It remains to show that $\mu(F) > 0$. Fix k' with

$$0 < k' < \min\{k, [F]_{\mathrm{b}}\},$$

and consider $G\colon \overline{\Omega} \to X$ with $[G|_{\overline{\Omega}}]_{\mathrm{A}} \leq k'$ and $[G]_{\mathrm{Q}} \leq k'$. The last estimate implies that we can find $R > 0$ such that $\|G(x)\| \leq k'\|x\|$ for $\|x\| \geq R$; in particular,

$$\sup_{\|x\|=R} \|G(x)\| \leq k'R < R[F]_{\mathrm{b}} \leq \inf_{\|x\|=R} \|F(x)\| = \mathrm{dist}(\theta, F(S_R(X))).$$

Moreover,

$$[G|_{B_R(X)}]_A = [G|_{\overline{\Omega}}]_A \le k' < k \le \nu_\Omega(F) = \nu_{B_R^o(X)}(F),$$

by our choice of k'. From Lemma 7.4 it follows that $F - G$ is strictly epi on $B_R(X)$, and so the equation $F(x) = G(x)$ has a solution in $B_R(X)$. This shows that F is k'-stably solvable, i.e., $\mu(F) > 0$ as claimed. □

The only example of an operator F whose AGV-spectrum is strictly smaller than its FMV-spectrum is (6.23) in Example 6.9. Of course, this operator is far from being 1-homogeneous. It would be interesting to have an example of a *homogeneous* operator F for which the inclusion in (8.39) is strict; we do not know any such example.

The following result shows that also the various point spectra and phantoms simplify for 1-homogeneous operators.

Theorem 8.12. *Let $F: X \to X$ be 1-homogeneous. Then we have*

$$\phi_p(F) = \sigma_p^0(F) = \sigma_p(F) \subseteq \phi_q(F) = \sigma_q(F). \tag{8.40}$$

Proof. In Theorem 8.6 we have proved the equality $\phi_q(L) = \sigma_q(L)$ for linear operators L. However, in the proof we have used only the 1-homogeneity of L, not the additivity, and so we see that the last equality in (8.40) is true.

We still have to show that $\sigma_p(F) \subseteq \phi_p(F)$ if F is 1-homogeneous. Given $\lambda \in \sigma_p(F)$, choose $x_0 \in X \setminus \{\theta\}$ such that $F(x_0) = \lambda x_0$. By homogeneity, we have then also $F(\mu x_0) = \lambda\mu x_0$ for every $\mu > 0$, which shows that $\lambda \in \phi_p(F)$, and so the left equalities in (8.40) are proved. □

In contrast to the inclusion in (8.39), it is easy to see that the inclusion in (8.40) may be strict even for linear operators, see (1.41) in Example 1.5 and Theorem 8.6. We illustrate Theorems 8.11 and 8.12 with the following

Example 8.6. Let X and F be defined as in Example 2.46. Clearly, F is compact and 1-homogeneous. A straightforward calculation shows that both $\sigma_{AGV}(F) = [0, 1]$ and $\sigma_F(F) = [0, 1]$. From Theorem 8.11 we conclude that

$$\sigma_{AGV}(F) = \phi(F) = \Phi(F) = \sigma_{FMV}(F) = \sigma_F(F) = [0, 1].$$

Moreover, another easy computation shows that the point spectra and phantoms are

$$\phi_p(F) = \sigma_p^0(F) = \sigma_p(F) = \phi_q(F) = \sigma_q(F) = \{\pm 1\}$$

in this example. ♡

8.5 A comparison of spectra and phantoms

In this section we want to take again a deep breath and summarize what we have done in this and the preceding two chapters. As we have seen, there are many different definitions of what we could call a "spectrum" for a continuous nonlinear operator. If we return for a moment to *linear* operators L, we may describe the spectrum as set of all scalars λ for which the operator $\lambda I - L$ does *not* have the following properties simultaneously:

(a) It is injective.

(a′) Its nullset is trivial.

(a″) Its nullset is bounded.

(b) It is surjective.

(b′) It maps some bounded neighborhood of θ onto a neighborhood of θ.

(c) Its inverse (resolvent operator) is continuous.

In the linear case, the conditions (a), (a′) and (a″) are mutually equivalent, and so are the conditions (b) and (b′). In fact, the equivalent conditions (a)/(a′)/(a″) lead to the point spectrum (1.21), the equivalent conditions (b)/(b′) to the defect spectrum (1.51), and the condition (c) to the continuous spectrum and the residual spectrum introduced at the beginning of Section 1.3. In the numerous generalizations of spectral theory to nonlinear operators F, we have chosen, loosely speaking, some or all of the above (or similar) properties, and have defined a spectrum to consist of all $\lambda \in \mathbb{K}$ for which the corresponding properties fail for $\lambda I - F$.

Of course, one should also require some natural properties for any nonlinear spectrum as a subset of the real line or complex plane. One essential property which is the thread running through the whole story is the *closedness* of any "reasonable" spectrum. In fact, it is the closedness of the spectrum which ensures that any of the properties in the above list is "stable" under suitable small perturbations.

Here it turns out that, if one considers a spectrum defined by properties like (a), (b) and (c) simultaneously, the spectrum is closed only under rather restrictive assumptions on F. Roughly speaking, to prove the closedness of such a spectrum, one has to apply *Banach's fixed point theorem*, and this in turn essentially restricts the class of admissible maps to Lipschitz continuous operators. As we have seen in Chapter 5 by means of the *Kachurovskij spectrum*, such a spectral theory leads to quite satisfactory results, but appears not very different from the linear case.

On the other hand, in Chapters 6–8 we have discussed other spectra which are more "accessible" to topological methods like, say, *Schauder's* or *Darbo's fixed point theorem*. This means, roughly speaking, that the various spectra covered in the last chapters deal mainly with properties similar to (b) or (b′); typically, some additional compactness conditions are involved here. The property (c) has been used only marginally, for instance, in the definition of the *Rhodius spectrum* in Chapter 4.

The most important spectra involving a condition of type (b) are of course the *FMV-spectrum* and its modification discussed in Section 6.6, the *AGV-spectrum*. Typical spectra involving a condition of type (b′) are the *Väth phantom* and the *large Väth phantom* introduced in this chapter. We think that phantoms provide a particular "stroke of luck" in nonlinear spectral theory, because they reflect very well the "local" character of nonlinear problems. As pointed out several times, only phantoms are "local", while the FMV-spectrum emphasizes "asymptotic" properties, and the Feng spectrum builds on "global" properties.

We start now with a more systematic comparison of the spectra and phantoms discussed in Chapters 6–8. First of all, let us compare the five regularity concepts introduced in these chapters.

Table 8.3

$$
\begin{array}{ccccc}
F \text{ F-regular} & \Longrightarrow & F \text{ FMV-regular} & \Longrightarrow & F \text{ AGV-regular} \\
 & & \Downarrow & & \Downarrow \\
 & & F \text{ V-regular} & \Longrightarrow & F \text{ v-regular}
\end{array}
$$

None of these implications can be inverted. For instance, Example 7.6 shows that FMV-regularity does not imply F-regularity, Example 6.9 shows that AGV-regularity does not imply FMV-regularity and v-regularity does not imply V-regularity, and Example 8.1 shows that v-regularity does not imply AGV-regularity and V-regularity does not imply FMV-regularity.

Of course, in case of a bounded linear operator L, all regularity notions coincide, as we have proved in Theorems 6.1, 6.10, 7.3, and 8.5. For the sake of completeness, we summarize this with the following table.

Table 8.4

$$
\begin{array}{ccccc}
L \text{ F-regular} & \Longleftrightarrow & L \text{ FMV-regular} & \Longleftrightarrow & L \text{ AGV-regular} \\
\Updownarrow & & \Updownarrow & & \Updownarrow \\
L \text{ isomorphism} & \Longleftrightarrow & L \text{ V-regular} & \Longleftrightarrow & L \text{ v-regular}
\end{array}
$$

In the last chapters we have proved various "Rouché type" perturbation theorems; such theorems ensure the closedness of spectra and phantoms. We collect them in the following table.

Table 8.5

If F is	and G satisfies	then $F+G$ is also
FMV-regular	$[G]_{\mathrm{A}} < [F]_{\mathrm{a}}$ $[G]_{\mathrm{Q}} < [F]_{\mathrm{q}}$	FMV-regular
AGV-regular	$[G]_{\mathrm{A}} < \mu(F)$ $[G]_{\mathrm{Q}} < [F]_{\mathrm{q}}$	AGV-regular
F-regular	$[G]_{\mathrm{A}} < \nu(F)$ $[G]_{\mathrm{B}} < [F]_{\mathrm{b}}$	F-regular
v-regular	$[G\vert_{\overline{\Omega}}]_{\mathrm{A}} < \nu_{\Omega}(F)$ $\sup_{x\in\partial\Omega} \Vert G(x)\Vert < \inf_{x\in\partial\Omega} \Vert F(x)\Vert$	v-regular
V-regular	$[G\vert_{\overline{\Omega}}]_{\mathrm{A}} < [F\vert_{\overline{\Omega}}]_{\mathrm{a}}$ $\Vert G(x)\Vert < \Vert F(x)\Vert$ on $\partial\Omega$	V-regular

As we have seen, each construction of nonlinear spectral theory requires a parallel construction of a corresponding eigenvalue theory. We give a schematic overview of this correspondence in the following table.

Table 8.6

FMV-spectrum (6.10)	$\longleftrightarrow$	asymptotic point spectrum (2.29)
AGV-spectrum (6.27)	$\longleftrightarrow$	unbounded point spectrum (6.36)
Feng spectrum (7.19)	$\longleftrightarrow$	classical point spectrum (3.18)
Väth phantom (8.5)	$\longleftrightarrow$	point phantom (8.18)
large Väth phantom (8.6)	$\longleftrightarrow$	point phantom (8.18)

As we have seen, the correspondence between spectra and eigenvalues also includes, under suitable hypotheses, certain "discreteness conditions" for the spectrum of a *compact* operator. We summarize with the following table.

Table 8.7

hypotheses on F	discreteness condition	see
compact, linear	$\sigma(L) \setminus \{0\} \subseteq \sigma_{\mathrm{p}}(L)$	Theorem 1.2
compact, asymptotically odd	$\sigma_{\mathrm{FMV}}(F) \setminus \{0\} \subseteq \sigma_{\mathrm{q}}(F)$	Theorem 6.12
compact, odd, 1-homogeneous	$\sigma_{\mathrm{F}}(F) \setminus \{0\} \subseteq \sigma_{\mathrm{p}}(F)$	Theorem 7.8
compact, odd	$\phi(F) \setminus \{0\} \subseteq \phi_{\mathrm{p}}(F)$	Theorem 8.10

We also give a scheme, for the reader's ease, of all possible inclusions between these spectra and phantoms in the following table which essentially completes Table 7.2.

Table 8.8

$$
\begin{array}{ccccccccc}
\sigma_{\mathrm{p}}(F) & & \Phi(F) & & & & \Phi(F) & & \\
\rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & & & & \rotatebox{-90}{$\subseteq$} & & \\
\sigma_{\mathrm{p}}^{0}(F) & & \phi(F) & \subseteq & \sigma_{\mathrm{AGV}}(F) & \subseteq & \sigma_{\mathrm{FMV}}(F) & \subseteq & \sigma_{\mathrm{F}}(F) \\
\rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & & & & \rotatebox{90}{$\subseteq$} \\
\phi_{\mathrm{p}}(F) & \subseteq & \phi_{\mathrm{q}}(F) & \subseteq & \sigma_{\mathrm{q}}(F) & & & & \sigma_{\mathrm{p}}(F)
\end{array}
$$

Of course, Table 8.8 drastically simplifies in case of a 1-homogeneous operator F, see Theorems 8.11 and 8.12.

Table 8.9

$$
\begin{array}{ccccccccc}
\sigma_{\mathrm{p}}(F) & & \Phi(F) & & & & \Phi(F) & & \\
\| & & \rotatebox{90}{$\subseteq$} & & & & \| & & \\
\sigma_{\mathrm{p}}^{0}(F) & \subseteq & \phi(F) & = & \sigma_{\mathrm{AGV}}(F) & \subseteq & \sigma_{\mathrm{FMV}}(F) & = & \sigma_{\mathrm{F}}(F) \\
\| & & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & & & & \rotatebox{90}{$\subseteq$} \\
\phi_{\mathrm{p}}(F) & \subseteq & \phi_{\mathrm{q}}(F) & = & \sigma_{\mathrm{q}}(F) & & & & \sigma_{\mathrm{p}}(F)
\end{array}
$$

If F is not only 1-homogeneous, but even linear, this table further simplifies; in fact, only three different sets may occur in the linear case.

Table 8.10

$$
\begin{array}{ccccccccc}
\sigma_{\mathrm{p}}(L) & & \Phi(L) & & & & \sigma(L) & & \\
\| & & \| & & & & \| & & \\
\sigma_{\mathrm{p}}^{0}(L) & \subseteq & \phi(L) & = & \sigma_{\mathrm{AGV}}(L) & = & \sigma_{\mathrm{FMV}}(L) & = & \sigma_{\mathrm{F}}(L) \\
\| & & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & & & & \rotatebox{90}{$\subseteq$} \\
\phi_{\mathrm{p}}(L) & \subseteq & \phi_{\mathrm{q}}(L) & = & \sigma_{\mathrm{q}}(L) & & & & \sigma_{\mathrm{p}}(L)
\end{array}
$$

We illustrate Table 8.8 with the two scalar examples which we already considered before. First, for the “seagull” (Example 6.6) the sets occurring in Table 8.8 are as follows.

Table 8.11

$\mathbb{R} \setminus \{0\}$		$\{0\}$				$\{0\}$		
$\cup$		$\parallel$				$\parallel$		
$\emptyset$		$\{0\}$	$=$	$\{0\}$	$=$	$\{0\}$	$\subset$	$\mathbb{R}$
$\parallel$		$\cup$		$\cup$				$\cup$
$\emptyset$	$=$	$\emptyset$	$\subset$	$\{0\}$				$\mathbb{R} \setminus \{0\}$

Second, for the “sawtooth” (Example 8.1) we get the following picture.

Table 8.12

$[0, 1]$		$\emptyset$				$\emptyset$		
$\cup$		$\parallel$				$\cap$		
$\{0\}$		$\emptyset$	$\subset$	$\{0\}$	$=$	$\{0\}$	$\subset$	$[0, 1]$
$\cup$		$\parallel$		$\parallel$				$\parallel$
$\emptyset$	$=$	$\emptyset$	$\subset$	$\{0\}$				$[0, 1]$

Finally, let us discuss our favoured “infinite dimensional” example which we already exploited several times.

Example 8.7. Consider again the operator

$$F(x) = \|x\|x$$

in an arbitrary infinite dimensional Banach space X. We already know that F is an isomorphism with $[F]_{\mathrm{a}} = 0$ (Example 2.33), that F is stably solvable but not strictly stably solvable (Example 6.4), and that F is epi on $B(X)$ but not k-epi for any $k > 0$ (Example 7.5).

Let us first calculate the phantoms for F. For $\lambda \neq 0$ and $\Omega := B^{o}_{|\lambda|/2}(X)$, the restriction $(\lambda I - F)|_{\overline{\Omega}}$ is open and injective with $\theta \in (\lambda I - F)(\overline{\Omega})$ and $[(\lambda I - F)_{\overline{\Omega}}]_{\mathrm{a}} > 0$; so $\lambda I - F$ is properly epi on $\overline{\Omega}$. From Theorem 8.7 we conclude that

$$\phi_{\mathrm{p}}(F) = \phi_{\mathrm{q}}(F) = \emptyset \tag{8.41}$$

and

$$\Phi(F) = \phi(F) = \{0\}. \tag{8.42}$$

The eigenvalue equation $F(x) = \lambda x$ has a nontrivial solution x for $\lambda = \|x\| > 0$, and hence

$$\sigma_p(F) = (0, \infty). \tag{8.43}$$

For the same reason we have $\sigma_b(F) = [0, \infty)$ but

$$\sigma_p^0(F) = \sigma_q(F) = \emptyset, \tag{8.44}$$

i.e., F has neither unbounded nor asymptotic eigenvalues. For $\lambda > 0$ we have $(\lambda I - F)(S_\lambda(X)) = \{\theta\}$ and thus $[\lambda I - F]_a = 0$. This together with $[F]_a = 0$ gives the inclusion $[0, \infty) \subseteq \sigma_a(F)$. Now let $\lambda < 0$, and consider the scalar function

$$f(t) := \frac{\lambda + \sqrt{\lambda^2 + 4t}}{2t} \quad (0 < t < \infty).$$

L'Hospital's rule shows that f admits a continuous extension to 0 by putting $f(0) := 1/|\lambda|$. A straightforward computation shows that

$$(\lambda I - F)^{-1}(y) = f(\|y\|)y$$

for every $y \in X$; moreover, we have $|f(t)| \leq M_0$ for some $M_0 > 0$ and all $t \geq 0$.

Now, the derivative

$$f'(t) = -\frac{2t + \lambda\sqrt{\lambda^2 + 4t} + \lambda^2}{2t^2\sqrt{\lambda^2 + 4t}} \quad (0 < t < \infty)$$

has the property that $t|f'(t)|$ remains bounded as $t \to 0$ or $t \to \infty$. For $\|x\| \leq \|y\|$ we have

$$\begin{aligned} \|f(\|x\|)x - f(\|y\|)y\| &= \|[f(\|x\|) - f(\|y\|)]x + f(\|y\|)(x - y)\| \\ &\leq |f'(\tau)(\|x\| - \|y\|)|\,\|x\| + M_0\|x - y\|, \end{aligned}$$

where $\|x\| \leq \tau \leq \|y\|$. Since the map $t \mapsto t|f'(t)|$ is bounded, say $t|f'(t)| \leq M_1$, we conclude that

$$\|(\lambda I - F)^{-1}(x) - (\lambda I - F)^{-1}(y)\| \leq (M_0 + M_1)\|x - y\|.$$

Interchanging the role of x and y we get the same estimates for $\|y\| \leq \|x\|$, and so we have proved that $(\lambda I - F)^{-1}$ is Lipschitz continuous for $\lambda < 0$. In particular, $\lambda I - F$ is a homeomorphism on X with $[\lambda I - F]_a > 0$. We conclude that $\sigma_a(F) = [0, \infty)$.

Now we calculate the remaining spectra for this operator. Suppose again that $\lambda < 0$ and that $G\colon X \to X$ is compact with quasinorm $[G]_Q = 0$; in particular, G maps some closed ball $B_R(X)$ into itself. Since $\|(\lambda I - F)(x)\| \geq \|x\|$, the (compact!) operator $(\lambda I - F)^{-1}G$ also maps the ball $B_R(X)$ into itself, and so has a fixed point

$\hat{x} \in B_R(X)$. But then $\lambda\hat{x} - F(\hat{x}) = G(\hat{x})$ which shows that $\lambda I - F$ is stably solvable. It follows that

$$0 \in \sigma_{\mathrm{AGV}}(F) \subseteq \sigma_{\mathrm{FMV}}(F) = [0, \infty). \tag{8.45}$$

Finally, fix $\Omega \in \mathfrak{OBC}(X)$, and let $G\colon \overline{\Omega} \to X$ be compact with $G(x) \equiv \theta$ on $\partial\Omega$. For $\lambda < 0$, the operator $(\lambda I - F)^{-1}G\colon \overline{\Omega} \to (\lambda I - F)^{-1}G(\overline{\Omega})$ is a compact operator which vanishes on the boundary $\partial\Omega$, and so has a fixed point in Ω. We conclude that $\lambda I - F$ is epi for such λ, and hence

$$\sigma_{\mathrm{F}}(F) = [0, \infty), \tag{8.46}$$

i.e., the FMV-spectrum and the Feng spectrum coincide for this operator. ♡

Unfortunately, we cannot say more than $0 \in \sigma_{\mathrm{AGV}}(F)$ about the AGV-spectrum of F in this example. The point is that we do not know whether or not $\lambda I - F$ is epi on any $\Omega \in \mathfrak{OBC}(X)$, let alone stably solvable on X, for $\lambda > 0$. So we get the following somewhat incomplete picture for this operator.

Table 8.13

$$\begin{array}{ccccccccc}
(0,\infty) & & \{0\} & & & & \{0\} & & \\
\cup & & \| & & & & \cap & & \\
\emptyset & & \{0\} & \subseteq & ??? & \subseteq & [0,\infty) & = & [0,\infty) \\
\| & & \cup & & \cup & & & & \cup \\
\emptyset & = & \emptyset & = & \emptyset & & & & (0,\infty)
\end{array}$$

8.6 Notes, remarks and references

As the title suggests, almost everything in this chapter is due to Väth ([259]–[262], see also [232], [233]). In particular, the short survey [261] provides an interesting comparison between the "asymptotic" Furi–Martelli–Vignoli spectrum and the "local" phantom.

Strictly epi operators have been introduced by Väth in [261] under the name *stably zero-epi*, properly epi operators under the name *stably* zero-epi*. All results of Section 8.1. may be found in [261]; the proof of the fundamental coincidence theorem used in Theorem 8.1 (see Theorem 7.1) is contained in the paper [262].

The phantom $\phi(F)$, the large phantom $\Phi(F)$, the point phantom $\phi_{\mathrm{p}}(F)$, and the approximate point phantom $\phi_{\mathrm{q}}(F)$ are discussed in [232], [233], [261], [262]. In particular, we mention that the paper [232] contains many alternative definitions of the term "eigenvalue" for nonlinear operators; we shall come back to this in Chapter 10. Väth's definition of the point phantom and the approximate point phantom seems to be quite natural, as may be seen, for instance, from Proposition 8.4 and Theorem 8.8. On the other hand, it is very subtle: even if one replaces "connected" in definition (8.18) by "pathwise connected", the results stated there are no longer true.

The discreteness result given in Theorem 8.10 also strongly indicates that (8.18) is the appropriate definition of "eigenvalue" for phantoms. We remark that all discreteness conditions in Table 8.7 admit a natural extension to operators F with $[F]_{\mathrm{A}} < \infty$. In the particular case of a bounded linear operator we have already seen in Theorem 1.3 (c) that every $\lambda \in \sigma(L)$ with $|\lambda| > [L]_{\mathrm{A}}$ is an eigenvalue. Similarly, for the other spectra we get the following generalization of Table 8.7 which contains the statements of Theorems 6.14 and 7.8.

Table 8.14

hypotheses on F	discreteness condition
linear	$\lambda \in \sigma(L),\ \lvert\lambda\rvert > [L]_{\mathrm{A}} \ \Rightarrow\ \lambda \in \sigma_{\mathrm{p}}(L)$
$F \in \mathfrak{A}(X)$, asymptotically odd	$\lambda \in \sigma_{\mathrm{FMV}}(F),\ \lvert\lambda\rvert > [F]_{\mathrm{A}} \ \Rightarrow\ \lambda \in \sigma_{\mathrm{q}}(F)$
$F \in \mathfrak{A}(X)$, odd, 1-homogeneous	$\lambda \in \sigma_{\mathrm{F}}(F),\ \lvert\lambda\rvert > [F]_{\mathrm{A}} \ \Rightarrow\ \lambda \in \sigma_{\mathrm{p}}(F)$
$F \in \mathfrak{A}(X)$, odd	$\lambda \in \phi(F),\ \lvert\lambda\rvert > [F]_{\mathrm{A}} \ \Rightarrow\ \lambda \in \phi_{\mathrm{p}}(F)$

We point out that Väth's original definition of phantoms and point phantoms is much more general than that discussed in this chapter. In fact, Väth throughout considers operators F also between *different* Banach spaces X and Y, and replaces $\lambda I - F$ by $\lambda J - F$, where $J : X \to Y$ is some fixed "well-behaved" operator. For example, in Chapter 12 we will use this more general approach for X and Y being two Sobolev spaces and J being the so-called *p-Laplace operator* which is a $(p-1)$-homogeneous isomorphism between these Sobolev spaces. In this case a natural choice for F is the Nemytskij operator (4.21) generated by the nonlinearity $f(u) = |u|^{p-2}u$.

The general definition of the various phantoms in [233] may also depend on an additional closed, bounded, connected set $K \subset X$ containing θ. So, $F : X \to Y$ is called *v-regular* (in our terminology) *with respect to* K if there exists some $\Omega \in \mathfrak{OBC}(X)$ containing K such that F is strictly epi on $\overline{\Omega}$, and the phantom $\phi(F; K)$ contains all scalars λ with the property that $\lambda J - F$ is not v-regular with respect to K. Likewise, the point phantom $\phi_{\mathrm{p}}(F; K)$ with respect to K is defined as the set of all $\lambda \in \mathbb{K}$ such that the nullset $N(\lambda J - F)$ contains an unbounded connected set *which meets* K. Of course, all definitions and results considered in this chapter refer to the special choice $Y = X$, $J = I$ and $K = \{\theta\}$. However, in some cases it may be useful to admit other choices of K, for example, $K = B(X)$.

Chapter 9

Other Spectra

In this chapter we discuss some other spectra which have been recently introduced in the literature but are somewhat beyond the scope of the spectral theory considered in the preceding chapters. First, we consider spectra for pairs (L, F) of a linear Fredholm operator L of index zero and a continuous nonlinear operator F. These spectra generalize the Furi–Martelli–Vignoli spectrum and the Feng spectrum, and reduce to them in case $L = I$. Moreover, we describe a spectrum which is based on the definition of some kind of linear adjoint to a Lipschitz continuous nonlinear operator. Afterwards, we present a spectrum which was introduced by Singhof in the seventieth and extended by Weyer to multivalued maps, and yet another spectrum due to Weber which describes, similarly as the FMV-spectrum, the asymptotic properties of a continuous nonlinear operator. In the following section we discuss a modification of the spectra and phantoms considered in Chapters 6–8 for homogeneous operators; an application will be given in the last chapter. Finally, we describe yet another spectrum which was introduced quite recently by Infante and Webb and is based on the notion of A-proper and related operators.

9.1 The semilinear Feng spectrum

The aim of this section is to extend the theory of the Feng spectrum discussed in Chapter 7 to a semilinear operator pair (L, F), where L is a linear Fredholm operator of index 0, and F is a continuous nonlinear operator. Such a situation arises frequently in applications to boundary value problems for both ordinary and partial differential equations. The spectrum of the semilinear operator pair (L, F), denoted by $\sigma_F(L, F)$, will be modelled on the Feng spectrum in such a way that for $L = I$ we get the usual Feng spectrum $\sigma_F(F)$ defined by (7.19).

Throughout we will suppose that the following hypotheses are satisfied. Let X and Y be two Banach spaces and $L: D(L) \to Y$, with $\overline{D(L)} = X$, a closed linear Fredholm operator of index zero. We write $X = N(L) \oplus X_0$ and $Y = Y_0 \oplus R(L)$, and denote by $P: X \to N(L)$ and $Q: Y \to Y_0$ the corresponding projections. Also let L_P denote the invertible operator L restricted to $D(L) \cap X_0$ into $R(L)$. Finally, we write $K_{PQ} := L_P^{-1}(I - Q): Y \to X_0$, let $\Pi: Y \to Y/R(L)$ be the quotient map, and denote by $\Lambda: Y/R(L) \to N(L)$ the natural linear isomorphism induced by L.

With this notation, we call an operator $F: X \to Y$ (L, α)-Lipschitz (and write $F \in \mathfrak{A}_L(X, Y)$ if $[K_{PQ}F]_A < \infty$, i.e. if $K_{PQ}F \in \mathfrak{A}(X)$. In particular, F is said to be L-*compact* if $[K_{PQ}F]_A = 0$, and (L, α)-*contractive* if $[K_{PQ}F]_A < 1$. Observe that we did not suppose L to be bounded or invertible. Of course, if $D(L) = X$ and $L: X \to Y$ is a bijection, then $X_0 = X$, $Y_0 = \{\theta\}$, $P = Q = \Theta$, $L_P = L$, and $K_{PQ} = L^{-1}$, and so F is (L, α)-Lipschitz if $[L^{-1}F]_A < \infty$, L-compact if $[L^{-1}F]_A = 0$, and (L, α)-contractive if $[L^{-1}F]_A < 1$.

Now we are going to generalize k-epi operators. Throughout this section, by $\mathfrak{OBC}_L(X)$ we denote the family of all sets $\Omega \in \mathfrak{OBC}(X)$ with the additional property that $\Omega_L := \Omega \cap D(L) \neq \emptyset$. Let us call $F: \overline{\Omega} \to Y$ an (L, k)-*epi operator* on $\overline{\Omega}_L$ if $F(x) \neq Lx$ on $\partial\Omega_L$ and, for any operator $G: \overline{\Omega} \to Y$ satisfying $[K_{PQ}G]_A \leq k$ and $G(x) \equiv \theta$ on $\partial\Omega_L$, the equation

$$Lx - F(x) = G(x)$$

has a solution $x \in \Omega_L$. In case $k = 0$ we call F simply an L-*epi operator*. Clearly, for the identity operator $L = I$ we get the definition of epi and k-epi operators (for $I - F$ instead of F) given at the beginning of Chapter 7. Moreover, the following three properties are analogous to those for k-epi operators stated in Section 7.1 and are proved in the same way.

Property 9.1 (Existence). *Suppose that F is L-epi on $\overline{\Omega}_L$. Then the equation $Lx = F(x)$ has a solution in Ω_L.*

Property 9.2 (Normalization). *Suppose that L is invertible, and $F: \overline{\Omega}_L \to Y$ is (L, α)-contractive with $F(x) \neq Lx$ on $\partial\Omega_L$. Then F is (L, k)-epi for every $k < 1 - [K_{PQ}F]_A$.*

Property 9.3 (Localization)). *Let $F: \overline{\Omega} \to Y$ be (L, k)-epi on $\overline{\Omega}_L$ and let $(L - F)^{-1}(\theta) \subseteq \Omega'_L$ for some $\Omega' \in \mathfrak{OBC}_L(X)$. Then F is also (L, k)-epi on $\overline{\Omega}'_L$.*

Property 9.4 (Homotopy). *Suppose that $F_0: \overline{\Omega} \to Y$ is (L, k_0)-epi on $\overline{\Omega}_L$, and $H: \overline{\Omega} \times [0, 1] \to Y$ is continuous with $H(x, 0) \equiv \theta$ and*

$$\alpha(H(M \times [0, 1]) \leq k\alpha(M) \quad (M \subseteq \overline{\Omega})$$

for some $k \leq k_0$. Let

$$S = \{x \in \overline{\Omega} : Lx - F_0(x) + H(x, t) = \theta \text{ for some } t \in [0, 1]\}.$$

If $S \cap \partial\Omega_L = \emptyset$ then the operator $F_1 := F_0 + H(\cdot, 1)$ is (L, k_1)-epi on $\overline{\Omega}_L$ for $k_1 \leq k_0 - k$.

Now, for fixed $\lambda \in \mathbb{K}$ we associate with (L, F) a new operator $\Phi_\lambda(L, F): X \to X$ defined by

$$\Phi_\lambda(L, F)(x) := \lambda(I - P)x - (\Lambda\Pi + K_{PQ})F(x). \tag{9.1}$$

For further reference, we mention the trivial, though useful identity

$$\Phi_\lambda(L, F) - \Phi_\mu(L, F) = (\lambda - \mu)(I - P) \quad (\lambda, \mu \in \mathbb{K}). \tag{9.2}$$

Moreover, the following lemma will be useful in the sequel.

Lemma 9.1. *Let $h\colon Y/R(L) \to Y_0$ be the natural linear isomorphism between the finite dimensional spaces $Y/R(L)$ and Y_0. Then $L + h\Lambda^{-1}P\colon D(L) \to Y$ is a linear isomorphism with inverse*

$$(L + h\Lambda^{-1}P)^{-1} = \Lambda\Pi + K_{PQ}. \tag{9.3}$$

Proof. Obviously, $h\Lambda^{-1}\colon N(L) \to Y_0$ is an isomorphism. Fix $x \in D(L)$ with $(L + h\Lambda^{-1}P)x = \theta$. Then $Lx = -h\Lambda^{-1}Px \in Y_0$, so $h\Lambda^{-1}Px = Lx = \theta$, $x \in N(L)$, and $h\Lambda^{-1}Px = h\Lambda^{-1}x = \theta$. This implies that $x = \theta$, which shows that $L + h\Lambda^{-1}P$ is invertible on $D(L)$.

Now, let $y \in Y$ and suppose that $(\Lambda\Pi + K_{PQ})y = \theta$. Then

$$\Lambda\Pi y = -L_P^{-1}(I - Q)y \in D(L) \cap X_0.$$

Consequently, $\Lambda\Pi y = \theta$ and $(I - Q)y = \theta$. Thus $y \in R(L) \cap Y_0$, so that $y = \theta$. We conclude that $\Lambda\Pi + K_{PQ}$ is one-to-one. For every $y \in Y$ we have

$$(L+h\Lambda^{-1}P)(\Lambda\Pi+K_{PQ})y = h\Lambda^{-1}P\Lambda\Pi y+(I-Q)y = h\Lambda^{-1}\Pi y-Qy+y = y.$$

Hence $L + h\Lambda^{-1}P$ is onto. Also, for every $x \in D(L)$ we have

$$(\Lambda\Pi + K_{PQ})(L + h\Lambda^{-1}P)x = (I - P)x + \Lambda h^{-1}\Pi h\Lambda^{-1}Px = x.$$

Hence $\Lambda\Pi + K_{PQ}$ is the (bounded) inverse of $L + h\Lambda^{-1}P$, and vice versa, as claimed. □

To illustrate Lemma 9.1 we give a simple example which comes from the theory of periodic boundary value problems for ordinary differential equations.

Example 9.1. For fixed $\omega > 0$, denote by $C_\omega = C_\omega(\mathbb{R})$ the space of all continuous ω-periodic functions $x\colon \mathbb{R} \to \mathbb{R}^n$ with the natural norm

$$\|x\|_{C_\omega} = \max_{0\le t\le\omega} |x(t)|.$$

Likewise, we write $C^1_\omega = C^1_\omega(\mathbb{R})$ for the space of all continuously differentiable ω-periodic functions $x\colon \mathbb{R} \to \mathbb{R}^n$ with norm

$$\|x\|_{C^1_\omega} = \max_{0\le t\le\omega} |x(t)| + \max_{0\le t\le\omega} |x'(t)|.$$

Moreover, we write $\hat{C}_\omega = \hat{C}_\omega(\mathbb{R})$ [respectively, $\hat{C}^1_\omega = \hat{C}^1_\omega(\mathbb{R})$] for the subspace of all $x \in C_\omega$ [respectively, C^1_ω] satisfying

$$Px := \frac{1}{\omega}\int_0^\omega x(t)\,dt = 0. \tag{9.4}$$

The operator P defined by (9.4) is a continuous projection which maps C_ω onto $\mathbb{R}^n$ and so induces decompositions $C_\omega = \hat{C}_\omega \oplus \mathbb{R}^n$ and $C^1_\omega = \hat{C}^1_\omega \oplus \mathbb{R}^n$.

Now, let $X = C^1_\omega$, $Y = C_\omega$, and define $L\colon X \to Y$ by $Lx = x'$. In the above notation, we have then $D(L) = X$, $N(L) = Y_0 = \mathbb{R}^n$, $R(L) = \hat{C}_\omega$, and $X_0 = \hat{C}^1_\omega$. So $L\colon X \to Y$ is a Fredholm operator with $\dim N(L) = \operatorname{codim} R(L) = n$, i.e. of index zero. The projection $P\colon C^1_\omega \to \mathbb{R}^n$ is given by (9.4), the projection $Q\colon C_\omega \to \mathbb{R}^n$ by the same formula. So the operator $L_P x = x'$ is a bijection between $\hat{C}^1_\omega$ and $\hat{C}_\omega$, the canonical quotient map Π associates to each $y \in C_\omega$ the class of all functions in C_ω with the same integral mean, and the canonical isomorphism Λ maps every such class onto this common integral mean.

The linear isomorphism $\Lambda\Pi + K_{PQ}\colon C_\omega \to C^1_\omega$ occurring in Lemma 9.1 is here

$$(\Lambda\Pi + K_{PQ})y(t) = \int_0^t y(s)\,ds - \frac{t}{\omega}\int_0^\omega y(s)\,ds + \frac{1}{\omega}\int_0^\omega \left(1 - \frac{\omega}{2} + s\right) y(s)\,ds,$$

its inverse $L + h\Lambda^{-1}P\colon C^1_\omega \to C_\omega$ is

$$(L + h\Lambda^{-1}P)x(t) = x'(t) + \frac{1}{\omega}\int_0^\omega x(s)\,ds.$$

The nonlinear operator F occurring in equation (9.1) is very often a Nemytskij operator of the form

$$F(x)(t) = f(t, x(t))$$

generated by some Carathéodory function $f\colon [0,\omega] \times \mathbb{R}^n \to \mathbb{R}^n$. For this operator it is possible to calculate the characteristics we are going to study below explicitly.

Finally, the operator $\Phi_\lambda(L, F)\colon C_\omega \to C_\omega$ has, for any $\lambda \in \mathbb{R}$, the form

$$\begin{aligned}\Phi_\lambda(L, F)(x)(t) = \lambda x(t) - \frac{\lambda}{\omega}\int_0^\omega x(s)\,ds - \int_0^t F(x)(s)\,ds \\ + \frac{t}{\omega}\int_0^\omega F(x)(s)\,ds - \frac{1}{\omega}\int_0^\omega \left(1 - \frac{\omega}{2} + s\right) F(x)(s)\,ds.\end{aligned}$$

So, already for the harmless differential operator $Lx = x'$, the auxiliary operator $\Phi_\lambda(L, F)$ may be quite complicated. ♡

Lemma 9.1 shows, roughly speaking, that the operator equation

$$\Phi_\lambda(L, F)(x) = y \tag{9.5}$$

is equivalent to the operator equation

$$\lambda Lx - F(x) = z, \tag{9.6}$$

where the equivalence is established by the transform $z = (L + h\Lambda^{-1}P)y$. Observe that (9.5) is, in contrast to (9.6), an operator equation in one and the same space; this fact will allow us to define a spectrum for the pair (L, F) through the auxiliary operator (9.1).

Actually, the analogy between the operators $\Phi_\lambda(L, F)$ and $\lambda L - F$ goes much further. We collect some natural relations between these operators in the following lemma.

Lemma 9.2. *Let $\Phi_\lambda(L, F)$ be defined as in* (9.1). *Then the following equivalences are true:*

(a) $\lambda L - F: X \to Y$ is surjective/injective/bijective if and only if $\Phi_\lambda(L, F)$: $X \to X$ is surjective/injective/bijective.

(b) $[\Phi_\lambda(L, F)]_{\mathrm{a}} > 0$ *implies* $[\lambda L - F]_{\mathrm{a}} > 0$*; the converse is true if L is bounded.*

(c) $[\Phi_\lambda(L, F)]_{\mathrm{b}} > 0$ *implies* $[\lambda L - F]_{\mathrm{b}} > 0$*; the converse is true if L is bounded.*

(d) $[\Phi_\lambda(L, F)]_{\mathrm{q}} > 0$ *implies* $[\lambda L - F]_{\mathrm{q}} > 0$*; the converse is true if L is bounded.*

(e) $\nu(\Phi_\lambda(L, F)) > 0$ *implies* $\nu(\lambda L - F) > 0$*; the converse is true if L is bounded.*

(f) $\lambda L - F$ *is stably solvable if* $\Phi_\lambda(L, F)$ *is stably solvable; the converse is true if L is bounded.*

Proof. The relation (a) is an immediate consequence of the above mentioned equivalence of the operator equations (9.5) and (9.6).

To prove (b), observe first that

$$[\Phi_\lambda(L, F)]_{\mathrm{a}} = [L_P^{-1}(\lambda L) - L_P^{-1}(I - Q)F]_{\mathrm{a}},$$

and so

$$[L_P^{-1}]_{\mathrm{a}}[\lambda L - F]_{\mathrm{a}} \le [\Phi_\lambda(L, F)]_{\mathrm{a}} \le [L_P^{-1}]_{\mathrm{A}}[\lambda L - F]_{\mathrm{a}}.$$

Since $[L_P^{-1}]_{\mathrm{A}} \le \|L_P^{-1}\| < \infty$, the positivity of $[\Phi_\lambda(L, F)]_{\mathrm{a}}$ implies the positivity of $[\lambda L - F]_{\mathrm{a}}$ which is the first part of (b). Conversely, if L is bounded, then $D(L) = X$ and $[L_P^{-1}]_{\mathrm{a}} > 0$, and so the positivity of $[\lambda L - F]_{\mathrm{a}}$ implies the positivity of $[\Phi_\lambda(L, F)]_{\mathrm{a}}$.

Next, suppose that $[\Phi_\lambda(L, F)]_{\mathrm{b}} > 0$, i.e.

$$\begin{aligned}\|\Phi_\lambda(L, F)(x)\| &= \|(L + h\Lambda^{-1}P)^{-1}[(\lambda L + \lambda h\Lambda^{-1}P)(I - P) - F](x)\| \\ &= \|(L + h\Lambda^{-1}P)^{-1}(\lambda L - F)(x)\| \\ &\ge [\Phi_\lambda(L, F)]_{\mathrm{b}}\|x\|,\end{aligned}$$

since $(L + h\Lambda^{-1}P)(I - P) = L$. But the linear operator $(L + h\Lambda^{-1}P)^{-1}$ is bounded, and so

$$\|(\lambda L - F)(x)\| \ge \frac{[\Phi_\lambda(L, F)]_{\mathrm{b}}}{\|(L + h\Lambda^{-1}P)^{-1}\|}\|x\|,$$

which proves the first part of (c). Conversely, if L is bounded, then $L + h\Lambda^{-1}P$ is bounded as well, and

$$\|L + h\Lambda^{-1}P\| \, \|\Phi_\lambda(L, F)(x)\| \le \|(\lambda L - F)(x)\| \le [\lambda L - F]_{\mathrm{b}}\|x\|.$$

The statement (d) is proved in the same way as (c).

Now suppose that $\nu(\Phi_\lambda(L, F)) > 0$. By definition, there exists $k > 0$ such that $\Phi_\lambda(L, F)$ is k-epi on $\overline{\Omega}$ for every $\Omega \in \mathfrak{O}(X)$. Let $G\colon \overline{\Omega} \to Y$ be continuous with $G|_{\partial\Omega} = \Theta$ and $[K_{PQ}G]_{\mathrm{A}} \le k$, and consider the operator $(\Lambda\Pi + K_{PQ})G\colon \overline{\Omega} \to X$. Obviously, this operator is zero on $\partial\Omega$ and satisfies

$$[(\Lambda\Pi + K_{PQ})G]_{\mathrm{A}} = [K_{PQ}G]_{\mathrm{A}} \le k,$$

since the operator $\Lambda\Pi G\colon \overline{\Omega} \to N(L)$ has a finite dimensional range. So by assumption the equation

$$\Phi_\lambda(L, F)(x) = (\Lambda\Pi + K_{PQ})G(x)$$

has a solution $\hat{x} \in \Omega$. Therefore

$$\lambda(I - P)\hat{x} = (\Lambda\Pi + K_{PQ})(G(\hat{x}) + F(\hat{x})) = (L + h\Lambda^{-1}P)^{-1}(G(\hat{x}) + F(\hat{x})),$$

which implies that $\lambda L\hat{x} - F(\hat{x}) = G(\hat{x})$, since $LP\hat{x} = \theta$. Since $(L+h\Lambda^{-1}P)^{-1}(Y) \subseteq D(L)$, we have

$$\lambda(I - P)\hat{x} \in (L + h\Lambda^{-1}P)^{-1}(Y) \subseteq D(L),$$

so that $\hat{x} \in \Omega_L$. Therefore the operator $\lambda L - F$ is (L, k)-epi on $\overline{\Omega}_L$, and so $\nu(\lambda L - F) \ge k > 0$.

Conversely, suppose now that L is bounded and $\nu(\lambda L - F) > 0$, i.e. there exists $k > 0$ such that $\lambda L - F$ is (L, k)-epi on $\overline{\Omega}_L$ for every $\Omega \in \mathfrak{OBC}_L(X)$. Let $G\colon \overline{\Omega}_L \to X$ be continuous with $G|_{\partial\Omega_L} = \Theta$ and $[G]_{\mathrm{A}} \le k$. Then

$$[K_{PQ}(L + h\Lambda^{-1}P)G]_{\mathrm{A}} = [(\Lambda\Pi + K_{PQ})(L + h\Lambda^{-1}P)G]_{\mathrm{A}} = [G]_{\mathrm{A}} \le k.$$

By assumption, the equation $\lambda Lx - F(x) = (L + h\Lambda^{-1}P)G(x)$ has a solution $\tilde{x} \in \Omega_L = \Omega$. Passing from (9.6) (with $z := (L + h\Lambda^{-1}P)G(\tilde{x})$) to (9.5) we see that

$$\Phi_\lambda(L, F)(\tilde{x}) = (L + h\Lambda^{-1}P)^{-1}z = G(\tilde{x}).$$

We conclude that $\Phi_\lambda(L, F)$ is k-epi on $\overline{\Omega}$, and so (e) is proved.

Finally, to prove (f) assume first that $\Phi_\lambda(L, F)$ is stably solvable. Let $G\colon X \to Y$ be compact with $[G]_{\mathrm{Q}} = 0$. Then the operator $(L + h\Lambda^{-1}P)^{-1}G\colon X \to X$ is compact and satisfies $[(L + h\Lambda^{-1}P)^{-1}G]_{\mathrm{Q}} = 0$. So by assumption the equation $\Phi_\lambda(L, F)(x) = (L+h\Lambda^{-1}P)^{-1}G(x)$ has a solution $\hat{x} \in X$. But then $\lambda L\hat{x} - F(\hat{x}) = G(\hat{x})$, and so the operator $\lambda L - F$ is stably solvable as claimed.

Conversely, suppose now that L is bounded and $\lambda L - F$ is stably solvable. Let $G\colon X \to X$ be compact with $[G]_{\mathrm{Q}} = 0$. Then the operator $(L + h\Lambda^{-1}P)G\colon X \to Y$ is compact and satisfies $[(L+h\Lambda^{-1}P)G]_{\mathrm{Q}} = 0$. So again by assumption the equation $\lambda Lx - F(x) = (L + h\Lambda^{-1}P)G(x)$ has a solution $\tilde{x} \in X$. But then $\Phi_\lambda(L, F)(\tilde{x}) = G(\tilde{x})$, and so the operator $\Phi_\lambda(L, F)$ is stably solvable as claimed. □

Lemma 9.2 shows that the auxiliary operator (9.1) is actually independent of P and Q if the linear part L is bounded. For instance, this is the case in Example 9.1.

Now we are ready to define the spectrum. Similarly as in the decomposition (7.20), we put

$$\sigma_\nu(L, F) = \{\lambda \in \mathbb{K} : \nu(\Phi_\lambda(L, F)) = 0\}, \tag{9.7}$$

$$\sigma_{\mathrm{a}}(L, F) = \{\lambda \in \mathbb{K} : [\Phi_\lambda(L, F)]_{\mathrm{a}} = 0\}, \tag{9.8}$$

$$\sigma_{\mathrm{b}}(L, F) = \{\lambda \in \mathbb{K} : [\Phi_\lambda(L, F)]_{\mathrm{b}} = 0\}, \tag{9.9}$$

and

$$\sigma_{\mathrm{F}}(L, F) = \sigma_\nu(L, F) \cup \sigma_{\mathrm{a}}(L, F) \cup \sigma_{\mathrm{b}}(L, F). \tag{9.10}$$

We call the set (9.10) the *semilinear Feng spectrum* of L and F in what follows. Of course, for $L = I$ we simply get $\Phi_\lambda(I, F) = \lambda I - F$, hence

$$\sigma_{\mathrm{F}}(I, F) = \sigma_{\mathrm{F}}(F)$$

(see (7.19)). On the other hand, for $F = I$ we have

$$\Phi_\lambda(L, I) = \lambda(I - P) - \Lambda\Pi - K_{PQ} = (\Lambda\Pi - K_{PQ})(\lambda L - I),$$

and so

$$\sigma_{\mathrm{F}}(L, I) = \left\{\tfrac{1}{\lambda} : \lambda \in \sigma(L) \setminus \{0\}\right\} \tag{9.11}$$

consists precisely of the (non-zero) *characteristic values* of the linear operator L.

In what follows, we assume without loss of generality that $L \neq \Theta$, and so the projection $P\colon X \to N(L)$ satisfies $\|I - P\| \neq 0$. The following result is analogous to Theorem 7.4.

Theorem 9.1. *The spectrum $\sigma_{\mathrm{F}}(L, F)$ is closed.*

Proof. Fix $\lambda \in \mathbb{K} \setminus \sigma_{\mathrm{F}}(L, F)$, i.e. $[\Phi_\lambda(L, F)]_{\mathrm{a}} > 0$, $[\Phi_\lambda(L, F)]_{\mathrm{b}} > 0$ and $\nu(\Phi_\lambda(L, F)) > 0$. Choose $\mu \in \mathbb{K}$ such that

$$|\lambda - \mu| < \min\left\{[\Phi_\lambda(L, F)]_{\mathrm{a}}, \nu(\Phi_\lambda(L, F)), \frac{[\Phi_\lambda(L, F)]_{\mathrm{b}}}{\|I - P\|}\right\}.$$

From (9.2) we get then

$$[\Phi_\mu(L, F)]_{\mathrm{a}} \geq [\Phi_\lambda(L, F)]_{\mathrm{a}} - [(\mu - \lambda)(I - P)]_{\mathrm{A}} = [\Phi_\lambda(L, F)]_{\mathrm{a}} - |\mu - \lambda| > 0.$$

Also, we have

$$\begin{aligned} \|\Phi_\mu(L, F)(x)\| &\geq \|\Phi_\lambda(L, F)(x)\| - |\mu - \lambda|\, \|(I - P)x\| \\ &\geq ([\Phi_\lambda(L, F)]_{\mathrm{b}} - |\mu - \lambda|\, \|(I - P)\|)\|x\|. \end{aligned}$$

Thus, $[\Phi_\mu(L, F)]_b > 0$. We apply Property 9.4 to $F_0 := \Phi_\lambda(L, F)$ and the homotopy $H(x,t) := t(\mu-\lambda)(I-P)x$. Then $[H]_A \le |\mu-\lambda| < \nu(\Phi_\lambda(L, F))$ and $H(x, 0) \equiv \theta$ for $x \in X$. If $\Phi_\lambda(L, F)(x) + t(\mu - \lambda)(I - P)x = \theta$ for some $t \in [0, 1]$, then

$$[\Phi_\lambda(L, F)]_b \|x\| \le \|\Phi_\lambda(L, F)(x)\| \le |\mu - \lambda| \, \|I - P\| \, \|x\|,$$

hence $x = \theta$. Since $F_0 = \Phi_\lambda(L, F)$ is k-epi on $\overline{\Omega}$ for every $\Omega \in \mathfrak{OBC}(X)$ and some $k > 0$, by Property 9.4 we conclude that $F_1 := F_0 + H(\,\cdot\,, 1) = \Phi_\lambda(L, F) + (\mu - \lambda)(I - P) = \Phi_\mu(L, F)$ is $(k - [H]_A)$-epi. Thus $\nu(\Phi_\mu(L, F)) > 0$, and so $\mu \notin \sigma_F(L, F)$. We have proved that $\mathbb{K} \setminus \sigma_F(L, F)$ is open and so $\sigma_F(L, F)$ is closed. □

In Theorem 7.5 we have shown that the Feng spectrum $\sigma_F(F)$ is bounded by the norm $\|F\|_{AB} = \max\{[F]_A, [F]_B\}$, hence compact, provided that the two characteristics $[F]_A$ and $[F]_B$ are finite, of course. An analogous result is not true for the semilinear spectrum (9.10). To see this, it suffices to take $F = I$ and L a linear operator with an unbounded sequence of characteristic values, by (9.11).

We give now some additional information on the topological character of the subspectra (9.8) and (9.9). Putting as in (7.25)

$$\sigma_\varphi(L, F) = \sigma_a(L, F) \cup \sigma_b(L, F) \tag{9.12}$$

we get the following analogue of Proposition 7.1.

Proposition 9.1. *The subspectrum* (9.12) *is closed. Moreover, the inclusion*

$$\partial\sigma_F(L, F) \subseteq \sigma_\varphi(L, F) \tag{9.13}$$

holds if L is bounded.

Proof. We show first that both subspectra (9.8) and (9.9) are closed. Let $\lambda_n \in \sigma_b(L, F)$ with $\lambda_n \to \lambda$; we have to show that $\lambda \in \sigma_b(L, F)$. Otherwise, there exists $m > 0$ such that $\|\Phi_\lambda(L, F)(x)\| \ge m\|x\|$ for every $x \in X$. Since $\Phi_{\lambda_n}(L, F) = \Phi_\lambda(L, F) + (\lambda_n - \lambda)(I - P)$, we have

$$\|\Phi_{\lambda_n}(L, F)(x)\| \ge (m - |\lambda - \lambda_n| \, \|I - P\|)\|x\|.$$

So for $|\lambda_n - \lambda| < m/\|I - P\|$ we obtain $\lambda_n \notin \sigma_b(L, F)$, a contradiction.

Similarly, let $\lambda_n \in \sigma_a(L, F)$ with $\lambda_n \to \lambda$. If $[\Phi_\lambda(L, F)]_a > 0$, then from

$$[\Phi_{\lambda_n}(L, F)]_a = [\Phi_\lambda(L, F) + (\lambda_n - \lambda)(I - P)]_a \ge [\Phi_\lambda(L, F)]_a - |\lambda - \lambda_n|$$

we obtain $\lambda_n \notin \sigma_a(L, F)$ when $|\lambda - \lambda_n| < [\Phi_\lambda(L, F)]_a$. Consequently we have $[\Phi_\lambda(L, F)]_a = 0$ and so $\lambda \in \sigma_a(L, F)$.

It remains to prove (9.13). Put

$$U := \sigma_F(L, F) \setminus \sigma_\varphi(L, F);$$

we claim that U is an open subset of $\mathbb{K}$. Suppose that there exists $\lambda \in U \setminus U^o$. Then we can find a sequence $(\lambda_n)_n$ in $\mathbb{K} \setminus U$ with $\lambda_n \to \lambda$. Since $\lambda \notin \sigma_\varphi(L, F)$ and $\sigma_\varphi(L, F)$ is closed, we may assume that $\lambda_n \notin \sigma_\varphi(L, F)$ for all n, and so $\lambda_n \notin \sigma_{\mathrm{F}}(L, F)$. In particular, the operator $\Phi_{\lambda_n}(L, F)$ is epi on $\overline{\Omega}$ for every $\Omega \in \mathfrak{OBC}(X)$, and the same is true for the operator $\lambda L - F$ on $\overline{\Omega}_L$, by Lemma 9.2 (e). So, for any compact operator $G\colon \overline{\Omega}_L \to Y$ with $G|_{\partial\Omega_L} = \Theta$, there exists some $x_n \in \Omega_L$ with

$$\lambda_n L x_n - F(x_n) = G(x_n).$$

As in the proof of Proposition 7.1, this implies that there exists some $\hat{x} \in \overline{\Omega}_L$ such that $\lambda L\hat{x} - F(\hat{x}) = G(\hat{x})$, and so $\lambda L - F$ is epi on $\overline{\Omega}_L$. Again from Lemma 9.2 (e) we conclude that $\Phi_\lambda(L, F)$ is epi on $\overline{\Omega}$. Since $\lambda \notin \sigma_\varphi(L, F)$, this implies $\lambda \notin \sigma_{\mathrm{F}}(L, F)$, a contradiction. The rest of the proof goes as that of Proposition 7.1. □

We close this section by proving a discreteness result for the semilinear Feng spectrum which in case $L = I$ reduces to the discreteness result in Theorem 7.8. A scalar $\lambda \in \mathbb{K}$ is called an *eigenvalue* of the pair (L, F) if the equation $F(x) = \lambda Lx$ has a nontrivial solution $x \in X$. The set of all eigenvalues,

$$\sigma_{\mathrm{p}}(L, F) = \{\lambda \in \mathbb{K} : F(x) = \lambda Lx \text{ for some } x \neq \theta\}, \tag{9.14}$$

will be called the *point spectrum* of L and F. Of course, in case $L = I$ we get the usual definition (3.18) of the point spectrum of F.

Theorem 9.2. *Let $F\colon X \to Y$ be L-compact, 1-homogeneous and odd. Then every $\lambda \in \sigma_{\mathrm{F}}(L, F) \setminus \{0\}$ is an eigenvalue of L and F.*

Proof. We show first that every nonzero $\lambda \in \sigma_{\mathrm{F}}(L, F)$ belongs to $\sigma_{\mathrm{b}}(L, F)$. In fact, suppose that $[\Phi_\lambda(L, F)]_{\mathrm{b}} > 0$, hence $\|\Phi_\lambda(L, F)(x)\| \geq [\Phi_\lambda(L, F)]_{\mathrm{b}}\|x\| > 0$ for all $x \in X, x \neq \theta$. Fix $\Omega \in \mathfrak{OBC}(X)$, and let $G\colon \overline{\Omega} \to X$ be compact with $G(x) \equiv \theta$ on $\partial\Omega$. We show that the equation $\Phi_\lambda(L, F)(x) = G(x)$ is solvable in Ω, and so $\Phi_\lambda(L, F)$ is epi on $\overline{\Omega}$. Define an operator $H\colon \overline{\Omega} \to X$ by

$$H(x) = Px + \frac{1}{\lambda}(\Lambda\Pi + K_{PQ})F(x) + \frac{1}{\lambda}G(x).$$

Obviously, H is compact and $x - H(x) = \frac{1}{\lambda}\Phi_\lambda(L, F)(x) \neq \theta$ on $\partial\Omega$. Moreover, the restriction $H|_{\partial\Omega}$ is odd. So, the degree $\deg(I - H, \Omega, \theta)$ is nonzero, by Borsuk's theorem, and hence there exists $\hat{x} \in \Omega$ such that

$$\hat{x} = P\hat{x} + \frac{1}{\lambda}(\Lambda\Pi + K_{PQ})F(\hat{x}) + \frac{1}{\lambda}G(\hat{x}).$$

We conclude that $\Phi_\lambda(L, F)$ is epi on $\overline{\Omega}$, and so $\lambda \notin \sigma_{\mathrm{F}}(L, F)$, contradicting our hypothesis.

Now, the relation $[\Phi_\lambda(L, F)]_b = 0$ means that there exists a sequence $(x_n)_n$ in X such that

$$\|\lambda(I - P)x_n - (L + h\Lambda^{-1}P)^{-1}F(x_n)\| \leq \frac{1}{n}\|x_n\|.$$

Putting $e_n := x_n/\|x_n\|$ we have, by the homogeneity of F,

$$\|\lambda(I - P)e_n - (L + h\Lambda^{-1}P)^{-1}F(e_n)\| \to 0 \quad (n \to \infty).$$

In addition, the set $M := \{e_1, e_2, e_3, \ldots\}$ satisfies

$$[\Phi_\lambda(L, F)]_a\alpha(M) \leq \alpha(\Phi_\lambda(L, F)(M)) = 0,$$

which shows that $(e_n)_n$ admits a convergent subsequence $(e_{n_k})_k$, say $e_{n_k} \to e$. By continuity, we have then $e \in S(X)$ and $\lambda(I - P)e = (L + h\Lambda^{-1}P)^{-1}F(e)$. Since $(L + h\Lambda^{-1}P)(I - P) = L$, we see that $\lambda Le = F(e)$, i.e. $\lambda \in \sigma_p(L, F)$. □

9.2 The semilinear FMV-spectrum

As before, we assume that X and Y are two Banach spaces, $L\colon X \to Y$ is a closed linear Fredholm operator of index zero, and $F\colon X \to Y$ is continuous and nonlinear. In contrast to the preceding section, we suppose now in addition that L is *bounded*; as a matter of fact, this assumption is not really restrictive, since every closed linear operator becomes bounded after a suitable renorming of X. Again, we have the decompositions $X = N(L) \oplus X_0$ and $Y = Y_0 \oplus R(L)$, where $N(L)$ and Y_0 have the same (finite) dimension. By $P\colon X \to N(L)$ we denote a (bounded) projection on the nullspace of L, and by $h\Lambda^{-1}\colon N(L) \to Y_0$ a fixed isomorphism.

For $\lambda \in \mathbb{K}$, let $\Phi_\lambda(L, F)\colon X \to X$ be defined as in (9.1), i.e.

$$\Phi_\lambda(L, F)(x) = \lambda(I - P)x - (L + h\Lambda^{-1}P)^{-1}F(x).$$

In addition to the subspectra (9.7)–(9.9), we still introduce the sets

$$\sigma_q(L, F) = \{\lambda \in \mathbb{K} : [\Phi_\lambda(L, F)]_q = 0\}, \tag{9.15}$$

and

$$\sigma_\delta(L, F) = \{\lambda \in \mathbb{K} : \Phi_\lambda(L, F) \text{ is not stably solvable}\}. \tag{9.16}$$

We define the *semilinear Furi–Martelli–Vignoli spectrum* (or *semilinear FMV-spectrum*, for short) of L and F by

$$\sigma_{FMV}(L, F) = \sigma_q(L, F) \cup \sigma_a(L, F) \cup \sigma_\delta(L, F). \tag{9.17}$$

Of course, for $L = I$ we simply get $L + h\Lambda^{-1}P = I$ and $\Phi_\lambda(I, F) = \lambda I - F$, hence

$$\sigma_{FMV}(I, F) = \sigma_{FMV}(F).$$

On the other hand, choosing $F = I$ we get again for the spectrum $\sigma_{\mathrm{FMV}}(L, I)$ the set of all nonzero characteristic values of L.

We give now a parallel discussion of the semilinear FMV-spectrum as we did before for the semilinear Feng spectrum. The following Theorem 9.3 is analogous to Theorem 9.1, the proof even simpler.

Theorem 9.3. *The spectrum $\sigma_{\mathrm{FMV}}(L, F)$ is closed.*

Proof. Fix $\lambda \in \mathbb{K} \setminus \sigma_{\mathrm{FMV}}(L, F)$, and let

$$0 < \delta < \frac{\min\{[\Phi_\lambda(L, F)]_{\mathrm{a}}, [\Phi_\lambda(L, F)]_{\mathrm{q}}\}}{\|I - P\|}.$$

We claim that $\mu \in \mathbb{K} \setminus \sigma_{\mathrm{FMV}}(L, F)$ for any μ satisfying $|\mu - \lambda| < \delta$.

First of all, from (9.2) we get

$$[\Phi_\mu(L, F)]_{\mathrm{a}} \geq [\Phi_\lambda(L, F)]_{\mathrm{a}} - |\lambda - \mu| \, \|I - P\| > 0$$

and

$$[\Phi_\mu(L, F)]_{\mathrm{q}} \geq [\Phi_\lambda(L, F)]_{\mathrm{q}} - |\lambda - \mu| \, \|I - P\| > 0,$$

by our choice of δ. It remains to show that $\Phi_\mu(L, F)$ is stably solvable for $|\mu - \lambda| < \delta$. But this follows from Lemma 6.3, since

$$\begin{aligned}\max\{[\Phi_\mu(L, F) - \Phi_\lambda(L, F)]_{\mathrm{A}}, [\Phi_\mu(L, F) - \Phi_\lambda(L, F)]_{\mathrm{Q}}\} \\ \leq |\mu - \lambda| \, \|I - P\| \\ < \min\{[\Phi_\lambda(L, F)]_{\mathrm{a}}, [\Phi_\lambda(L, F)]_{\mathrm{q}}\}.\end{aligned}$$

We conclude that λ is an interior point of $\mathbb{K} \setminus \sigma_{\mathrm{FMV}}(L, F)$, and so $\mathbb{K} \setminus \sigma_{\mathrm{FMV}}(L, F)$ is open. □

Let us now take a closer look at the structure of the subspectra $\sigma_{\mathrm{q}}(L, F)$, $\sigma_{\mathrm{a}}(L, F)$, and $\sigma_\delta(L, F)$. Putting as in (6.16)

$$\sigma_\pi(L, F) = \sigma_{\mathrm{a}}(L, F) \cup \sigma_{\mathrm{q}}(L, F), \tag{9.18}$$

we get the following analogue to Proposition 9.1.

Proposition 9.2. *The subspectrum* (9.18) *is closed, and the inclusion*

$$\partial\sigma_{\mathrm{FMV}}(L, F) \subseteq \sigma_\pi(L, F)$$

holds.

Proof. The closedness of $\sigma_\pi(L, F)$ may be proved as before. We show that $\sigma_{\text{FMV}}(L, F) \setminus \sigma_\pi(L, F)$ is open. So fix $\lambda \in \sigma_{\text{FMV}}(L, F)$ such that both $\lambda \notin \sigma_{\text{a}}(L, F)$ and $\lambda \notin \sigma_{\text{q}}(L, F)$. Suppose that there exists a sequence $(\lambda_n)_n$ with $\lambda_n \to \lambda$ such that $\Phi_{\lambda_n}(L, F)$ is stably solvable. Since $\lambda \in \sigma_\delta(L, F)$, there exists a compact operator G such that $[G]_{\text{Q}} = 0$ and $\Phi_\lambda(L, F)(x) \neq G(x)$ for all $x \in X$. On the other hand, by the stable solvability of each $\Phi_{\lambda_n}(L, F)$ we find a sequence $(x_n)_n$ in X with $\Phi_{\lambda_n}(L, F)(x_n) = G(x_n)$.

We claim that this sequence $(x_n)_n$ is bounded. To see this, suppose that $\|x_n\| \to \infty$. Then

$$\begin{aligned}\frac{\|\Phi_\lambda(L, F)(x_n)\|}{\|x_n\|} &\leq \frac{\|\Phi_\lambda(L, F)(x_n) - \Phi_{\lambda_n}(L, F)(x_n)\|}{\|x_n\|} + \frac{\|G(x_n)\|}{\|x_n\|} \\ &\leq |\lambda - \lambda_n| \, \|I - P\| + \frac{\|G(x_n)\|}{\|x_n\|} \to 0 \quad (n \to \infty).\end{aligned}$$

But this means that $[\Phi_\lambda(L, F)]_{\text{q}} = 0$, contradicting our choice $\lambda \notin \sigma_{\text{q}}(L, F)$. Now, the boundedness of $(x_n)_n$ implies that

$$\|\Phi_\lambda(L, F)(x_n) - G(x_n)\| \leq |\lambda_n - \lambda| \, \|I - P\| \, \|x_n\| \to 0 \quad (n \to \infty).$$

From this and the fact that $[\Phi_\lambda(L, F) - G]_{\text{a}} = [\Phi_\lambda(L, F)]_{\text{a}} > 0$ it follows that there exists a subsequence $(x_{n_k})_k$ of $(x_n)_n$ with $x_{n_k} \to \hat{x}$. By continuity, $\hat{x}$ is a solution of the equation $\Phi_\lambda(L, F)(\hat{x}) = G(\hat{x})$. This contradiction shows that our assumption was false, and so the above assertion is true.

Now let $\lambda \in \partial\sigma_{\text{FMV}}(L, F)$ and suppose that $\lambda \notin \sigma_\pi(L, F)$. Then $\lambda \in \sigma_{\text{FMV}}(L, F) \setminus \sigma_\pi(L, F)$ which is open in $\sigma_{\text{FMV}}(L, F)$. This contradicts $\lambda \in \partial\sigma_{\text{FMV}}(L, F)$. □

Finally, the following discreteness result holds for the semilinear FMV-spectrum in the same way as Theorem 9.2 for the semilinear Feng spectrum.

Theorem 9.4. *Let F be L-compact and odd. Then the inclusion*

$$\sigma_{\text{FMV}}(L, F) \setminus \{0\} \subseteq \sigma_{\text{q}}(L, F)$$

is true, i.e. every nonzero spectral point is an asymptotic eigenvalue of L and F.

Proof. Suppose that $\lambda \neq 0$ does not belong to $\sigma_{\text{q}}(L, F)$, i.e. $[\Phi_\lambda(L, F)]_{\text{q}} > 0$. Since F is compact, we have $[\Phi_\lambda(L, F)]_{\text{a}} = 0$ if and only if $\lambda = 0$; consequently, λ does not belong to $\sigma_{\text{a}}(L, F)$ either. So we have to show only that $\lambda \notin \sigma_\delta(L, F)$. By Lemma 9.2 (f), this is equivalent to proving that $\lambda L - F$ is stably solvable.

Let $G\colon X \to Y$ be compact with $[G]_{\text{Q}} = 0$. By assumption, $\lambda \neq 0$ and $[\Phi_\lambda(L, F)]_{\text{q}} > 0$, and hence there exists $R_1 > 0$ and $\delta > 0$ such that

$$\|\lambda^{-1}(L + h\Lambda^{-1}P)^{-1}(\lambda L - F)(x)\| \geq \delta\|x\|$$

whenever $\|x\| \geq R_1$. On the other hand, we can find $R_2 > 0$ such that

$$\|\lambda^{-1}(L + h\Lambda^{-1}P)^{-1}G(x)\| \leq \frac{\delta}{2}\|x\|$$

whenever $\|x\| \geq R_2$. Thus, for $\|x\| \geq R := \max\{R_1, R_2\}$ and $0 \leq \mu \leq 1$ we have

$$\|\lambda^{-1}(L + h\Lambda^{-1}P)^{-1}(\lambda L - F - \mu G)(x)\| \geq \frac{\delta}{2}\|x\|. \tag{9.19}$$

Now

$$\lambda^{-1}(L + h\Lambda^{-1}P)^{-1}(\lambda L - F - \mu G) = (I - P) - \lambda^{-1}(L + h\Lambda^{-1}P)^{-1}(F + \mu G)$$

is a compact perturbation of the identity. Therefore, from (9.19) and the homotopy invariance of the Leray–Schauder degree and Borsuk's theorem (see Section 3.5), we obtain

$$\begin{aligned}\deg(&\lambda^{-1}(L + h\Lambda^{-1}P)^{-1}(\lambda L - F - G), B_R^o(X), \theta)\\ &= \deg(\lambda^{-1}(L + h\Lambda^{-1}P)^{-1}(\lambda L - F), B_R^o(X), \theta) \equiv 1 \ (\mathrm{mod}\, 2),\end{aligned}$$

and so the equation $\lambda Lx = F(x) + G(x)$ has a solution in $B_R(X)$. □

In Chapter 6 we have obtained some very precise results on the "topological disposition" of several subspectra of the FMV-spectrum. The same may be done for the semilinear FMV-spectrum. Recall that, given a closed subset Σ of the complex plane, by $c_0[\Sigma]$ we denote the connected component of $\mathbb{C} \setminus \Sigma$ containing zero, and by $c_\infty[\Sigma]$ the unbounded connected component of $\mathbb{C} \setminus \Sigma$.

Theorem 9.5. *Suppose that X and Y are infinite dimensional and $F: X \to Y$ is compact. Then the following is true*:

(a) *F is not onto; in particular,* $0 \in \sigma_\delta(L, F)$.

(b) *Either* $0 \in \sigma_q(L, F)$*, or* $c_0[\sigma_q(L, F)] \subseteq \sigma_\delta(L, F)$.

(c) *If* $\sigma_{\mathrm{FMV}}(L, F) \neq \mathbb{K}$*, then* $\sigma_q(L, F) \neq \emptyset$.

(d) *If $0 \notin \sigma_q(L, F)$ and $\sigma_{\mathrm{FMV}}(L, F)$ is bounded, then $c_0[\sigma_q(L, F)]$ is bounded; consequently, $\sigma_q(L, F)$ contains a positive and a negative value.*

(e) *If $\mathbb{K} = \mathbb{C}$ and $\sigma_{\mathrm{FMV}}(L, F)$ is bounded, then $c_\infty[\sigma_\pi(L, F)] \cap \sigma_{\mathrm{FMV}}(L, F) = \emptyset$.*

Proof. Being a compact operator between two Banach spaces, F cannot be onto, by Baire's category theorem. Consequently, $\Phi_0(L, F) = -(L + h\Lambda^{-1}P)^{-1}F$ is not onto either, and thus $0 \in \sigma_\delta(L, F)$, which proves (a).

(b) Suppose that $0 \notin \sigma_q(L, F)$. This means that $[\Phi_0(L, F)]_q > 0$. From the fact that $\sigma_a(L, F) = \{0\}$ it follows that 0 is then an isolated point of $\sigma_a(L, F) \cup \sigma_q(L, F)$. Therefore it suffices to show that $\Phi_\lambda(L, F)$ is not surjective for λ small enough. In fact, assume that the set $c_0[\sigma_q(L, F)]\backslash\sigma_\delta(L, F)$ is nonempty. Since this set has no boundary in $c_0[\sigma_q(L, F)]$, by Proposition 9.2, it is both open and closed in $c_0[\sigma_q(L, F)]$. But $c_0[\sigma_q(L, F)]$ is connected, by definition, and so $\sigma_\delta(L, F) = \emptyset$.

Now, to show that $\Phi_\lambda(L, F)$ is not surjective for λ small enough, assume that this is false. Then there exists a sequence $(\lambda_n)_n$, $\lambda_n \to 0$, such that $\Phi_{\lambda_n}(L, F) = \lambda_n I - P - (L + h\Lambda^{-1}P)^{-1}F$ is onto for all n. Given $a > 0$ with $2a < [\Phi_0(L, F)]_q$, we may find $R > 0$ such that $\|\Phi_0(L, F)(x)\| \geq 2a\|x\|$ for $\|x\| \geq R$. Taking $b := 2aR$ we have then $\|\Phi_0(L, F)(x)\| \geq 2a\|x\| - b$ for all $x \in X$.

Fix $y \in X$ with $\|y\| \leq 1$. By assumption, we find a sequence $(x_n)_n$ in X such that $\Phi_{\lambda_n}(L, F)(x_n) = y$. Without loss of generality we may assume that $|\lambda_n| \leq a/\|I - P\|$ for all n, where a is as above. Consequently,

$$\begin{aligned} 1 \geq \|y\| &= \|\Phi_{\lambda_n}(L, F)(x_n)\| \\ &= \|\lambda_n(I - P)x_n + \Phi_0(L, F)(x_n)\| \\ &\geq 2a\|x_n\| - b - |\lambda_n|\,\|I - P\|\,\|x_n\| \\ &\geq a\|x_n\| - b, \end{aligned}$$

hence

$$\|x_n\| \leq \frac{1 + b}{a},$$

i.e. the sequence $(x_n)_n$ is bounded. We conclude that $\lambda_n x_n \to 0$, and thus $F(x_n) \to -(L + h\Lambda^{-1}P)y$ as $n \to \infty$. Since y with $\|y\| \leq 1$ was arbitrary, we have actually shown that the closure of the set $F(\{x : \|x\| \leq (1 + b)/a\})$ contains a ball (of radius $\left| L + h\Lambda^{-1}P \right| = \|(L + h\Lambda^{-1}P)^{-1}\|^{-1}$), and so has nonempty interior. But this is impossible, because F is a compact operator.

To prove (c) we distinguish the cases $0 \in \sigma_q(L, F)$ and $0 \notin \sigma_q(L, F)$. In the first case the assertion is trivially true. In the second case it follows from (b) that 0 is an interior point of $\sigma_{\mathrm{FMV}}(L, F)$. On the other hand, again the equality $\sigma_a(L, F) = \{0\}$ and Proposition 9.2 imply that

$$\partial\sigma_{\mathrm{FMV}}(L, F) \subseteq \sigma_\pi(L, F) = \{0\} \cup \sigma_q(L, F),$$

and $\sigma_{\mathrm{FMV}}(L, F)$ has nonempty boundary, since $0 \in \sigma_{\mathrm{FMV}}(L, F)$ and $\sigma_{\mathrm{FMV}}(L, F) \neq \mathbb{K}$. So we have $0 \in \partial\sigma_{\mathrm{FMV}}(L, F)$, a contradiction.

Let us now prove (d). If $0 \notin \sigma_q(L, F)$, from (b) we conclude that

$$c_0[\sigma_q(L, F)] \subseteq \sigma_\delta(L, F) \subseteq \sigma_{\mathrm{FMV}}(L, F),$$

and the assertion follows from the assumed boundedness of $\sigma_{\mathrm{FMV}}(L, F)$.

Finally, to prove (e) put $C_\infty := c_\infty[\sigma_\pi(L, F)]$ and $C := C_\infty \setminus \sigma_{\mathrm{FMV}}(L, F)$; we have to show that $C = C_\infty$. Since the relative boundary of C with respect to C_∞ is empty, by Proposition 9.2, the set C is both open and closed in C_∞. From the connectedness of C_∞ it follows that either $C = C_\infty$ or $C = \emptyset$. But the latter is impossible if $\sigma_{\mathrm{FMV}}(L, F) = C_\infty \setminus C$ is bounded. □

Observe that Theorem 9.5 (d) implies the following alternative on the “size” of the subspectrum $\sigma_q(L, F)$: *if* $\mathbb{K} = \mathbb{C}$ *then either* $\sigma_q(L, F) = \emptyset$ *or* $0 \in \sigma_q(L, F)$ *or* $\sigma_q(L, F)$ *is infinite*. This is false in case $\mathbb{K} = \mathbb{R}$. For example, for the operator $F(x_1, x_2, x_3, \dots) := (\|x\|, x_1, x_2, \dots)$ in the real sequence space $X = l_2$ we have $\sigma_q(I, F) = \{\pm\sqrt{2}\}$, see Example 3.15.

9.3 The pseudo-adjoint spectrum

In this section we discuss a new spectrum which is defined in a rather “strange” way, but seems to be at least of theoretical interest. The idea is quite simple and may be summarized as follows. Given two Banach spaces X and Y and a bounded *linear* operator $L: X \to Y$, recall that the *adjoint* $L^*: Y^* \to X^*$ of L is defined by

$$(L^*\ell)x := \ell(Lx) \quad (\ell \in Y^*, x \in X), \tag{9.20}$$

where X^* denotes, as usual, the *dual space* of X. It is then well known that L^* is also bounded and linear, and $\sigma(L^*) = \sigma(L)$, i.e. the original and the adjoint operator have the same spectrum.

So it might be a good idea to associate also to *nonlinear* operators F some kind of “adjoint” for which a spectrum is already defined, and then to define the spectrum of F through the spectrum of its adjoint. This is in fact possible for Lipschitz continuous operators, as we shall show in this section.

So throughout the following we will use the Banach space $\mathfrak{Lip}_0(X, Y)$ of all Lipschitz continuous operators $F: X \to Y$ satisfying $F(\theta) = \theta$, equipped with the norm

$$[F]_{\mathrm{Lip}} = \sup_{x \neq y} \frac{\|F(x) - F(y)\|}{\|x - y\|}$$

(see (2.1)). It is easy to see that the space $\mathfrak{L}(X, Y)$ of all bounded linear operators from X to Y is a closed subspace of $\mathfrak{L}ip_0(X, Y)$. In particular, we set

$$\mathfrak{Lip}_0(X, \mathbb{K}) =: X^\# \tag{9.21}$$

and call $X^\#$ the *pseudo-dual space* of X; this space contains the usual dual space X^* as closed subspace.

Now, given $F \in \mathfrak{Lip}_0(X, Y)$ let us define the *pseudo-adjoint* $F^\#: Y^\# \to X^\#$ of F by

$$F^\#(g)(x) := g(F(x)) \quad (g \in Y^\#,\ x \in X). \tag{9.22}$$

This is of course a straightforward generalization of (9.20); in fact, for linear operators L we have $L^\#|_{Y^*} = L^*$, i.e. the restriction of the pseudo-adjoint to the dual space is the classical adjoint.

Before defining a new kind of spectrum for operators $F \in \mathfrak{Lip}_0(X, X)$, let us discuss some simple properties of the pseudo-adjoint (9.22). The following observation is rather trivial, but useful.

Lemma 9.3. *The pseudo-adjoint $F^\#$ of $F \in \mathfrak{Lip}_0(X, Y)$ is bounded and linear with $\|F^\#\| = [F]_{\mathrm{Lip}}$.*

Proof. For $g \in Y^\#$ we have

$$[F^\#(g)]_{\mathrm{Lip}} = \sup_{\|x\| \le 1} |g(F(x))| \le [g]_{\mathrm{Lip}}[F]_{\mathrm{Lip}},$$

which implies the estimate $\|F^\#\| \le [F]_{\mathrm{Lip}}$. The converse estimate is somewhat less trivial. Given $\varepsilon > 0$, choose $x, y \in X$ with

$$\|F(x) - F(y)\| \ge (1 - \varepsilon)\,[F]_{\mathrm{Lip}}\|x - y\|.$$

By the Hahn–Banach theorem we may find a functional $\ell \in Y^*$ such that $\|\ell\| = 1$ and

$$\ell(F(x) - F(y)) = \|F(x) - F(y)\|,$$

hence

$$\|F^\#(\ell)(x) - F^\#(\ell)(y)\| = \ell(F(x) - F(y)) \ge (1 - \varepsilon)\,[F]_{\mathrm{Lip}}\|x - y\|.$$

This shows that $\|F^\#\| \ge [F]_{\mathrm{Lip}}$, and so we are done. □

The next lemma illustrates, similarly as in the linear case, the "interaction" between the mapping properties of F and $F^\#$.

Lemma 9.4. *If $F^\# \colon Y^\# \to X^\#$ is injective, then $F \colon X \to Y$ has a dense range $R(F) \subseteq Y$. If $F^\# \colon Y^\# \to X^\#$ is surjective, then $F \colon X \to Y$ is injective with Lipschitz-continuous inverse on $R(F)$.*

Proof. Let $F^\#$ be injective, and suppose that the range $R(F)$ of F is not dense in Y. Then we find $y_0 \in Y$ and $r > 0$ such that $\|y - y_0\| \ge r$ for all $y \in \overline{R(F)}$. The function $g \colon Y \to \mathbb{K}$ defined by

$$g(y) := \max\{0, r - \|y - y_0\|\}$$

belongs to $Y^\#$ (with $[g]_{\mathrm{Lip}} = 1$) and vanishes on $\overline{R(F)}$, but is not identically zero on Y. Since $F^\#$ is injective, by assumption, the function $F^\#(g) \in X^\#$ cannot be identically zero on X either. On the other hand, we have $F^\#(g)(x) = g(F(x)) \equiv 0$, a contradiction.

Assume now that $F^\#$ is surjective, i.e. for each $f \in X^\#$ we find $g \in Y^\#$ with $F^\#(g) = f$. From the open mapping theorem we conclude that then $\|g\| \le c\|f\|$ for some $c > 0$ independent of f. For fixed $x, y \in X$ we may find, by the Hahn–Banach theorem, a functional $\ell \in X^* \subseteq X^\#$ such that $\|\ell\| = 1$ and $|\ell(x - y)| = \|x - y\|$. For

$g \in Y^\#$ with $\ell = F^\#(g)$ we have then

$$\begin{aligned} \|x-y\| &= |\ell(x-y)| \\ &= |\ell(x)-\ell(y)| \\ &= |g(F(x))-g(F(y))| \\ &\le [g]_{\mathrm{Lip}}\|F(x)-F(y)\| \\ &\le c\|F(x)-F(y)\|. \end{aligned}$$

Since c does not depend on x and y, we see that F is invertible on $R(F)$ with $[F^{-1}|_{R(F)}]_{\mathrm{Lip}} \le c$. □

The well known example $X = Y = l_2$ and $L(x_1, x_2, x_3, \dots) = (x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, \dots)$ shows that one cannot expect surjectivity in the first statement of Lemma 9.4. In fact, the operator $L^\# = L^* = L$ is injective in $X^* = X$ in this example, but $R(L)$ cannot coincide with the whole space l_2, since L^{-1} would then be bounded, by the closed graph theorem.

As in Chapter 2, we call a bijection $F\colon X \to Y$ a *lipeomorphism* if both $F \in \mathfrak{Lip}_0(X, Y)$ and $F^{-1} \in \mathfrak{Lip}_0(Y, X)$.

Proposition 9.3. *An operator $F\colon X \to Y$ is a lipeomorphism if and only if its pseudo-adjoint $F^\#\colon Y^\# \to X^\#$ is a linear isomorphism; in this case one has $(F^\#)^{-1} = (F^{-1})^\#$.*

Proof. If $F\colon X \to Y$ is a lipeomorphism, the linear equation $F^\#(g) = f$ has, for each $f \in X^\#$, a unique solution $g \in Y^\#$, namely $g(y) = f(F^{-1}(y))$. Conversely, if $F^\#\colon Y^\# \to X^\#$ is a bijection, then F is a lipeomorphism between X and $R(F)$, by Lemma 9.4. Since X is a Banach space, $R(F)$ is also a Banach space, and the fact that $R(F)$ is dense in Y implies that F is onto.

Finally, the equality $(F^\#)^{-1} = (F^{-1})^\#$ follows from the functorial properties of the map $F \mapsto F^\#$, i.e. $(GF)^\# = F^\# G^\#$ and $I^\# = I$. □

Proposition 9.3 suggests the following canonical definition. Given $F \in \mathfrak{Lip}_0(X)$, we put

$$\sigma^\#(F) := \sigma(F^\#) = \{\lambda \in \mathbb{K} : \lambda I - F^\# \text{ is not a bijection}\} \tag{9.23}$$

and call $\sigma^\#(F)$ the *pseudo-adjoint spectrum* of F. By construction, the spectrum (9.23) is always compact and nonempty, being the spectrum of the bounded linear operator $F^\#$. This implies, in particular, that this spectrum cannot coincide with the Kachurovskij spectrum (5.9). To illustrate this, let us return to our famous example of an operator for which all spectra considered so far have been empty.

Example 9.2. Let $X = \mathbb{C}^2$ and $F \in \mathfrak{Lip}_0(X)$ be defined as in Example 3.18. We already know that $\sigma_{\mathrm{K}}(F) = \emptyset$ (see Section 5.2). On the other hand, $\sigma^\#(F)$ cannot be empty by what we have observed above. Indeed, put

$$\lambda_\pm := \pm\frac{1}{\sqrt{2}}(1+i), \qquad \mu_\pm := \pm\frac{1}{\sqrt{2}}(1-i),$$
$$f_\pm(z,w) := \lambda_\pm \overline{z} + w, \qquad g_\pm(z,w) := \mu_\pm z + \overline{w}.$$

Then $f_\pm, g_\pm \in X^\#$, $F^\#(f_\pm) = \lambda_\pm f_\pm$, and $F^\#(g_\pm) = \mu_\pm g_\pm$. Consequently, the spectrum $\sigma^\#(F)$ contains at least the four points λ_+, λ_-, μ_+ and μ_-. ♡

As in the linear case (1.8), we define the #-*spectral radius* of F by

$$r^\#(F) := \sup\{|\lambda| : \lambda \in \sigma^\#(F)\}. \tag{9.24}$$

The following theorem shows that there is some kind of “Gel’fand formula” (1.9) for the spectral radius (9.24). Example 6.5 shows that this is *not* true for the Kachurovskij spectrum.

Proposition 9.4. *For $F \in \mathfrak{Lip}_0(X)$ we have*

$$r^\#(F) = \lim_{n\to\infty} \sqrt[n]{[F^n]_{\mathrm{Lip}}}.$$

Proof. Let $r(F^\#)$ denote the usual spectral radius (1.8) of the linear operator $F^\#$ in $X^\#$. From the equality $(F^n)^\# = (F^\#)^n$ and from Lemma 9.3 we obtain the chain of equalities

$$r^\#(F) = r(F^\#) = \lim_{n\to\infty} \sqrt[n]{\|(F^\#)^n\|} = \lim_{n\to\infty} \sqrt[n]{\|(F^n)^\#\|} = \lim_{n\to\infty} \sqrt[n]{[F^n]_{\mathrm{Lip}}}$$

which proves the assertion. □

All results proved so far suggest that the spectrum (9.23) has better properties than the Kachurovskij spectrum (5.9). Unfortunately, the spectrum (9.23) has a serious drawback: it does not give the familiar spectrum in the linear case!

Example 9.3. Consider the linear operator $L(x) = -x$ in $X = \mathbb{R}$, hence $(L^\# f)(x) = f(-x)$ for $f \in \mathbb{R}^\#$. Obviously, the (Kachurovskij) spectrum of L is simply $\sigma_{\mathrm{K}}(L) = \sigma(L) = \{-1\}$. Clearly, the scalar $\lambda = -1$ also belongs to $\sigma(L^\#)$, because the function $f(x) = x$ satisfies $L^\#(f) = \lambda f$. On the other hand, the scalar $\mu = 1$ belongs to $\sigma(L^\#)$ as well, because the function $g(x) = |x|$ satisfies $L^\#(g) = \mu g$. So we see that $\pm 1 \in \sigma^\#(L)$.

Now, the trivial equality $(L^\#)^2 = I$ implies that $\sigma((L^\#)^2) = \{1\}$. From Theorem 1.1 (h) we conclude that $\sigma^\#(L) = \{\pm 1\}$. ♡

Example 9.3 contains a Lipschitz continuous operator for which the spectrum (9.23) is strictly larger than the Kachurovskij spectrum (5.9). A converse example is given by the operator (3.26): for this operator we have $\sigma_{\mathrm{K}}(F) = [0,1]$, but $\sigma^\#(F) = \{0\}$, since F^2 is the zero operator.

9.4 The Singhof–Weyer spectrum

Now we discuss another spectrum introduced by Singhof in the seventieth. The motivation for the definition of this spectrum will be more transparent in the multivalued case which was considered by Weyer.

Given two Banach spaces X and Y, we denote by $\mathfrak{Cl}(X, Y)$ the linear space of all (not necessarily continuous) nonlinear operators $F\colon X \to Y$ which satisfy $F(\theta) = \theta$ and whose graph

$$\Gamma(F) = \{(x, y) \in X \times Y : y = F(x)\} \tag{9.25}$$

is a closed subset of $X \times Y$. As in Section 5.7, we equip $X \times Y$ with the norm $\|(x, y)\| := (\|x\|^2 + \|y\|^2)^{1/2}$, and we write again $\mathfrak{Cl}(X, X) =: \mathfrak{Cl}(X)$ for brevity.

Now, given $F \in \mathfrak{Cl}(X)$, we call the set

$$\rho_{\mathrm{SW}}(F) := \{\lambda \in \mathbb{K} : \lambda I - F \text{ is bijective and } R(\lambda; F) \in \mathfrak{Lip}_0(X)\}, \tag{9.26}$$

where $R(\lambda; F) = (\lambda I - F)^{-1}$ as usual, the *Singhof–Weyer resolvent set* of F and its complement

$$\sigma_{\mathrm{SW}}(F) := \mathbb{K} \setminus \rho_{\mathrm{SW}}(F) \tag{9.27}$$

the *Singhof–Weyer spectrum* of F. Observe that there is a certain “asymmetry” in the definition (9.26), inasmuch as $\lambda I - F$ belongs only to $\mathfrak{Cl}(X)$, but $(\lambda I - F)^{-1}$ is assumed to belong to $\mathfrak{Lip}_0(X)$. This asymmetry is explained by certain monotonicity properties of the resolvent operator which will become more transparent in the multivalued case discussed below (see Lemma 9.5).

The following theorem shows that the Singhof–Weyer spectrum shares the most important property with all other spectra.

Theorem 9.6. *The spectrum $\sigma_{\mathrm{SW}}(F)$ is closed.*

Proof. We show that $\rho_{\mathrm{SW}}(F)$ is open for $F \in \mathfrak{Cl}(X)$. Fix $\lambda \in \rho_{\mathrm{SW}}(F)$ and choose $\mu \in \mathbb{K}$ such that $|\mu - \lambda| < 1/[R(\lambda; F)]_{\mathrm{Lip}}$. We use the trivial equality

$$\mu I - F = \lambda I - F + (\mu - \lambda) I$$

to obtain for $x, y \in X$ the estimate

$$\begin{aligned}\|\mu x - F(x) - \mu y + F(y)\| &\geq \|\lambda x - F(x) - \lambda y + F(y)\| - |\mu - \lambda|\, \|x - y\| \\ &\geq [R(\lambda; F)]_{\mathrm{Lip}}^{-1} \|x - y\| - |\mu - \lambda|\, \|x - y\|,\end{aligned}$$

which shows that $\mu I - F$ is invertible on its range with

$$[R(\mu; F)]_{\mathrm{Lip}} \leq \frac{1}{[R(\lambda; F)]_{\mathrm{Lip}}^{-1} - |\mu - \lambda|}.$$

The proof of the surjectivity goes precisely as in the proof of Theorem 5.1. □

In the following elementary example we compare the Singhof–Weyer spectrum with those we studied in Chapter 4.

Example 9.4. Let $X = \mathbb{R}$ and $F(x) = \arctan x$. We have seen in Example 4.2 that the Rhodius spectrum of F is $\sigma_R(F) = [0, 1)$. By Theorem 9.6, the Singhof–Weyer spectrum $\sigma_{SW}(F)$ must be different. In fact, it is clear that $[0, 1) \subseteq \sigma_{SW}(F) \subseteq [0, 1]$, since the map $x \mapsto \lambda x - \arctan x$ is a bijection with derivative bounded away from zero for $\lambda < 0$ or $\lambda > 1$. Since $\sigma_{SW}(F)$ is closed, we have $\sigma_{SW}(F) = [0, 1]$. ♡

We discuss now an extension of the spectrum (9.27) to multivalued operators which is due to Weyer. To this end, we recall some notions and results on multivalued maps. By 2^M we denote the family of all nonempty subsets of a set M. Given two Banach spaces X and Y, a *multivalued operator* $F\colon X \to 2^Y$ is called *surjective* if its range

$$R(F) = \bigcup_{x \in X} F(x) \tag{9.28}$$

is the whole space Y, and *injective* if $F(x_1) \cap F(x_2) \neq \emptyset$ implies $x_1 = x_2$. Of course, this gives the usual definitions if one identifies a singlevalued operator $x \mapsto F(x)$ with the multivalued operator $x \mapsto \{F(x)\}$. An injective operator $F\colon X \to 2^Y$ admits an *inverse* F^{-1} on $R(F)$ defined by $F^{-1}(y) = x$ if and only if $y \in F(x)$; observe that the inverse F^{-1} is always singlevalued. As before, by $R(\lambda; F) = (\lambda I - F)^{-1}$ we denote the (singlevalued) *resolvent operator* of an injective operator $F\colon X \to 2^X$. Finally, the *graph* of an operator $F\colon X \to 2^Y$ is defined by

$$\Gamma(F) := \{(x, y) \in X \times Y : y \in F(x)\}. \tag{9.29}$$

Given a multivalued operator $F\colon X \to 2^X$ with closed graph, we define its *Singhof–Weyer resolvent set* $\rho_{SW}(F)$ as in (9.26) and its *Singhof–Weyer spectrum* $\sigma_{SW}(F)$ as in (9.27). The following example is the main motivation for considering the spectrum (9.27) in the multivalued case.

Let H be a real Hilbert space with scalar product $\langle \cdot, \cdot \rangle$. A multivalued operator $F\colon H \to 2^H$ is called *monotone* if

$$\langle x - y, u - v \rangle \geq 0 \quad (u \in F(x), v \in F(y)),$$

and *maximal monotone* if F has no proper monotone extension. In the following lemma we collect some fundamental properties of monotone and maximal monotone operators.

Lemma 9.5. *Let H be a real Hilbert space and $F\colon H \to 2^H$ a multivalued operator. Then the following holds.*

(a) *F is monotone if and only if the estimate*

$$\|x - y\| \leq \|x - y + \mu(p - q)\| \tag{9.30}$$

holds for all $(x, p) \in \Gamma(F)$, $(y, q) \in \Gamma(F)$, *and* $\mu > 0$.

(b) F *is maximal monotone if and only if the property*

$$\langle x - y, u - v\rangle \geq 0 \quad (y \in H, v \in F(y)) \tag{9.31}$$

is equivalent to $u \in F(x)$.

(c) *If* F *is monotone, then the resolvent operator* $R(\lambda; F)\colon R(\lambda I - F) \to H$ *exists for all* $\lambda < 0$ *and is Lipschitz continuous on* $R(\lambda I - F)$ *with*

$$[R(\lambda; F)|_{R(\lambda I - F)}]_{\mathrm{Lip}} \leq \frac{1}{|\lambda|}.$$

(d) *A monotone operator* F *is maximal monotone if and only if* $\lambda I - F$ *is onto for all* $\lambda < 0$.

Proof. To prove (a) observe first that

$$\|x - y + \mu(p - q)\|^2 = \|x - y\|^2 + \mu^2\|p - q\|^2 + 2\mu\langle x - y, p - q\rangle. \tag{9.32}$$

So, if F is monotone we get for $\mu > 0$

$$\begin{aligned}\|x - y + \mu(p - q)\|^2 - \|x - y\|^2 &= \mu^2\|p - q\|^2 + 2\mu\langle x - y, p - q\rangle \\ &\geq \mu^2\|p - q\|^2 \geq 0\end{aligned}$$

which is (9.30). Conversely, from (9.30) and (9.32) we deduce that

$$\mu^2\|p - q\|^2 + 2\mu\langle x - y, p - q\rangle \geq 0.$$

Diving by $\mu > 0$ and letting μ tend to zero we see that F is monotone.

The assertion (b) is simply a reformulation of the fact that the graph $\Gamma(F)$ of a maximal monotone operator F is a maximal subset of $H \times H$ with respect to inclusion.

Now let $F\colon H \to 2^H$ be monotone and $\lambda < 0$. Fix $x \in R(\lambda; F)(u)$ and $y \in R(\lambda; F)(v)$; then for $\mu := -1/\lambda > 0$ we have $p := -(u + x/\mu) \in F(x)$ and $q := -(v + y/\mu) \in F(y)$. Applying to this choice of μ, p and q the estimate (9.30) we obtain

$$\begin{aligned}\|x - y\| &\leq \|x - y - \mu(u + \frac{1}{\mu}x) + \mu(v + \frac{1}{\mu}y)\| \\ &= |\mu|\,\|u - v\| \\ &= \frac{1}{|\lambda|}\|u - v\|\end{aligned}$$

which shows that $R(\lambda; F)$ is in fact singlevalued and $[R(\lambda; F)]_{\mathrm{Lip}} \leq 1/|\lambda|$. So we have proved (c).

Finally, for the proof of (d) we point out first that it suffices to show that the maximal monotonicity of F is equivalent to the surjectivity of $I + F$. Indeed, this follows from the trivial equality $\lambda I - F = I + \mu F$ with $\mu := -1/\lambda$ and the fact that F is monotone if and only if μF is monotone for all $\mu > 0$.

Assume first that F is monotone and $I+F$ is onto. We use (b) to prove the maximal monotonicity of F. So suppose that (9.31) holds for all $y \in H$ and $v \in F(y)$; we have to show that $u \in F(x)$. Since $I + F$ is surjective we may choose y as a solution of the inclusion

$$u + x \in y + F(y),$$

i.e. $u + x = y + v$ for some $v \in F(y)$. Then (9.31) implies

$$\|x - y\|^2 = \langle x - y, x - y\rangle = -\langle x - y, u - v\rangle \leq 0,$$

hence $y = x$. It follows that $u = v \in F(y) = F(x)$ as claimed.

The converse implication is more complicated, and we only sketch the idea of the proof. Assume that F is maximal monotone and fix $y_0 \in H$; we have to show that $y_0 \in x_0 + F(x_0)$ for some $x_0 \in H$. Passing from F to the maximal monotone operator $x \mapsto F(x) - y_0$, if necessary, we may suppose without loss of generality that $y_0 = \theta$, i.e. we must show that $-x_0 \in F(x_0)$. Now, the quadratic functional $\varphi_{y,v} \colon H \to \mathbb{R}$ defined for fixed $(y, v) \in \Gamma(F)$ by

$$\varphi_{y,v}(x) := \langle x + v, x - y\rangle$$

is convex and weakly lower semicontinuous. Therefore we can find some $x_0 \in H$ such that $\varphi_{y,v}(x_0) \leq 0$ for *all* $(y, v) \in \Gamma(F)$, hence

$$\langle -x_0 - v, x_0 - y\rangle = -\varphi_{y,v}(x_0) \geq 0 \quad ((y, v) \in \Gamma(F)).$$

By what we have proved in (b), this means precisely that $-x_0 \in F(x_0)$. □

Lemma 9.5 (c) and (d) imply that, for a maximal monotone operator F in a real Hilbert space, we always have $(-\infty, 0) \subseteq \rho_{\mathrm{SW}}(F)$, i.e. $\sigma_{\mathrm{SW}}(F) \subseteq [0, \infty)$.

We point out that Lemma 9.5 (c) provides a striking similarity to a well-known result in linear spectral theory: *a bounded linear operator L in a complex Hilbert space is selfadjoint if and only if every scalar $\lambda \in \mathbb{K} \setminus \mathbb{R}$ belongs to $\rho(L)$ and*

$$\|(\lambda I - F)^{-1}\| \leq \frac{1}{|\mathrm{Im}\,\lambda|}.$$

This analogy between linear and monotone operators suggests to introduce a class of operators which have some spectral properties in common. For a multivalued operator $F \colon H \to 2^H$ in a Hilbert space H over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and $\lambda \in \mathbb{K}$ we put

$$\begin{aligned}\gamma(\lambda; F) := \inf\{\gamma \in \mathbb{R} :\ 0 \leq \gamma\|x - y\|^2 - 2\,\mathrm{Re}\langle\lambda(x - y), u - v\rangle \\ + \|u - v\|^2 \text{ for all } (x, u), (y, v) \in \Gamma(F)\},\end{aligned} \tag{9.33}$$

and call F in the sequel *λ-polytone* if $\gamma(\lambda; F) < |\lambda|^2$. Moreover, a λ-polytone operator F is called *maximal λ-polytone* if F has no proper λ-polytone extension. It is easy to see that every maximal λ-polytone operator has a closed graph. In the next example we see what the characteristic (9.33) gives for linear (singlevalued) operators.

Example 9.5. Let H be a Hilbert space, $L \in \mathfrak{L}(H)$, and $\lambda \in \rho(L)$. By linearity, we may then restrict ourselves to $x \in S(H)$, $u = Lx$, and $y = v = \theta$ in the calculation of (9.33) and get

$$\begin{aligned}\gamma(\lambda; L) &:= \inf\{\gamma \in \mathbb{R} : \; 0 \le \gamma\|x\|^2 - 2\operatorname{Re}\langle\lambda x, Lx\rangle + \|Lx\|^2 \text{ for all } x \in X\} \\ &= \sup_{\|x\|=1}\left[2\operatorname{Re}\langle\lambda x, Lx\rangle - \|Lx\|^2\right] \\ &= \sup_{\|x\|=1}\left[|\lambda|^2 - \|\lambda x - Lx\|^2\right] \\ &= |\lambda|^2 - \inf_{\|x\|=1}\|\lambda x - Lx\|^2 = |\lambda|^2 - \|(\lambda I - L)^{-1}\|^{-2} \\ &= |\lambda|^2 - \|R(\lambda; L)\|^{-2},\end{aligned}$$

since $\lambda I - L$ is a linear isomorphism. ♡

The following Lemma 9.6 generalizes the previous result and emphasizes the importance of polytone operators in the nonlinear theory. It implies, in particular, that an operator F is monotone if and only if $\gamma(\lambda; F) \le 0$ for each $\lambda < 0$.

Lemma 9.6. *An operator $F \colon H \to 2^H$ is λ-polytone if and only if the resolvent operator $R(\lambda; F) = (\lambda I - F)^{-1}$ exists on $R(\lambda I - F)$ and is Lipschitz continuous. In this case the equality*

$$\gamma(\lambda; F) = |\lambda|^2 - \frac{1}{[R(\lambda; F)]^2_{\mathrm{Lip}}} \tag{9.34}$$

holds.

Proof. Suppose first that F is λ-polytone, i.e. we can find $\gamma < |\lambda|^2$ such that

$$\begin{aligned}0 &\le \gamma\|x-y\|^2 - 2\operatorname{Re}\langle\lambda(x-y), u-v\rangle + \|u-v\|^2 \\ &= \|\lambda(x-y) - (u-v)\|^2 + \left(\gamma - |\lambda|^2\right)\|x-y\|^2,\end{aligned}$$

hence

$$\|\lambda(x-y) - (u-v)\|^2 \ge (|\lambda|^2 - \gamma)\|x-y\|^2 \tag{9.35}$$

for all $x, y \in X$, $u \in F(x)$, and $v \in F(y)$. We have to show that $\lambda I - F$ is injective and $(\lambda I - F)^{-1}$ is Lipschitz continuous on $R(\lambda I - F)$. So assume that $z \in (\lambda I - F)(x) \cap (\lambda I - F)(y)$, i.e. $z = \lambda x - u = \lambda y - v$ for some $u \in F(x)$ and $v \in F(y)$. Putting this into (9.35) yields

$$0 = \|z - z\|^2 = \|(\lambda x - u) - (\lambda y - v)\|^2 \ge (|\lambda|^2 - \gamma)\|x - y\|^2.$$

Consequently, $x = y$, since $|\lambda|^2 - \gamma > 0$, and so $\lambda I - F$ is in fact injective. Now, putting $\lambda x - u =: z$ and $\lambda y - v =: w$ for $u \in F(x)$ and $v \in F(y)$, we have $z, w \in R(\lambda I - F)$ and

$$\frac{1}{|\lambda|^2 - \gamma}\|z - w\|^2 \geq \|x - y\|^2 = \|R(\lambda; F)(z) - R(\lambda; F)(w)\|^2,$$

by (9.35). This shows that

$$[R(\lambda; F)]_{\mathrm{Lip}}^2 \leq \frac{1}{|\lambda|^2 - \gamma},$$

hence $\gamma \geq |\lambda|^2 - [R(\lambda; F)]_{\mathrm{Lip}}^{-2}$. Since $\gamma(\lambda; F)$ is the infimum of all possible values for γ, we have

$$\gamma(\lambda; F) \geq |\lambda|^2 - \frac{1}{[R(\lambda; F)]_{\mathrm{Lip}}^2}$$

as well. But one may also take $\gamma = |\lambda|^2 - [R(\lambda; F)]_{\mathrm{Lip}}^{-2}$ in (9.35), and so in fact the equality (9.34) holds.

Conversely, if the resolvent operator $R(\lambda; F)$ exists on $R(\lambda I - F)$ and is Lipschitz continuous, one may show by the same reasoning that F is λ-polytone with (9.34). □

We summarize some monotonicity and polytonicity properties of $F\colon H \to 2^H$ in the following Table 9.1, where throughout $\lambda < 0$. Afterwards we illustrate the previous two lemmas by means of two simple scalar examples.

Table 9.1

$$\begin{array}{ccc}
F \text{ maximal monotone} & \Longrightarrow & F \text{ monotone} \\
\Downarrow & & \Downarrow \\
R(\lambda; F) \in \mathfrak{Lip}(H, H) & \Longrightarrow & R(\lambda; F) \in \mathfrak{Lip}(R(\lambda I - F), H) \\
\Updownarrow & & \Updownarrow \\
F \text{ maximal } \lambda\text{-polytone} & \Longrightarrow & F\ \lambda\text{-polytone}
\end{array}$$

Example 9.6. In $H = \mathbb{R}$ consider the multivalued operator

$$F(x) = \begin{cases} \{x - 1\} & \text{if } x < 0, \\ \{-1, 1\} & \text{if } x = 0, \\ \{x + 1\} & \text{if } x > 0. \end{cases} \tag{9.36}$$

It is easy to see that $F\colon \mathbb{R} \to 2^{\mathbb{R}}$ is monotone, and that (9.30) is true for all $(x, p) \in \Gamma(F)$, $(y, q) \in \Gamma(F)$, and $\mu > 0$. For $\lambda < 0$, the resolvent operator $R(\lambda; F) = (\lambda I - F)^{-1}$ exists on $R(\lambda I - F) = \mathbb{R} \setminus (-1, 1)$ and has the form

$$R(\lambda; F)(u) = \begin{cases} \frac{u+1}{\lambda-1} & \text{if } u \le -1, \\ \frac{u-1}{\lambda-1} & \text{if } u \ge 1. \end{cases} \tag{9.37}$$

Moreover, one easily sees that

$$[R(\lambda; F)]_{\text{Lip}} = \frac{1}{1-\lambda} \le \frac{1}{|\lambda|} \quad (\lambda < 0),$$

in accordance with Lemma 9.5 (c).

On the other hand, F is *not* maximal monotone, as may be seen by checking either (b) or (d) in Lemma 9.5. For instance, this follows from the fact that $R(I + F) = \mathbb{R} \setminus (-1, 1)$.

Let us now calculate the characteristic $\gamma(\lambda; F)$ for the operator (9.36). For $x, y \in \mathbb{R}$ we have

$$\begin{aligned} \gamma(\lambda; F) &= \inf\{\gamma \in \mathbb{R} : 0 \le \gamma(x-y)^2 - 2\lambda(x-y)^2 + (x-y)^2\} \\ &= \inf\{\gamma \in \mathbb{R} : \gamma - 2\lambda + 1 \ge 0\} \\ &= 2\lambda - 1. \end{aligned}$$

The same result may be obtained from (9.34); in fact,

$$\gamma(\lambda; F) = \lambda^2 - (1-\lambda)^2 = 2\lambda - 1.$$

In particular, $\gamma(\lambda; F) < -1 < 0$ for $\lambda < 0$. ♡

Example 9.7. We modify the previous example by "filling the gap" at zero, i.e. by putting

$$F(x) = \begin{cases} \{x-1\} & \text{if } x < 0, \\ [-1, 1] & \text{if } x = 0, \\ \{x+1\} & \text{if } x > 0. \end{cases} \tag{9.38}$$

Now $F\colon \mathbb{R} \to 2^{\mathbb{R}}$ is even maximal monotone, as may be proved by checking condition (b) of Lemma 9.5. Alternatively, it suffices to observe that $R(I + F) = \mathbb{R}$ and use Lemma 9.5 (d). The resolvent operator $R(\lambda; F) = (\lambda I - F)^{-1}$ is now defined on the whole real axis and has the form

$$R(\lambda; F)(u) = \begin{cases} \frac{u+1}{\lambda-1} & \text{if } u \le -1, \\ 0 & \text{if } -1 < u < 1, \\ \frac{u-1}{\lambda-1} & \text{if } u \ge 1. \end{cases} \tag{9.39}$$

Moreover, the formulas for $[R(\lambda; F)]_{\text{Lip}}$ and $\gamma(\lambda; F)$ are the same as in Example 9.6. ♡

Lemma 9.6 may be used to characterize the Singhof–Weyer spectrum explicitly in terms of maximal polytonicity.

Theorem 9.7. *A scalar $\lambda \in \mathbb{K}$ belongs to $\rho_{SW}(F)$ if and only if F is maximal λ-polytone.*

Proof. Let $\lambda \in \rho_{SW}(F)$. We already know that the operator F is λ-polytone for these λ, by Lemma 9.6, but we still have to prove the maximality of F. So let $\tilde{F}$ be a λ-polytone extension of F which means that $\Gamma(\tilde{F}) \supseteq \Gamma(F)$; we claim that $\Gamma(\tilde{F}) = \Gamma(F)$.

Fix $(\tilde{x}, \tilde{u}) \in \Gamma(\tilde{F})$. Since $\lambda I - F$ is surjective, we find $(x, u) \in \Gamma(F)$ such that $\lambda x - u = \lambda\tilde{x} - \tilde{u}$. Moreover, the Lipschitz continuity of $R(\lambda; F)$ implies

$$0 = [R(\lambda; F)]_{\text{Lip}} \|\lambda x - u - \lambda\tilde{x} + \tilde{u}\| \geq \|x - \tilde{x}\|,$$

so $\tilde{x} = x$ and $\tilde{u} = u$, hence $(\tilde{x}, \tilde{u}) \in \Gamma(F)$ as claimed.

Conversely, suppose now that F is maximal λ-polytone. We already know from Lemma 9.6 that $\lambda I - F$ is then invertible with Lipschitz continuous inverse $R(\lambda; F)$, so it remains to show that $\lambda I - F$ is surjective. Let $u \in H$ be arbitrary; we have to find $x \in H$ such that $u \in \lambda x - F(x)$. We may choose $x \in H$ such that

$$0 \leq \gamma(\lambda; F)\|x - y\|^2 - 2\,\mathrm{Re}\langle \lambda(x - y), \lambda x - u - v\rangle + \|\lambda x - u - v\|^2$$

for all $(y, v) \in \Gamma(F)$. But this means that the operator $\tilde{F}$ defined by

$$\tilde{F}(y) := \begin{cases} F(x) & \text{if } y \neq x, \\ F(x) \cup \{\lambda x - u\} & \text{if } y = x \end{cases}$$

is also λ-polytone with $\gamma(\lambda; \tilde{F}) = \gamma(\lambda; F)$, by definition (9.33). Since F is maximal, by assumption, we conclude that $\tilde{F} = F$, hence $F(x) \cup \{\lambda x - u\} = F(x)$ or $\lambda x - u \in F(x)$. □

Example 9.8. Let F be again the maximal monotone operator (9.38) from Example 9.7. Since $\gamma(\lambda; F) = 2\lambda - 1$, we see that F is λ-polytone if and only if $2\lambda - 1 < \lambda^2$, i.e. $\lambda \neq 1$. For these values of λ, F is even maximal λ-polytone, so that

$$\sigma_{SW}(F) = \{1\}.$$

Of course, one may get the same conclusion by adopting directly the definition (9.27) of the Singhof–Weyer spectrum. Indeed, for $\lambda \neq 1$ the operator $\lambda I - F$ is onto, and its inverse (9.39) is Lipschitz continuous on the whole real axis. ♡

We close this section with a further example involving a non-monotone operator.

Example 9.9. Let $H = \mathbb{R}$ and let $F : \mathbb{R} \to 2^{\mathbb{R}}$ be the "pitchfork" defined by

$$F(x) = \begin{cases} \{0\} & \text{if } x < 0, \\ \{\pm\sqrt{x}\} & \text{if } 0 \le x \le 1, \\ \{\pm 1\} & \text{if } x > 1. \end{cases}$$

Then F is not monotone, since $\langle x, -\sqrt{x}\rangle < 0$ for $0 < x < 1$. However, a straightforward calculation shows that F is 0-polytone with $\gamma(0; F) = -1/4$.

Now, an easy calculation shows that $\gamma(\lambda; F) = \infty$ for all $\lambda \neq 0$, and so F cannot be maximal λ-polytone for these λ. Since the Singhof–Weyer spectrum is closed, we have $\sigma_{SW}(F) = \mathbb{R}$. The same result may be obtained by observing that $\lambda I - F$ is not injective for $\lambda \neq 0$ and not surjective for $\lambda = 0$. However, F is invertible on $R(F) = [-1, 1]$ with inverse $F^{-1}(y) = y^2$. Putting $\lambda = 0$ and $[R(0; F)]_{\mathrm{Lip}} = [F^{-1}]_{\mathrm{Lip}} = 2$ into (9.34) yields $\gamma(0; F) = -1/4$ as before. ♡

9.5 The Weber spectrum

In the previous three chapters we have studied various spectra for operators of the form $\lambda I - F$, where F is a continuous operator with some additional properties. As we have seen, for each spectral theory there is some associated eigenvalue theory dealing, in a sense to be made precise, with nontrivial solutions of the equation

$$F(x) = \lambda x. \tag{9.40}$$

A comparison of these different notions of spectral values and eigenvalues may be found in Table 8.6, all possible inclusions between them in Table 8.8.

In this connection, the following question arises quite naturally: *why should we study equation* (9.40) *for nonlinear operators*? After all, this equation is too much modelled on the linear case, and there is no reasonable justification for using the identity operator in (9.40) when F is nonlinear. Instead, it seems more reasonable to replace (9.40) by an equation of the form

$$F(x) = \lambda J(x), \tag{9.41}$$

where J is some operator which is "well-behaved" (e.g., a homeomorphism), on the one hand, but takes into account the specific properties of the nonlinearity F, on the other. Passing from (9.40) to (9.41) has two essential advantages: it takes into account the nonlinear structure of the operator F, and it makes it possible to study even operators between different spaces. Of course, one should then also modify the numerical characteristics for F which occur in the description of the different spectra we studied so far. In what follows we will call every $\lambda \in \mathbb{K}$ for which (9.41) has a nontrivial solution, an *eigenvalue* of the pair (J, F).

The first attempt to define such a modification for the Furi–Martelli–Vignoli spectrum is apparently due to Weber; we will describe the corresponding spectrum in this section. Throughout, let $\varphi\colon [0,\infty) \to [0,\infty)$ a continuous function such that $\varphi(t) = 0$ if and only if $t = 0$. Given $F \in \mathfrak{C}(X, Y)$, we put

$$[F]_Q^\varphi := \limsup_{\|x\|\to\infty} \frac{\|F(x)\|}{\varphi(\|x\|)} \tag{9.42}$$

and

$$[F]_q^\varphi := \liminf_{\|x\|\to\infty} \frac{\|F(x)\|}{\varphi(\|x\|)}. \tag{9.43}$$

In the special case when $\varphi(t) = t^\tau$ for some $\tau > 0$ we write τ instead of φ, i.e.

$$[F]_Q^\tau := \limsup_{\|x\|\to\infty} \frac{\|F(x)\|}{\|x\|^\tau} \tag{9.44}$$

and

$$[F]_q^\tau := \liminf_{\|x\|\to\infty} \frac{\|F(x)\|}{\|x\|^\tau}. \tag{9.45}$$

Moreover, we denote by $\mathfrak{Q}_\varphi(X, Y)$ and $\mathfrak{Q}_\tau(X, Y)$ the class of all continuous operators $F\colon X \to Y$ for which $[F]_Q^\varphi < \infty$ and $[F]_Q^\tau < \infty$, respectively. Of course, in case $\tau = 1$ we get the old characteristics (2.3) and (2.4) and the class $\mathfrak{Q}_1(X, Y) = \mathfrak{Q}(X, Y)$.

Throughout this section, we will assume that F and J are continuous operators from $\mathfrak{Q}_\varphi(X, Y)$ satisfying $F(\theta) = J(\theta) = \theta$. Given two such operators, we call the set

$$\rho_W^\varphi(J, F) := \{\lambda \in \mathbb{K} : [\lambda J - F]_q^\varphi > 0\} \tag{9.46}$$

the *Weber resolvent set* and its complement

$$\sigma_W^\varphi(J, F) := \mathbb{K} \setminus \rho_W^\varphi(J, F) \tag{9.47}$$

the *Weber spectrum* of the pair (J, F) (with respect to φ). In case $X = Y$, $J = I$, and $\varphi(t) = t$ we have of course

$$\sigma_W^\varphi(J, F) = \sigma_q(F),$$

i.e. the Weber spectrum reduces to the set of asymptotic eigenvalues of F (see (2.29)). In case $\varphi(t) = t^\tau$ with some $\tau > 0$, we write $\rho_W^\tau(J, F)$ instead of $\rho_W^\varphi(J, F)$ and $\sigma_W^\tau(J, F)$ instead of $\sigma_W^\varphi(J, F)$; so, by definition,

$$\sigma_W^\tau(J, F) = \{\lambda \in \mathbb{K} : \liminf_{\|x\|\to\infty} \frac{\|\lambda J(x) - F(x)\|}{\|x\|^\tau} = 0\}. \tag{9.48}$$

Finally, it is reasonable to call the set

$$\sigma_p(J, F) := \{\lambda \in \mathbb{K} : \lambda \text{ eigenvalue of } (J, F)\} \tag{9.49}$$

the *point spectrum* of the pair (J, F). Clearly, $\sigma_p(I, F) = \sigma_p(F)$ is simply the point spectrum (3.18) of F in the classical sense.

It is not hard to prove that the estimates

$$|\lambda|\,[J]_{\mathrm{q}}^{\varphi} \le [F]_{\mathrm{Q}}^{\varphi}, \quad [F]_{\mathrm{q}}^{\varphi} \le |\lambda|\,[J]_{\mathrm{Q}}^{\varphi} \tag{9.50}$$

hold for every $\lambda \in \sigma_{\mathrm{W}}^{\varphi}(J, F)$. Consequently, the Weber spectrum (9.47) is always contained in the bounded annular region with inner radius $[F]_{\mathrm{q}}^{\varphi}/[J]_{\mathrm{Q}}^{\varphi}$ (if $[J]_{\mathrm{Q}}^{\varphi} > 0$) and outer radius $[F]_{\mathrm{Q}}^{\varphi}/[J]_{\mathrm{q}}^{\varphi}$ (if $[J]_{\mathrm{q}}^{\varphi} > 0$). In the special case $X = Y$, $J = I$, and $\varphi(t) = t$ this is of course nothing else but the inclusion (2.33), since $[I]_{\mathrm{Q}} = [I]_{\mathrm{q}} = 1$.

The following Theorem 9.8 is a straightforward generalization of a part of Theorem 2.4. In the subsequent Theorem 9.9 we establish some connections between the spectrum (9.47) and the point spectrum (9.49).

Theorem 9.8. *The spectrum $\sigma_{\mathrm{W}}^{\varphi}(J, F)$ is closed, and even compact in case $[J]_{\mathrm{q}}^{\varphi} > 0$.*

Proof. It is easy to see that Proposition 2.2 (d) carries over to the characteristics (9.42) and (9.43) in the sense that $|[F]_{\mathrm{q}}^{\varphi} - [G]_{\mathrm{q}}^{\varphi}| \le [F - G]_{\mathrm{Q}}^{\varphi}$ for every $F, G \in \mathfrak{Q}_{\varphi}(X, Y)$. In particular,

$$|[\lambda J - F]_{\mathrm{q}}^{\varphi} - [\mu J - F]_{\mathrm{q}}^{\varphi}| \le |\lambda - \mu|\,[J]_{\mathrm{Q}}^{\varphi}$$

for $F, J \in \mathfrak{Q}_{\varphi}(X, Y)$, and this immediately implies that the resolvent set $\rho_{W}^{\varphi}(J, F)$ is open, by definition (9.46).

As (9.50) shows, under the hypothesis $[J]_{\mathrm{q}}^{\varphi} > 0$ every element $\lambda \in \sigma_{\mathrm{W}}^{\varphi}(J, F)$ satisfies the upper estimate $|\lambda| \le [F]_{\mathrm{Q}}^{\varphi}/[J]_{\mathrm{q}}^{\varphi}$, and so $\sigma_{\mathrm{W}}^{\varphi}(J, F)$ is even compact. □

Theorem 9.9. *Let $F, J \in \mathfrak{Q}_{\tau}(X, Y)$ be τ-homogeneous operators. Then the following holds.*

(a) *Every eigenvalue of (J, F) is a spectral value, i.e.*

$$\sigma_{\mathrm{p}}(J, F) \subseteq \sigma_{\mathrm{W}}^{\tau}(J, F).$$

(b) *Conversely, if F is compact and J is a homeomorphism between X and Y, then*

$$\sigma_{\mathrm{W}}^{\tau}(J, F) \setminus \{0\} \subseteq \sigma_{\mathrm{p}}(J, F), \tag{9.51}$$

i.e. every nonzero spectral value is an eigenvalue of (J, F).

Proof. Fix $\lambda \in \sigma_{\mathrm{p}}(J, F)$, and let $x \neq \theta$ be a solution of (9.41). Then the sequence $(nx)_n$ is unbounded and

$$\frac{\|\lambda J(nx) - F(nx)\|}{\|nx\|^{\tau}} = \frac{\|\lambda J(x) - F(x)\|}{\|x\|^{\tau}} = 0,$$

by homogeneity. This shows that $\lambda \in \sigma_{\mathrm{W}}^{\tau}(J, F)$.

Now suppose that F is compact, J is a homeomorphism, and $\lambda \in \sigma_{\mathrm{W}}^{\tau}(J, F) \setminus \{0\}$, which means that $[\lambda J - F]_{\mathrm{q}}^{\tau} = 0$. We may find an unbounded sequence $(x_n)_n$ such that

$$\frac{\|\lambda J(x_n) - F(x_n)\|}{\|x_n\|^{\tau}} \to 0 \quad (n \to \infty).$$

Then the sequence $(e_n)_n$ defined by $e_n := x_n/\|x_n\|$ belongs to $S(X)$, and satisfies

$$\|\lambda J(e_n) - F(e_n)\| = \frac{\|\lambda J(x_n) - F(x_n)\|}{\|x_n\|^{\tau}} \to 0, \tag{9.52}$$

as $n \to \infty$, again by homogeneity. Since F is compact, we find a subsequence $(e_{n_k})_k$ of $(e_n)_n$ such that $F(e_{n_k}) \to y$ for some $y \in Y$. From (9.52) and $\lambda \neq 0$ it follows that $J(e_{n_k}) \to \frac{1}{\lambda} y$ as $k \to \infty$. Let $x := J^{-1}(\frac{1}{\lambda} y)$. Then $x \neq \theta$ and $e_{n_k} \to x$ as $k \to \infty$, because J is a homeomorphism, and so

$$\|\lambda J(x) - F(x)\| \leq |\lambda| \, \|J(x) - J(e_{n_k})\| + \|\lambda J(e_{n_k}) - F(e_{n_k})\| + \|F(e_{n_k}) - F(x)\| \to 0$$

as $k \to \infty$. We conclude that λ is an eigenvalue of (J, F), and so we are done. □

In case $X = Y$, $J = I$, and $\tau = 1$, Theorem 9.9 (b) is of course (a weak form of) the discreteness result obtained earlier for the Furi–Martelli–Vignoli spectrum, see Theorem 6.12.

9.6 Spectra for homogeneous operators

During our study of various spectra and phantoms in Chapters 6–8 we have seen that one gets particularly sharp results for operators which are τ-homogeneous in the sense of (7.29). In this section we will develop a more systematic spectral theory for such operators. To this end, we first have to modify the characteristics from Chapter 2 in order to take into account the τ-homogeneity of the operators involved.

For the characteristics (2.3) and (2.4) this has been done already in (9.44) and (9.45). Similarly, let us put

$$[F]_{\mathrm{B}}^{\tau} := \sup_{x \neq \theta} \frac{\|F(x)\|}{\|x\|^{\tau}}, \tag{9.53}$$

$$[F]_{\mathrm{b}}^{\tau} := \inf_{x \neq \theta} \frac{\|F(x)\|}{\|x\|^{\tau}}, \tag{9.54}$$

$$[F]_{\mathrm{A}}^{\tau} := \inf\{k > 0 : \alpha(F(M)) \leq k\alpha(M)^{\tau} \ (M \subset X \text{ bounded})\}, \tag{9.55}$$

and

$$[F]_{\mathrm{a}}^{\tau} := \sup\{k > 0 : \alpha(F(M)) \geq k\alpha(M)^{\tau} \ (M \subset X \text{ bounded})\}. \tag{9.56}$$

Of course, for $\tau = 1$ we get the old characteristics (2.6), (2.7), (2.13) and (2.15). The following example shows, however, that this extension is not as harmless as it looks like at first glance; in fact, calculating the characteristics (9.55) and (9.56), say, may be nontrivial in case $\tau \neq 1$.

Example 9.10. Let X be an infinite dimensional Banach space and $F_p\colon X \to X$ be defined by

$$F_p(x) := \|x\|^{p-1}x \quad (p \geq 1).$$

In Example 2.33 we have considered this operator for $p = 2$ and have shown that $[F_2]_{\mathrm{a}} = [F_2]_{\mathrm{a}}^1 = 0$. More generally, from the obvious relation $F_p(S_r(X)) = S_{r^p}(X)$ it follows that

$$[F_p]_{\mathrm{a}}^{\tau} = 0, \quad [F_p]_{\mathrm{A}}^{\tau} = \infty, \quad (\tau \neq p).$$

Moreover, since $F_1 = I$ we clearly have $[F_1]_{\mathrm{a}}^1 = [F_1]_{\mathrm{A}}^1 = 1$. So we have to calculate only the case $\tau = p > 1$. We claim that

$$[F_p]_{\mathrm{a}}^p = 1, \quad [F_p]_{\mathrm{A}}^p = \infty \quad (p > 1).$$

First of all, it is not hard to see that $[F_p]_{\mathrm{A}}^p = \infty$ for $p > 1$. Indeed, fix $x_0 \in S(X)$ and consider for $\varepsilon > 0$ the (noncompact) set

$$M_\varepsilon := \{x \in S(X) : \|x - x_0\| \leq \varepsilon\}.$$

Since $F_p(M_\varepsilon) = M_\varepsilon$ we see that

$$[F_p]_{\mathrm{A}}^p \geq \sup_{\varepsilon>0} \frac{\alpha(F_p(M_\varepsilon))}{\alpha(M_\varepsilon)^p} = \sup_{\varepsilon>0} \frac{1}{\alpha(M_\varepsilon)^{p-1}} \geq \lim_{\varepsilon\downarrow 0} \varepsilon^{-(p-1)} = \infty.$$

The proof of the equality $[F_p]_{\mathrm{a}}^p = 1$ is more involved. Let $M \subseteq X$ be bounded by $1/N$ for some $N \in \mathbb{N}$. For $n = N, N+1, \ldots$ put

$$R_n := \{x \in X : \tfrac{1}{n+1} \leq \|x\| \leq \tfrac{1}{n}\}$$

and $M_n := M \cap R_n$. The inclusion

$$\frac{1}{(n+1)^{p-1}} M_n \subseteq \mathrm{co}(F_p(M_n) \cup \{\theta\})$$

implies that

$$\alpha(F_p(M_n)) = \alpha(\mathrm{co}(F_p(M_n) \cup \{\theta\})) \geq \frac{1}{(n+1)^{p-1}}\alpha(M_n).$$

Since $\alpha(M_n) \leq \alpha(R_n) \leq \frac{1}{n}$, we obtain

$$\alpha(F_p(M)) \geq \alpha(F_p(M_n)) \geq \frac{n^{p-1}}{(n+1)^{p-1}}\alpha(M_n)^p \geq \frac{N^{p-1}}{(N+1)^{p-1}}\alpha(M_n)^p.$$

Putting $s := \sup\{\alpha(M_n) : n = N, N+1, \dots\}$, we thus have

$$\alpha(F_p(M)) \geq \left(\frac{N}{N+1}\right)^{p-1} s^p.$$

Now, for given $\varepsilon > s$ choose $\hat{N} > N$ such that $\hat{N}\varepsilon > 1$. For each $n = N, \dots, \hat{N}$ there is a finite ε-net $\mathscr{N}_n$ for M_n (by definition of s). Then $\mathscr{N} = \mathscr{N}_N \cup \dots \cup \mathscr{N}_{\hat{N}} \cup \{\theta\}$ is a finite ε-net for $M_N \cup M_{N+1} \cup \dots = M$. Consequently, $\alpha(M) \leq \varepsilon$, and so also $\alpha(M) \leq s$. It follows that

$$\alpha(M)^p \leq s^p \leq \left(\frac{N+1}{N}\right)^{p-1} \alpha(F_p(M)).$$

We thus have proved that

$$\inf\left\{\frac{\alpha(F_p(M))}{\alpha(M)^p} : M \subseteq B_{1/N}(X)\right\} \geq \left(\frac{N}{N+1}\right)^{p-1}.$$

Since F_p is p-homogeneous, the left-hand side of this inequality is precisely $[F_p]_{\mathrm{a}}^p$, i.e.

$$[F_p]_{\mathrm{a}}^p \geq \left(\frac{N}{N+1}\right)^{p-1} \qquad (N = 1, 2, \dots),$$

and so we may conclude that $[F_p]_{\mathrm{a}}^p \geq 1$. On the other hand, the fact that F_p maps the unit sphere $S(X)$ into itself trivially implies that $[F_p]_{\mathrm{a}}^p = 1$, and so we are done. ♡

Next, we modify some of the regularity properties of the preceding chapters by means of the above characteristics. We call an operator $F \in \mathfrak{C}(X, Y)$ (k, τ)-*stably solvable* if, given any continuous operator $G\colon X \to Y$ with $[G]_{\mathrm{A}}^{\tau} \leq k$ and $[G]_{\mathrm{Q}}^{\tau} \leq k$, the equation $F(x) = G(x)$ has a solution $x \in X$. In case $k = 0$ we will call F simply τ-*stably solvable*. Generalizing (6.4) we put

$$\mu^{\tau}(F) := \inf\{k : k \geq 0,\ F \text{ is not } (k, \tau)\text{-stably solvable}\} \tag{9.57}$$

Moreover, denoting as in Chapter 7 by $\mathfrak{OBC}(X)$ the family of all open, bounded, connected subsets Ω of X with $\theta \in \Omega$, we call an operator $F \in \mathfrak{C}(X, Y)$ (k, τ)-*epi* on $\overline{\Omega}$ if $F(x) \neq \theta$ on $\partial\Omega$ and, for any continuous operator $G\colon \overline{\Omega} \to Y$ satisfying $[G]_{\mathrm{A}}^{\tau} \leq k$ and $G(x) \equiv \theta$ on $\partial\Omega$, the equation $F(x) = G(x)$ has a solution $x \in \Omega$. In case $k = 0$ this gives of course the old definition of epi operators F, since $[G]_{\mathrm{A}}^{\tau} = 0$ for all $\tau > 0$ if and only if G is compact. Generalizing (7.2) and (7.3) we put

$$\nu_{\Omega}^{\tau}(F) := \inf\{k : k \geq 0,\ F \text{ is not } (k, \tau)\text{-epi on } \overline{\Omega}\} \tag{9.58}$$

and

$$\nu^{\tau}(F) = \inf_{\Omega \in \mathfrak{OBC}(X)} \nu_{\Omega}^{\tau}(F). \tag{9.59}$$

Now we are ready to define the modified spectra. As orientation, we imitate the definition (9.49) of the point spectrum of a pair (J, F). So for $J, F \in \mathfrak{C}(X, Y)$ we put

$$\sigma_{\mathrm{q}}^{\tau}(J, F) := \{\lambda \in \mathbb{K} : [\lambda J - F]_{\mathrm{q}}^{\tau} = 0\}, \tag{9.60}$$

$$\sigma_{\mathrm{b}}^{\tau}(J, F) := \{\lambda \in \mathbb{K} : [\lambda J - F]_{\mathrm{b}}^{\tau} = 0\}, \tag{9.61}$$

$$\sigma_{\mathrm{a}}^{\tau}(J, F) := \{\lambda \in \mathbb{K} : [\lambda J - F]_{\mathrm{a}}^{\tau} = 0\}, \tag{9.62}$$

$$\sigma_{\delta}^{\tau}(J, F) := \{\lambda \in \mathbb{K} : \lambda J - F \text{ is not } \tau\text{-stably solvable}\}, \tag{9.63}$$

$$\sigma_{\mu}^{\tau}(J, F) := \{\lambda \in \mathbb{K} : \mu^{\tau}(\lambda J - F) = 0\}, \tag{9.64}$$

and

$$\sigma_{\nu}^{\tau}(J, F) := \{\lambda \in \mathbb{K} : \nu^{\tau}(\lambda J - F) = 0\}. \tag{9.65}$$

Moreover, we call the set

$$\sigma_{\mathrm{FMV}}^{\tau}(J, F) := \sigma_{\delta}^{\tau}(J, F) \cup \sigma_{\mathrm{a}}^{\tau}(J, F) \cup \sigma_{\mathrm{q}}^{\tau}(J, F) \tag{9.66}$$

the *FMV-τ-spectrum*, the set

$$\sigma_{\mathrm{AGV}}^{\tau}(J, F) := \sigma_{\mu}^{\tau}(J, F) \cup \sigma_{\mathrm{q}}^{\tau}(J, F) \tag{9.67}$$

the *AGV-τ-spectrum*, and the set

$$\sigma_{\mathrm{F}}^{\tau}(J, F) := \sigma_{\nu}^{\tau}(J, F) \cup \sigma_{\mathrm{a}}^{\tau}(J, F) \cup \sigma_{\mathrm{b}}^{\tau}(J, F) \tag{9.68}$$

the *Feng τ-spectrum* of the operator pair (J, F). Of course, in case $X = Y$, $J = I$, and $\tau = 1$ these are the old spectra (6.10), (6.27), and (7.19), i.e.

$$\sigma_{\mathrm{FMV}}^{1}(I, F) = \sigma_{\mathrm{FMV}}(F), \quad \sigma_{\mathrm{AGV}}^{1}(I, F) = \sigma_{\mathrm{AGV}}(F), \quad \sigma_{\mathrm{F}}^{1}(I, F) = \sigma_{\mathrm{F}}(F).$$

Moreover, the subspectrum (9.60) is nothing else than the Weber spectrum (9.47) for $\varphi(t) = t^{\tau}$.

Now we are going to extend the phantoms (8.5) and (8.6); as before, we get the old definition in case $X = Y$, $J = I$, and $\tau = 1$.

Given $F, J \in \mathfrak{C}(X, Y)$, we call the set

$$\phi^{\tau}(J, F) = \Big\{\lambda \in \mathbb{K} : \nu_{\Omega}^{\tau}(\lambda J - F) = 0 \text{ or } \inf_{x \in \partial\Omega} \|(\lambda J - F)(x)\| = 0 \text{ for all } \Omega \in \mathfrak{OBC}(X)\Big\} \tag{9.69}$$

the *τ-phantom* and the set

$$\Phi^{\tau}(J, F) = \Big\{\lambda \in \mathbb{K} : \lambda J - F \text{ is not epi on } \overline{\Omega} \text{ or } [(\lambda J - F)|_{\overline{\Omega}}]_{\mathrm{a}}^{\tau} = 0 \text{ for all } \Omega \in \mathfrak{OBC}(X)\Big\} \tag{9.70}$$

the *large τ-phantom* of the pair (J, F). As in Chapter 8, one may then show that both τ-phantoms are closed, and that the inclusions

$$\phi^\tau(J, F) \subseteq \sigma^\tau_{\mathrm{AGV}}(J, F), \tag{9.71}$$

$$\Phi^\tau(J, F) \subseteq \sigma^\tau_{\mathrm{FMV}}(J, F) \subseteq \sigma^\tau_{\mathrm{F}}(J, F) \tag{9.72}$$

and

$$\phi^\tau(J, F) \subseteq \Phi^\tau(J, F) \tag{9.73}$$

are true; compare this with (8.7)–(8.9).

It turns out that the properties of the above general spectra and phantoms, as well as the relations between them, are essentially the same as in the special case $X = Y$ and $J = I$. One has only to add some technical assumptions on the operator J which are automatically fulfilled for $J = I$. We just summarize without proof in the following

Theorem 9.10. *Let $J, F \in \mathfrak{C}(X, Y)$ with $[J]^\tau_{\mathrm{A}} < \infty$ for some $\tau > 0$ be given. Then the following holds.*

(a) *The spectra $\sigma^\tau_{\mathrm{FMV}}(J, F)$ and $\sigma^\tau_{\mathrm{AGV}}(J, F)$ are closed if $[J]^\tau_{\mathrm{Q}} < \infty$.*

(b) *If J is τ-stably solvable, the spectrum $\sigma^\tau_{\mathrm{FMV}}(J, F)$ is bounded by the number*

$$R_1 := \max\left\{\frac{[F]^\tau_{\mathrm{A}}}{[J]^\tau_{\mathrm{a}}}, \frac{[F]^\tau_{\mathrm{Q}}}{[J]^\tau_{\mathrm{q}}}\right\},$$

hence compact if $R_1 < \infty$ and $[J]^\tau_{\mathrm{Q}} < \infty$.

(c) *The spectrum $\sigma^\tau_{\mathrm{AGV}}(J, F)$ is bounded by the number*

$$R_2 := \max\left\{\frac{[F]^\tau_{\mathrm{A}}}{\mu^\tau(J)}, \frac{[F]^\tau_{\mathrm{Q}}}{[J]^\tau_{\mathrm{q}}}\right\},$$

hence compact if $R_2 < \infty$ and $[J]^\tau_{\mathrm{Q}} < \infty$.

(d) *The spectrum $\sigma^\tau_{\mathrm{F}}(J, F)$ is closed if $[J]^\tau_{\mathrm{B}} < \infty$.*

(e) *If J is epi on every $\Omega \in \mathfrak{OBC}(X)$, the spectrum $\sigma^\tau_{\mathrm{F}}(J, F)$ is bounded by the number*

$$R_3 := \max\left\{\frac{[F]^\tau_{\mathrm{A}}}{[J]^\tau_{\mathrm{a}}}, \frac{[F]^\tau_{\mathrm{B}}}{[J]^\tau_{\mathrm{b}}}\right\},$$

hence compact if $R_3 < \infty$ and $[J]^\tau_{\mathrm{B}} < \infty$.

(f) *The phantoms $\phi^\tau(J, F)$ and $\Phi^\tau(J, F)$ are closed.*

(g) *If J is epi on some $\Omega \in \mathfrak{OBC}(X)$, the phantoms $\phi^\tau(J, F)$ and $\Phi^\tau(J, F)$ are bounded by the number*

$$R_4 := \max\left\{\frac{[F|_{\overline{\Omega}}]_A^\tau}{[J|_{\overline{\Omega}}]_a^\tau}, \frac{\sup_{x\in\partial\Omega}\|F(x)\|}{\inf_{x\in\partial\Omega}\|J(x)\|}\right\},$$

hence compact if $R_4 < \infty$.

We point out that the conditions in Theorem 9.10 essentially simplify in the case of classical spectra, i.e. for $J = I$. In fact, in this case we have $[J]_A^1 = [J]_Q^1 = [J]_B^1 = [J]_a^1 = [J]_q^1 = [J]_b^1 = \mu^1(J) = 1$, and so the "spectral radii" in Theorem 9.10 simply become

$$R_1 = R_2 = \max\{[F]_A, [F]_Q\}, \quad R_3 = \max\{[F]_A, [F]_B\},$$
$$R_4 = \max\left\{[F|_{\overline{\Omega}}]_A, \frac{\sup_{x\in\partial\Omega}\|F(x)\|}{\operatorname{dist}(\theta, \partial\Omega)}\right\}.$$

Finally, the various point spectra and asymptotic point spectra also carry over in an obvious way. We have already defined the point spectrum (9.49) and the asymptotic point spectrum (9.60) of (J, F). Analogously to (8.18) we define the *point phantom* of the pair (J, F) by

$$\phi_p(J, F) := \{\lambda \in \mathbb{K} : \lambda \text{ connected eigenvalue of } (J, F)\}, \tag{9.74}$$

where we call λ a *connected eigenvalue* of (J, F) if the nullset $N(\lambda J - F)$ (i.e. the solution set of equation (9.41)) contains an unbounded connected set containing θ. Similarly, we define the *τ-approximate point phantom* of (J, F) by

$$\begin{aligned}\phi_q^\tau(J, F) := \{\lambda \in \mathbb{K} : &\text{ there exist sequences } (F_n)_n \text{ and } (\lambda_n)_n \text{ with} \\ &d_{\mathfrak{V}}^\tau(F_n, F) \to 0, \lambda_n \to \lambda, \text{ and } \lambda_n \in \phi_p(J, F_n)\},\end{aligned} \tag{9.75}$$

where $d_{\mathfrak{V}}^\tau$ denotes the *Väth τ-metric*

$$d_{\mathfrak{V}}^\tau(F, G) := \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{q_k^\tau(F - G)}{1 + q_k^\tau(F - G)} \tag{9.76}$$

with

$$q_k^\tau(F) := \max\left\{[F|_{B_k(X)}]_A^\tau, \sup_{\|x\|\le k}\|F(x)\|\right\}; \tag{9.77}$$

compare this with (8.13) and (8.14). Observe that the set $\sigma_q^\tau(J, F)$ depends on τ, while the sets $\sigma_p(J, F)$ and $\phi_p(J, F)$ do not. It is remarkable that also the $\phi_q^\tau(J, F)$ does not depend on τ, since an analogous characterization as Proposition 8.3 is valid also for $\phi_q^\tau(J, F)$ instead of $\phi_q(F)$. We will use this fact in the proof of Theorem 9.11 below.

Analogously to Table 8.8, we get the following Table 9.2 which illustrates the various connections between all these spectra and phantoms.

Table 9.2

		$\Phi^\tau(J, F)$				$\Phi^\tau(J, F)$		
		∪|				|∩		
$\sigma_{\mathrm{p}}(J, F)$		$\phi^\tau(J, F)$	$\subseteq$	$\sigma^\tau_{\mathrm{AGV}}(J, F)$	$\subseteq$	$\sigma^\tau_{\mathrm{FMV}}(J, F)$	$\subseteq$	$\sigma^\tau_{\mathrm{F}}(J, F)$
∪|		∪|		∪|				∪|
$\phi_{\mathrm{p}}(J, F)$	$\subseteq$	$\phi^\tau_{\mathrm{q}}(J, F)$	$\subseteq$	$\sigma^\tau_{\mathrm{q}}(J, F)$				$\sigma_{\mathrm{p}}(J, F)$

Now we suppose that both F and J are τ-homogeneous for some $\tau > 0$, i.e.

$$F(tx) = t^\tau F(x), \quad J(tx) = t^\tau J(x) \quad (t > 0, x \in X).$$

Operators of this type are the motivation for introducing all the spectra of this section. Indeed, for such operators some of the spectra and phantoms coincide, as we shall show in the next two theorems. Since these theorems are in part new, we shall also give their proofs.

Theorem 9.11. *Let $F, J\colon X \to Y$ be τ-homogeneous for some $\tau > 0$. Then we have*

$$\sigma^\tau_{\mathrm{AGV}}(J, F) = \phi^\tau(J, F) \subseteq \Phi^\tau(J, F) = \sigma^\tau_{\mathrm{FMV}}(J, F) = \sigma^\tau_{\mathrm{F}}(J, F) \tag{9.78}$$

and

$$\sigma_{\mathrm{p}}(J, F) = \phi_{\mathrm{p}}(J, F) \subseteq \phi^\tau_{\mathrm{q}}(J, F) = \sigma^\tau_{\mathrm{q}}(J, F). \tag{9.79}$$

Proof. A comparison with (9.71)–(9.73) shows that for the proof of (9.79) we only have to show that

$$\sigma^\tau_{\mathrm{F}}(J, F) \subseteq \Phi^\tau(J, F), \quad \sigma^\tau_{\mathrm{AGV}}(J, F) \subseteq \phi^\tau(J, F). \tag{9.80}$$

Suppose first that $\lambda \notin \Phi^\tau(J, F)$. Then there exists $\Omega \in \mathfrak{OBC}(X)$ such that $\lambda J - F$ is epi on $\overline{\Omega}$ and $[(\lambda J - F)|_{\overline{\Omega}}]^\tau_{\mathrm{a}} > 0$. As in the proof of Theorem 8.11 and Proposition 7.2 we may conclude that $\lambda J - F$ is also epi on $B(X)$ and $[(\lambda J - F)|_{B(X)}]^\tau_{\mathrm{a}} > 0$. We still have to show that $\lambda \notin \sigma^\tau_{\mathrm{b}}(J, F)$, i.e. $[\lambda J - F]^\tau_b > 0$.

Assume that this is false, i.e. there exists a sequence $(x_n)_n$ in $X \setminus \{\theta\}$ such that $\|\lambda J(x_n) - F(x_n)\| \leq \|x_n\|^\tau/n$. Putting $e_n := x_n/\|x_n\|$ and $M := \{e_1, e_2, e_3, \dots\} \subset S(X)$ we see that

$$\|\lambda J(e_n) - F(e_n)\| = \frac{\|\lambda J(x_n) - F(x_n)\|}{\|x_n\|^\tau} \leq \frac{1}{n} \to 0 \quad (n \to \infty),$$

and thus

$$[(\lambda J - F)|_{B(X)}]^\tau_{\mathrm{a}} \alpha(M)^\tau \leq \alpha((\lambda J - F)(M)) = 0.$$

This shows that $\alpha(M) = 0$, so we may find a subsequence $(e_{n_k})_k$ of $(e_n)_n$ such that $e_{n_k} \to e \in S(X)$ as $k \to \infty$. By continuity, we have $\lambda J(e) - F(e) = \theta$, contradicting the fact that $\lambda J - F$ is epi on $B(X)$. So we have proved that $\lambda \notin \sigma_{\mathrm{b}}^{\tau}(J, F)$, and thus $\lambda \notin \sigma_{\mathrm{F}}^{\tau}(J, F)$, i.e. the left inclusion in (9.80).

Suppose now that $\lambda \notin \phi^{\tau}(J, F)$. Then there exists $\Omega \in \mathfrak{OBC}(X)$ such that $\nu_{\Omega}^{\tau}(\lambda J - F) > 0$ and

$$\inf_{x \in \partial\Omega} \|\lambda J(x) - F(x)\| > 0. \tag{9.81}$$

We claim that (9.81) implies $[\lambda J - F]_q^{\tau} > 0$. In fact, assume that there exists an unbounded sequence $(x_n)_n$ such that, without loss of generality, $\|\lambda J(x_n) - F(x_n)\| \leq \|x_n\|^{\tau}/n$. Since $\Omega \in \mathfrak{OBC}(X)$, we can choose a sequence $(t_n)_n$ of positive real numbers such that $t_n x_n \in \partial\Omega$ for each $n \in \mathbb{N}$. Then

$$\|\lambda J(t_n x_n) - F(t_n x_n)\| = t_n^{\tau} \|\lambda J(x_n) - F(x_n)\| \leq \frac{t_n^{\tau}}{n} \|x_n\|^{\tau} = \frac{1}{n} \|t_n x_n\|^{\tau} \to 0$$

as $n \to \infty$, contradicting (9.81). It remains to prove that $\mu^{\tau}(\lambda J - F) > 0$. Fix a real number k with

$$0 < k < \min\left\{[\lambda J - F]_q^{\tau}, \nu_{\Omega}^{\tau}(\lambda J - F)\right\},$$

and let $G\colon X \to Y$ be continuous with $[G]_A^{\tau} \leq k$ and $[G]_{\mathrm{Q}}^{\tau} \leq k$. Moreover, define for each $n \in \mathbb{N}$ an operator G_n by $G_n(x) := d_n(\|x_n\|)G(x)$, where d_n is the truncation (6.8). Then still $[G_n]_A^{\tau} \leq k$, but in addition $G_n(x) \equiv \theta$ on $S_{2n}(X)$. Since $\nu^{\tau}(\lambda J - F) > k$, by our choice of k, we find for each $n \in \mathbb{N}$ a solution $x_n \in B_{2n}(X)$ of the equation $\lambda J(x) - F(x) = G_n(x)$. We claim that $\|x_m\| < m$ for some $m \in \mathbb{N}$. In fact, otherwise we would have

$$\frac{\|\lambda J(x_n) - F(x_n)\|}{\|x_n\|^{\tau}} = d_n(\|x_n\|)\frac{\|G(x_n)\|}{\|x_n\|^{\tau}} \leq k < [\lambda J - F]_q^{\tau}$$

for sufficiently large n, a contradiction. So we have $\|x_m\| < m$ for some m which implies that $\lambda J(x_m) - F(x_m) = G_m(x_m) = G(x_m)$. We conclude that $\mu^{\tau}(\lambda J - F) > 0$, and so the proof of (9.80) is complete.

Let us now prove (9.79). The inclusions $\phi_{\mathrm{p}}(J, F) \subseteq \sigma_{\mathrm{p}}(J, F)$ and $\phi_{\mathrm{p}}(J, F) \subseteq \phi_{\mathrm{q}}^{\tau}(J, F)$ are of course trivial. Let $\lambda \in \sigma_{\mathrm{p}}(J, F)$, and let $x \neq \theta$ be a solution of equation (9.41). Then the ray $R := \{tx : t \geq 0\}$ is unbounded and connected, contains θ, and belongs to the nullset $N(\lambda J - F)$ of $\lambda J - F$, and so $\lambda \in \phi_{\mathrm{p}}(J, F)$.

Finally we show that $\phi_{\mathrm{q}}^{\tau}(J, F) \subseteq \sigma_{\mathrm{q}}^{\tau}(J, F)$. If λ does not belong to $\sigma_{\mathrm{q}}^{\tau}(J, F)$, we may find $\delta > 0$ and $m \in \mathbb{N}$ such that

$$\|\lambda J(x) - F(x)\| \geq \delta\|x\|^{\tau} \quad (\|x\| \geq m).$$

This estimate guarantees that the operator $\lambda J - F$ is bounded away from zero on $\partial\Omega = S_m(X)$. From Proposition 8.3 (for $\phi_{\mathrm{q}}^{\tau}(J, F)$ rather than $\phi_{\mathrm{q}}(F)$) we conclude that $\lambda \notin \phi_{\mathrm{q}}^{\tau}(J, F)$.

Finally, we have to show that $\sigma_q^\tau(J,F) \subseteq \phi_q^\tau(J,F)$. Fix $\lambda \in \sigma_q^\tau(J,F)$, and let $(x_n)_n$ be an unbounded sequence in $X \setminus \{\theta\}$ such that $\|\lambda J(x_n) - F(x_n)\| \leq \|x_n\|/n$. Then the elements $e_n := x_n/\|x_n\| \in S(X)$ satisfy $\|\lambda J(e_n) - F(e_n)\| \leq \frac{1}{n} \to 0$ as $n \to \infty$. Define $F_n \colon X \to Y$ by

$$F_n(x) := F(x) + \|x\|^\tau(\lambda J(e_n) - F(e_n)).$$

For fixed $m \in \mathbb{N}$ we have then, on the one hand,

$$\sup_{\|x\|\leq m} \|F_n(x) - F(x)\| \leq m^\tau \|\lambda J(e_n) - F(e_n)\| \to 0 \quad (n \to \infty).$$

But we also have $[(F_n - F)|_{B_m(X)}]_A^\tau = 0$, on the other, since $(F_n - F)(B_m(X))$ is one-dimensional and bounded, hence precompact. This shows that $q_m^\tau(F_n - F) \to 0$ as $n \to \infty$, with q_m^τ given by (9.77). It remains to show that $\lambda \in \phi_p(J, F_n)$ for all $n \in \mathbb{N}$. But the ray $R_n := \{te_n : t \geq 0\}$ belongs to the nullset $N(\lambda J - F_n)$, because

$$F_n(te_n) = F(te_n) + \|te_n\|^\tau(\lambda J(e_n) - F(e_n)) = \lambda J(te_n).$$

We conclude that $\lambda \in \phi_q^\tau(J,F)$, and so we are done. □

By Theorem 9.11, Table 9.2 essentially simplifies for τ-homogeneous operators as follows.

Table 9.3

$$\begin{array}{ccccccccc}
 & & \Phi^\tau(J,F) & & & & \Phi^\tau(J,F) & & \\
 & & \cup\!\vert & & & & \Vert & & \\
\sigma_p(J,F) & & \phi^\tau(J,F) & = & \sigma_{AGV}^\tau(J,F) & \subseteq & \sigma_{FMV}^\tau(J,F) & = & \sigma_F^\tau(J,F) \\
\Vert & & \cup\!\vert & & \cup\!\vert & & & & \cup\!\vert \\
\phi_p(J,F) & \subseteq & \phi_q^\tau(J,F) & = & \sigma_q^\tau(J,F) & & & & \sigma_p(J,F)
\end{array}$$

To see what happens with point spectra in the homogeneous case, we extract the sets $\sigma_p(J,F)$, $\sigma_q^\tau(J,F)$, $\phi_p(J,F)$ and $\phi_q^\tau(J,F)$ from Table 9.2 and Table 9.3 and compare them in the general case (left) and the homogeneous case (right).

Table 9.4

$$\begin{array}{ccc}
\sigma_p(J,F) & & \sigma_q^\tau(J,F) \\
\cup\!\vert & & \cup\!\vert \\
\phi_p(J,F) & \subseteq & \phi_q^\tau(J,F)
\end{array}
\qquad\qquad
\begin{array}{ccc}
\sigma_p(J,F) & \subseteq & \sigma_q^\tau(J,F) \\
\Vert & & \Vert \\
\phi_p(J,F) & \subseteq & \phi_q^\tau(J,F)
\end{array}$$

We illustrate Table 9.4 with the 1-homogeneous operator which we considered in Examples 3.15 and 7.11.

Example 9.11. In $X = l_2$, consider the operators $J = I$ and

$$F(x_1, x_2, x_3, \dots) := (\|x\|, x_1, x_2, \dots).$$

We have already proved in Example 7.11 that $\sigma_{\mathrm{p}}(F) = \mathbb{S}_{\sqrt{2}}$, and so we only have to calculate $\sigma_{\mathrm{q}}^1(I, F) = \sigma_{\mathrm{q}}(F)$. But from the obvious equality $[F]_{\mathrm{q}} = [F]_{\mathrm{Q}} = \sqrt{2}$ it follows that $\sigma_{\mathrm{q}}(F) \subseteq \mathbb{S}_{\sqrt{2}}$, by (2.33). So we see that

$$\sigma_{\mathrm{p}}(F) = \phi_{\mathrm{p}}(F) = \sigma_{\mathrm{q}}^1(F) = \phi_{\mathrm{q}}^1(F) = \mathbb{S}_{\sqrt{2}}$$

in this example. ♡

To conclude, we prove some kind of discreteness result for the spectra and phantoms; putting $X = Y$, $J = I$, and $\tau = 1$ we regain Theorems 6.11, 7.8 and 8.10. This discreteness result will be quite useful in applications to the so-called p-Laplace operator, see Section 12.5.

Theorem 9.12. *Suppose that J is an odd τ-homogeneous homeomorphism with $[J]_{\mathrm{a}}^\tau > 0$, and F is odd, τ-homogeneous and compact. Then*

$$\begin{aligned}\sigma_{\mathrm{FMV}}^\tau(J, F) \setminus \{0\} &= \sigma_{\mathrm{AGV}}^\tau(J, F) \setminus \{0\} \\ &= \sigma_{\mathrm{F}}^\tau(J, F) \setminus \{0\} \\ &= \Phi^\tau(J, F) \setminus \{0\} \\ &= \phi^\tau(J, F) \setminus \{0\} \\ &= \phi_{\mathrm{p}}(J, F) \setminus \{0\} \\ &= \sigma_{\mathrm{b}}^\tau(J, F) \setminus \{0\} \\ &= \sigma_{\mathrm{q}}^\tau(J, F) \setminus \{0\} \\ &= \sigma_{\mathrm{p}}(J, F) \setminus \{0\}.\end{aligned} \tag{9.82}$$

Proof. To prove (9.82) it obviously suffices to show that

$$\sigma_{\mathrm{F}}^\tau(J, F) \setminus \{0\} \subseteq \sigma_{\mathrm{p}}(J, F),$$

by Theorem 9.11. Thus, let $\lambda \neq 0$ in the complement of $\sigma_{\mathrm{p}}(J, F)$ be given; we have to show that $\lambda \notin \sigma_{\mathrm{F}}^\tau(J, F)$. Since F is compact and $[J]_{\mathrm{a}}^\tau > 0$, we have $[\lambda J - F]_{\mathrm{a}}^\tau = |\lambda|[J]_{\mathrm{a}}^\tau > 0$. Hence, the same argument as in the proof of Theorem 9.11 shows that $[\lambda J - F]_{\mathrm{b}}^\tau > 0$. So it remains to prove that $\lambda J - F$ is epi on $\overline{\Omega}$ for every $\Omega \in \mathfrak{OBC}(X)$. By Proposition 7.2, it suffices to show this for $\Omega = B^o(X)$.

Thus, let $H\colon B(X) \to Y$ be compact with $H|_{S(X)} = \Theta$. The operator $G_0 := J^{-1}(\frac{1}{\lambda}F)\colon B(X) \to X$ is compact and odd with $G_0(x) \neq x$ on $S(X)$, while the operator $G_1 := J^{-1}(\frac{1}{\lambda}(F + H)): B(X) \to X$ is compact and satisfies $G_1(x) = G_0(x)$ on $S(X)$. From Borsuk's theorem and Property 3.7 of the Leray–Schauder degree (see Section 3.5) it follows that

$$\deg(I - G_1, B^o(X), \theta) = \deg(I - G_0, B^o(X), \theta) \equiv 1 \pmod 2,$$

and so there exists $x \in B^o(X)$ such that $x = G_1(x)$, i.e. $\lambda J(x) = F(x) + H(x)$. But this means precisely that $\lambda J - F$ is epi on $B(X)$, and so we are done. □

9.7 The Infante–Webb spectrum

In this section we discuss another spectrum which is defined through a finite dimensional approach and relies on the notion of so-called A-proper maps. This spectrum "almost" reduces to the familiar one in the linear case (for a precise formulation see Theorem 9.16 below), and it still preserves many interesting properties in case of a 1-homogeneous operator. We also point out that this spectrum may be defined for certain discontinuous operators, and so we do not suppose in this section, as we did throughout before, that all maps are continuous.

Throughout the following X denotes an infinite dimensional Banach space endowed with a fixed *approximation scheme* $(X_n, P_n)_n$. By this we mean an increasing sequence of finite dimensional subspaces X_n and corresponding linear projections $P_n : X \to X_n$ satisfying $\|P_n\| = 1$ and $P_n x \to x$ as $n \to \infty$. We call an operator $F : X \to X$ *finitely continuous* at $x \in X$ if, for any finite dimensional subspace X_0 of X and every sequence $(x_n)_n$ in X_0 with $x_n \to x$, we have $F(x_n) \rightharpoonup F(x)$ (weak convergence). This occurs, for example, if F is *demicontinuous* at x, i.e., $x_n \to x$ implies $F(x_n) \rightharpoonup F(x)$.

Some more definitions are in order. Given an operator $F : X \to X$, we say that the equation

$$F(x) = y \tag{9.83}$$

is *approximation solvable* (or *A-solvable*, for short) with respect to an approximation scheme $(X_n, P_n)_n$ if there exists $n_0 \in \mathbb{N}$ such that the equation

$$P_n F(x) = P_n y \quad (n \geq n_0) \tag{9.84}$$

has a solution $x_n \in X_n$, and the sequence $(x_n)_n$ admits a subsequence which converges to some solution of (9.83). So A-solvability means that not only we can find a solution of the infinite dimensional problem (9.83), but we can construct this solution via a limit of solutions of the finite dimensional approximate problem (9.84). Note that the A-solvability of an equation implies its solvability, by definition, but the converse is not true even for linear operators, as the following example shows.

Example 9.12. Consider the sequence space $X = l_2$ with the canonical basis $\{e_n : n \in \mathbb{N}\}$, the subspaces $X_n := \mathrm{span}\{e_1, e_2, \ldots, e_n\}$, and the natural projections

$$P_n(x_1, x_2, x_3, \ldots) = (x_1, x_2, \ldots, x_n, 0, \ldots).$$

Define $L \in \mathfrak{L}(X)$ by

$$L(x_1, x_2, x_3, x_4, \ldots) = (x_2, x_1, x_4, x_3, \ldots). \tag{9.85}$$

Since L is linear isomorphism with $L^{-1} = L$, equation (9.83) is clearly solvable for each $y \in X$. On the other hand, this equation is not A-solvable for $y = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots)$, say. In fact, (9.84) is then solvable only for n even, but not for n odd. ♡

In view of Example 9.12 the question arises if one may at least characterize the *linear* operators $L\colon X \to X$ for which the equation $Lx = y$ is A-solvable for each $y \in X$. This is in fact possible, but requires some definitions.

Recall that an operator F is called *A-proper* (with respect to an approximation scheme $(X_n, P_n)_n$) if the operator $P_n F|_{X_n}\colon X_n \to X_n$ is continuous for each n and, for any $y \in X$ and any bounded sequence $(x_{n_j})_j$ with $x_{n_j} \in X_{n_j}$ and

$$\|P_{n_j} F(x_{n_j}) - P_{n_j} y\| \to 0 \quad (j \to \infty), \tag{9.86}$$

there exist a subsequence $(x_{n_{j(k)}})_k$ of $(x_{n_j})_j$ and an element $x \in X$ such that $x_{n_{j(k)}} \to x$ and $F(x) = y$. One may show that every continuous A-proper operator is proper on closed bounded sets; a certain converse of this is contained in the following

Lemma 9.7. *Let $(X_n, P_n)_n$ be an approximation scheme, and let $F\colon X \to X$ be continuous and proper. If $F(X_n) \subseteq X_n$ for all n, then F is A-proper with respect to $(X_n, P_n)_n$.*

Proof. Let $y \in X$ and $x_{n_j} \in X_{n_j}$ be given such that (9.86) holds. By assumption, we have then $F(x_{n_j}) \in X_{n_j}$, hence

$$\|F(x_{n_j}) - P_{n_j} y\| \to 0 \quad (j \to \infty).$$

From $P_{n_j} y \to y$ it follows that also $F(x_{n_j}) \to y$, which shows that the set $\{F(x_{n_j}) : j \in \mathbb{N}\}$ is precompact. The properness of F implies that $(x_{n_j})_j$ admits a convergent subsequence, whose limit x satisfies (9.83), by continuity. □

Now we return to the problem of characterizing the linear operators L for which the equation $Lx = y$ is A-solvable. We say that $L \in \mathfrak{L}(X)$ is A-*stable* (with respect to an approximation scheme $(X_n, P_n)_n$) if there exists $n_0 \in \mathbb{N}$ and $c > 0$ such that

$$[P_n L|_{X_n}]_{\mathrm{b}} = \inf_{x \in X_n \setminus \{\theta\}} \frac{\|P_n Lx\|}{\|x\|} \geq c \quad (n \geq n_0).$$

Roughly speaking, this means that all operators $P_n L|_{X_n} : X_n \to X_n$ are uniformly invertible on their ranges. The following proposition which we state without proof gives a necessary and sufficient condition for the (unique) A-solvability of the linear equation $Lx = y$.

Proposition 9.5. *Let X be a Banach space, $(X_n, P_n)_n$ an approximation scheme on X, and $L \in \mathfrak{L}(X)$. Then the following three conditions are equivalent.*

(a) *L is injective and A-proper with respect to $(X_n, P_n)_n$.*

(b) *L is surjective and A-stable with respect to $(X_n, P_n)_n$.*

(c) *The equation $Lx = y$ is uniquely A-solvable.*

In what follows, an operator $F: X \to X$ will be called *A-proper stable* if there exists some $\varepsilon > 0$ such that $F + \mu I$ is A-proper for $|\mu| < \varepsilon$. The following example shows that this is in fact stronger than just A-properness.

Example 9.13. In the real Banach space $X = l_2$, consider the usual approximation scheme $(X_n, P_n)_n$, where X_n is spanned by the first n basis elements. Let F be the operator from Example 4.9, i.e.,

$$F(x) = e^{-\|x\|}x.$$

Clearly, for any $\mu \in \mathbb{R}$, the operator $\mu I - F$ is continuous and satisfies $(\mu I - F)(X_n) \subseteq X_n$ for all $n \in \mathbb{N}$. So from Lemma 9.7 it follows that properness and A-properness are the same for this operator. But we have already seen in Example 4.9 that F is proper, while $\mu I - F$ is not proper for $0 < \mu < 1$. ♡

The definition of A-proper stable operators suggests to introduce the characteristics

$$\tau(F) := \sup\{\varepsilon > 0 : \mu I - F \text{ is A-proper for every } \mu \text{ with } |\mu| < \varepsilon\} \tag{9.87}$$

and

$$\upsilon(F) := \inf\{\varepsilon > 0 : \mu I - F \text{ is A-proper for every } \mu \text{ with } |\mu| > \varepsilon\}. \tag{9.88}$$

So $\tau(F) > 0$ if and only if F is A-proper stable. On the other hand, $|\lambda| > \upsilon(F)$ implies that $\mu I - F$ is A-proper for $|\mu - \lambda| < \frac{1}{2}(|\lambda| - \upsilon(F))$, and thus $\lambda I - F$ itself is A-proper stable.

Finally, we say that F is *A-stably solvable* if there exists $n_0 \in \mathbb{N}$ such that the operator $P_n F$ is stably solvable for $n \geq n_0$ in the sense of Section 6.1.

For the definition of the next spectrum we still need the characteristics

$$d_R(F) := \liminf_{n\to\infty} \inf \left\{ \frac{\|P_n F(x)\|}{\|x\|} : x \in X_n,\ \|x\| \geq R \right\} \quad (R > 0) \tag{9.89}$$

and

$$d(F) := \sup_{R>0} d_R(F). \tag{9.90}$$

The characteristic (9.90) is related to the lower quasinorm (2.4) by the estimate

$$d(F) \leq [F]_{\mathrm{q}}.$$

To see this, let $\varepsilon > 0$ and $R > 0$ be fixed. Then there exists $x_\varepsilon \in X$ such that $\|x_\varepsilon\| > R$ and

$$[F]_q - \varepsilon < \frac{\|F(x_\varepsilon)\|}{\|x_\varepsilon\|} < [F]_q + \varepsilon.$$

From $P_n x_\varepsilon \to x_\varepsilon$ and $F(P_n x_\varepsilon) \to F(x_\varepsilon)$, as $n \to \infty$, it follows that

$$\begin{aligned} d_R(P_nF) &\leq \inf\left\{\frac{\|P_nF(x_n)\|}{\|x_n\|} : x_n \in X_n,\ \|x_n\| \geq R\right\} \\ &\leq \frac{\|P_nF(P_nx_\varepsilon)\|}{\|P_nx_\varepsilon\|} \\ &\leq \frac{\|F(P_nx_\varepsilon)\|}{\|P_nx_\varepsilon\|} \\ &\leq \frac{\|F(x_\varepsilon)\|}{\|x_\varepsilon\|} + \varepsilon < [F]_q + 2\varepsilon. \end{aligned}$$

As $\varepsilon > 0$ is arbitrary this shows that $d_R(F) \leq [F]_q$, and so also $d(F) \leq [F]_q$.

The usefulness of the characteristic (9.90) is explained by the following lemma.

Lemma 9.8. *Suppose that $d(F) > 0$. Moreover, assume that $x_n \in X_n$ ($n \in \mathbb{N}$) is a solution of the approximate equation* (9.84) *for some fixed $y \in X$. Then the sequence $(x_n)_n$ is bounded.*

Proof. By definition, we may choose $R > 0$ and $N \in \mathbb{N}$ such that

$$\inf_{\|x\|\geq R} \frac{\|P_nF(x)\|}{\|x\|} > \frac{1}{2} d_R(F) \quad (n \geq N). \tag{9.91}$$

So for $\|x_n\| \geq R$ we have

$$\|y\| \geq \|P_ny\| = \|P_nF(x_n)\| \geq \frac{1}{2} d_R(F)\|x_n\| \quad (n \geq N).$$

Consequently,

$$\|x_n\| \leq \max\left\{R, \frac{2\|y\|}{d_R(F)}, \|x_1\|, \ldots, \|x_{N-1}\|\right\}$$

for all $n \in \mathbb{N}$, which shows that $(x_n)_n$ is bounded. □

Given a finitely continuous operator $F\colon X \to X$, we say that F is *IW-regular* (where IW stands for Infante–Webb) if F is A-stably solvable and if both $\tau(F) > 0$ and $d(F) > 0$. It is not hard to see that, if F is IW-regular, the equation (9.83) is A-solvable for each $y \in X$. In fact, since P_nF is stably solvable for n sufficiently large, by assumption, we can find solutions x_n of (9.84) which form a bounded sequence, by Lemma 9.8. For this sequence we have that $\|P_nF(x_n) - P_ny\| = 0$ for sufficiently large n. Since F is A-proper, we find a subsequence of $(x_n)_n$ which converges to a solution of equation (9.83).

Now we are ready for the definition of a new spectrum. We call the set

$$\rho_{\text{IW}}(F) = \{\lambda \in \mathbb{K} : \lambda I - F \text{ is IW-regular}\} \tag{9.92}$$

the *Infante–Webb resolvent set* and its complement

$$\sigma_{\text{IW}}(F) = \mathbb{K} \setminus \rho_{\text{IW}}(F) \tag{9.93}$$

the *Infante–Webb spectrum* (or *IW-spectrum*, for short) of F.

The following two theorems are parallel to corresponding results for the Furi–Martelli–Vignoli spectrum (Theorems 6.2 and 6.3) and Feng spectrum (Theorems 7.4 and 7.5).

Theorem 9.13. *The spectrum $\sigma_{\text{IW}}(F)$ is closed.*

Proof. Let $\lambda \in \rho_{\text{IW}}(F)$ be fixed, so we know that $\tau(\lambda I - F) > 0$, $d_R(\lambda I - F) > 0$ for some $R > 0$, and $\lambda P_n - P_n F$ is stably solvable for n sufficiently large. Choose $\mu \in \mathbb{K}$ with

$$|\mu - \lambda| < \min\{d_R(\lambda I - F), \tau(\lambda I - F)\}; \tag{9.94}$$

we show that $\mu \in \rho_{\text{IW}}(F)$. It is not hard to see that (9.94) implies

$$d_R(\mu I - F) \geq d_R(\lambda I - F) - |\mu - \lambda| > 0$$

and

$$\tau(\mu I - F) \geq \tau(\lambda I - F) - |\mu - \lambda| > 0.$$

So it remains to show that $\mu P_n - P_n F$ is stably solvable for n sufficiently large. But this is a consequence of the trivial identity

$$\mu P_n - P_n F = (\mu - \lambda) P_n + \lambda P_n - P_n F$$

and Lemma 6.4, applied to $G = (\mu - \lambda) P_n$. We conclude that λ is an interior point of $\rho_{\text{IW}}(F)$, and so $\sigma_{\text{IW}}(F)$ is closed. □

Theorem 9.14. *Suppose that F satisfies $[F]_{\text{B}} < \infty$ and $\upsilon(F) < \infty$. Then the spectrum $\sigma_{\text{IW}}(F)$ is bounded, hence compact.*

Proof. We show that every $\lambda \in \mathbb{K}$ with $|\lambda| > \max\{[F]_{\text{B}}, \upsilon(F)\}$ belongs to $\rho_{\text{IW}}(F)$. As observed above, the estimate $|\lambda| > \upsilon(F)$ implies that $\lambda I - F$ is A-proper stable, while the estimate $|\lambda| > [F]_{\text{B}}$ implies that $d(\lambda I - F) \geq |\lambda| - [F]_{\text{B}} > 0$. So we only have to show that $\lambda P_n - P_n F$ is stably solvable for n sufficiently large.

Let $G\colon X_n \to X_n$ be a compact operator with $[G]_{\text{Q}} = 0$; we have to show that the equation $\lambda P_n x - P_n F(x) = G(x)$ has a solution $\hat{x}_n \in X_n$ for every fixed $n \in \mathbb{N}$. Now, the restriction of the operator $H := (P_n F + G)/\lambda$ to X_n is trivially compact and satisfies

$$[H]_{\text{Q}} = \frac{1}{|\lambda|}[P_n F + G]_{\text{Q}} = \frac{[P_n F]_{\text{Q}}}{|\lambda|} \leq \frac{[P_n F]_{\text{B}}}{|\lambda|} \leq \frac{[F]_{\text{B}}}{|\lambda|} < 1,$$

by our choice of λ. From Theorem 2.2 it follows that H has a fixed point $\hat{x}_n \in X_n$ which is of course a solution of the equation $\lambda P_n x - P_n F(x) = G(x)$. □

Theorem 9.14 shows that the *Infante–Webb spectral radius*

$$r_{\mathrm{IW}}(F) = \sup\{|\lambda| : \lambda \in \sigma_{\mathrm{IW}}(F)\}, \tag{9.95}$$

satisfies the upper estimate

$$r_{\mathrm{IW}}(F) \leq \max\{[F]_{\mathrm{B}}, \upsilon(F)\}. \tag{9.96}$$

This is of course completely analogous to the estimates (6.12) and (7.23). As one could expect, the Infante–Webb spectrum can be described more easily for operators with additional properties. The simplest such class which turned out to be useful already in the preceding chapters, is that of 1-homogeneous operators. Thus, the IW-spectrum of such an operator contains the classical eigenvalues in the sense of (3.18), as we show now.

Proposition 9.6. *Let F be continuous and* 1*-homogeneous. Then the inclusion*

$$\sigma_{\mathrm{p}}(F) \subseteq \sigma_{\mathrm{IW}}(F) \tag{9.97}$$

holds true.

Proof. Observe, first of all, that for a 1-homogeneous operator F the characteristic (9.90) may be calculated in the simpler form

$$d(F) = \liminf_{n\to\infty} [P_n F]_{\mathrm{b}}. \tag{9.98}$$

Given $\lambda \in \sigma_{\mathrm{p}}(F)$, by homogeneity we may choose $e \in S(X)$ such that $F(e) = \lambda e$. Then the sequence $(e_n)_n$ defined for sufficiently large n by $e_n := P_n e/\|P_n e\|$ belongs to $S(X)$ and satisfies $e_n \to e$ (since the projection sequence $(P_n)_n$ converges pointwise to the identity) and $P_n F(e_n) \to F(e)$ (since we supposed F to be continuous). So by (9.98) we obtain

$$\begin{aligned} d(\lambda I - F) &= \liminf_{n\to\infty} [\lambda P_n - P_n F]_{\mathrm{b}} \\ &= \liminf_{n\to\infty} \inf_{\|x\|=1} \|\lambda P_n x - P_n F(x)\| \\ &\leq \liminf_{n\to\infty} \|\lambda e_n - P_n F(e_n)\| \\ &\leq |\lambda| \lim_{n\to\infty} \|e_n - e\| + \lim_{n\to\infty} \|F(e) - P_n F(e_n)\| = 0. \end{aligned}$$

This shows that $\lambda \in \sigma_{\mathrm{IW}}(F)$, by definition of the Infante–Webb spectrum. □

The following example shows that Proposition 9.6 is not true without the homogeneity requirement on F.

Example 9.14. Let X and $(X_n, P_n)_n$ be as in Example 9.13, and let $F\colon X \to X$ be defined by

$$F(x_1, x_2, x_3, \dots) = (f(x_1), x_2, x_3, \dots), \tag{9.99}$$

where

$$f(t) = \begin{cases} t & \text{if } t \le \frac{1}{2}, \\ 1-t & \text{if } \frac{1}{2} < t \le 1, \\ 2t-2 & \text{if } 1 < t \le 2, \\ t & \text{if } t > 2. \end{cases}$$

Since $F(1, 0, 0, \dots) = (0, 0, 0, \dots)$, we obviously have $0 \in \sigma_{\mathrm{p}}(F)$. On the other hand, we claim that $0 \notin \sigma_{\mathrm{IW}}(F)$. In fact, the restriction of $P_n F$ to $X_n = \mathbb{R}^n$ has the form

$$F(x_1, x_2, \dots, x_n) = (f(x_1), x_2, \dots, x_n).$$

Since the scalar function $f\colon \mathbb{R} \to \mathbb{R}$ is onto, it has a right inverse $g\colon \mathbb{R} \to \mathbb{R}$, and so F has in $\mathbb{R}^n$ the right inverse

$$G(y_1, y_2, \dots, y_n) = (g(y_1), y_2, \dots, y_n).$$

From Example 6.1 we conclude that $P_n F$ is stably solvable on X_n, and so F is A-stably solvable.

To see that $\tau(F) > 0$, observe that $(\mu - 1)I$ is A-proper for $\mu \neq 1$ and $I - F$ has a one-dimensional range, and so $\mu I - F = (\mu - 1)I + I - F$ is A-proper for $|\mu| < 1$, being a compact perturbation of an A-proper operator. This shows that $\tau(F) = 1$, by the definition (9.87). Moreover, a straightforward calculation shows that $d(F) = 1$, and so F is IW-regular as claimed. ♡

The following discreteness result for the IW-spectrum is parallel to Theorems 6.14 and 7.8.

Theorem 9.15. *Suppose that F is finitely continuous, 1-homogeneous, and odd. Let $\lambda \in \sigma_{\mathrm{IW}}(F)$ with $|\lambda| > \upsilon(F)$. Then $\lambda \in \sigma_{\mathrm{p}}(F)$, i.e., λ is an eigenvalue of F.*

Proof. From $|\lambda| > \upsilon(F)$ it follows that $\lambda I - F$ is A-proper stable. We distinguish two cases.

Suppose first that $d(\lambda I - F) = 0$, and so

$$\liminf_{n\to\infty} [\lambda P_n - P_n F]_{\mathrm{b}} = \liminf_{n\to\infty} \inf_{x_n \in S(X_n)} \|\lambda x_n - P_n F(x_n)\| = 0. \tag{9.100}$$

Since F is finitely continuous we may choose elements $e_n \in S(X_n)$ such that

$$\|\lambda e_n - P_n F(e_n)\| = \inf_{x_n \in S(X_n)} \|\lambda x_n - P_n F(x_n)\|.$$

From (9.100) it follows that there exists a subsequence $(e_{n_j})_j$ of $(e_n)_n$ such that

$$\|\lambda e_{n_j} - P_{n_j} F(e_{n_j})\| \to 0 \quad (j \to \infty).$$

Moreover, the A-properness of $\lambda I - F$ implies that we may find a further subsequence $(e_{n_{j(k)}})_k$ of $(e_{n_j})_j$ which converges to some element $e \in S(X)$ as $k \to \infty$. But then we have $F(e) = \lambda e$, and so $\lambda \in \sigma_{\mathrm{p}}(F)$ as claimed.

Second, suppose that $d(\lambda I - F) > 0$. We claim that $\lambda I - F$ is then A-stably solvable, contradicting our choice of λ, and so this case cannot occur. In fact, let $G\colon X_n \to X_n$ be continuous with $[G]_{\mathrm{Q}} = 0$: we have to show that the equation $\lambda x - P_n F(x) = G(x)$ is solvable in X_n.

Given $0 < \varepsilon < d(\lambda I - F)$, we can find $R > 0$ such that

$$d_R(\lambda I - F) = \liminf_{n\to\infty} \inf_{\|x\|\geq R} \frac{\|\lambda P_n x - P_n F(x)\|}{\|x\|} \geq \varepsilon, \tag{9.101}$$

by the definition (9.90) of $d(\lambda I - F)$. Define a homotopy $H\colon B_R(X_n) \times [0,1] \to X_n$ by

$$H(x_n, t) = \lambda x_n - P_n F(x_n) - tG(x_n).$$

Then (9.101) implies that, for n large enough,

$$\frac{1}{R}\|\lambda x_n - P_n F(x_n) - tG(x_n)\| \geq d_R(\lambda I - F) - t\varepsilon > 0 \quad (x_n \in S_R(X_n),\ 0 \leq t \leq 1).$$

Consequently, by the homotopy invariance of the Brouwer degree, the oddness of $\lambda I - P_n F$, and Borsuk's theorem we have

$$\begin{aligned}\deg(\lambda I - P_n F - G, B_R^o(X_n), \theta) &= \deg(H(\,\cdot\,,1), B_R^o(X_n), \theta)\\ &= \deg(H(\,\cdot\,,0), B_R^o(X_n), \theta)\\ &= \deg(\lambda I - P_n F, B_R^o(X_n), \theta) \equiv 1 \ (\mathrm{mod}\, 2),\end{aligned}$$

and so there exists $\hat{x}_n \in X_n$ such that $\lambda \hat{x}_n - P_n F(\hat{x}_n) = G(\hat{x}_n)$. We have shown that $\lambda I - F$ is A-stably solvable, and so $\lambda \in \rho_{\mathrm{IW}}(F)$, contradicting our hypothesis. □

The Theorems 9.13 and 9.14 show that the Infante–Webb spectrum shares some important properties with the Furi–Martelli–Vignoli spectrum and Feng spectrum. Let us now discuss the problem whether or not this spectrum may be empty. Of course, if F is a compact operator in a finite dimensional Banach space X, then 0 always belongs to the spectrum $\sigma_{\mathrm{IW}}(F)$; this is completely analogous to all the other spectra we considered so far. On the other hand, we have used the operator (3.16) throughout to show that all spectra may be empty. Surprisingly, this example *fails* in case of the IW-spectrum, as we show now.

Example 9.15. Consider the infinite dimensional analogue of (3.16), i.e.,

$$F(z_1, z_2, z_3, z_4, \ldots) = (\bar{z}_2, i\bar{z}_1, \bar{z}_4, i\bar{z}_3, \ldots) \tag{9.102}$$

in the complex space $X = l_2$ with the standard approximation scheme $(X_n, P_n)_n$. Since F is 1-homogeneous, we have

$$[\lambda I - P_n F]_{\mathrm{q}} = \inf_{z\in S(X_n)} \|\lambda z - P_n F(z)\|$$

for every $\lambda \in \mathbb{C}$. Now we distinguish the cases of even and odd n.

Suppose first that $n = 2k$ is even. Then we have for $z = (z_1, z_2, \ldots, z_{2k-1}, z_{2k}) \in X_{2k}$

$$\begin{aligned}&(\lambda I - P_{2k}F)(z_1, z_2, \ldots, z_{2k-1}, z_{2k})\\ &\quad = (\lambda z_1 - \overline{z}_2, \lambda z_2 - i\overline{z}_1, \ldots, \lambda z_{2k-1} - \overline{z}_{2k}, \lambda z_{2k} - i\overline{z}_{2k-1}).\end{aligned} \tag{9.103}$$

Writing out this in components one can show that $[\lambda I - P_{2k}F]_{\mathrm{q}} > 0$ for every $\lambda \in \mathbb{C}$. Moreover, $[\lambda I - P_{2k}F]_{\mathrm{a}} > 0$ for every $\lambda \in \mathbb{C}$, since $P_{2k}F$ acts in the finite dimensional space $X_n = \mathbb{C}^n$. Finally, the operator (9.103) is stably solvable in X_{2k} for every $\lambda \in \mathbb{C}$, and so we see that the FMV-spectrum $\sigma_{\mathrm{FMV}}(P_{2k}F)$ of $P_{2k}F$ is empty.

Now suppose that $n = 2k + 1$ is odd. For $z = (z_1, z_2, \ldots, z_{2k-1}, z_{2k}, z_{2k+1}) \in X_{2k+1}$ we then have

$$\begin{aligned}&(\lambda I - P_{2k+1}F)(z_1, z_2, \ldots, z_{2k-1}, z_{2k}, z_{2k+1})\\ &\quad = (\lambda z_1 - \overline{z}_2, \lambda z_2 - i\overline{z}_1, \ldots, \lambda z_{2k-1} - \overline{z}_{2k}, \lambda z_{2k} - i\overline{z}_{2k-1}, \lambda z_{2k+1}).\end{aligned} \tag{9.104}$$

Considering the basis element $e_{2k+1} = (\delta_{2k+1,n})_n \in S(X_{2k+1})$ we see that

$$[P_{2k+1}F]_{\mathrm{q}} = \inf_{z \in S(X_{2k+1})} \|P_n F(z)\| \leq \|P_{2k+1}F(e_{2k+1})\| = 0, \tag{9.105}$$

and hence $0 \in \sigma_{\mathrm{q}}(P_{2k+1}F) \subseteq \sigma_{\mathrm{FMV}}(P_{2k+1}F)$. But this implies that $0 \in \sigma_{\mathrm{IW}}(F)$ as well. In fact, suppose that $0 \in \rho_{\mathrm{IW}}(F)$. Then $d_R(F) > 0$, and there exists $k_0 \in \mathbb{N}$ such that $P_{2k+1}F$ is stably solvable for $k \geq k_0$. It follows that

$$[P_{2k+1}F]_{\mathrm{q}} \geq d_R(P_{2k+1}F) \geq \frac{1}{2} d_R(F) > 0 \quad (k \geq k_0),$$

contradicting (9.105). Therefore the spectrum $\sigma_{\mathrm{IW}}(F)$ is *nonempty* for the operator (9.102), in contrast to all the other spectra we considered in Chapters 6–8. ♡

Now let us see how the Infante–Webb spectrum of a *linear* operator looks like. The following proposition gives a partial answer.

Proposition 9.7. *For $L \in \mathfrak{L}(X)$, the inclusion*

$$\sigma(L) \subseteq \sigma_{\mathrm{IW}}(L) \tag{9.106}$$

holds.

Proof. Let $\lambda \in \rho_{\mathrm{IW}}(L)$; we have to show that $\lambda I - L$ is a linear isomorphism. To see that $\lambda I - L$ is onto, fix $y \in X$. Since $\lambda P_n - P_n L$ is stably solvable for n large enough, we find $x_n \in X_n$ such that $\lambda P_n x_n - P_n L x_n = P_n y$. From $d(\lambda I - L) > 0$ it follows, by Lemma 9.8, that the sequence $(x_n)_n$ is bounded. But $\lambda I - L$ is also A-proper, and so there exists a subsequence $(x_{n_k})_k$ of $(x_n)_n$ and an element $x \in X$ such that $x_{n_k} \to x$ and $\lambda x - Lx = y$.

So we have proved that $\lambda I - L$ is surjective. The injectivity of $\lambda I - L$ is an immediate consequence of Proposition 9.5. □

Unfortunately, the inclusion in (9.106) may be strict, as the following example shows.

Example 9.16. Let X and L be as in Example 9.12. It is easy to see that both $\lambda = 1$ and $\lambda = -1$ are eigenvalues of L. Conversely, the fact that $L^2 = I$ implies that $\sigma(L) = \{\pm 1\}$. On the other hand, a similar reasoning as in Example 9.12 shows that L is not A-proper, and so $0 \in \sigma_{\mathrm{IW}}(L)$. ♡

In spite of the somewhat disappointing Example 9.16, one may give a precise result relating the IW-spectrum $\sigma_{\mathrm{IW}}(L)$ and the classical spectrum $\sigma(L)$ of a bounded linear operator. In order to state this result we consider, in analogy to the set $\pi(F)$ defined in Section 4.3, the set

$$\pi_A(F) := \{\lambda \in \mathbb{K} : \lambda I - F \text{ is not A-proper}\}. \tag{9.107}$$

Since every continuous A-proper operator is proper on closed bounded sets, we have the inclusion $\pi(F) \subseteq \pi_A(F)$ for coercive F. It is also clear that $\pi_A(F) \subseteq \sigma_{\mathrm{IW}}(F)$. For example, we have $0 \in \pi_A(L)$ for the operator (9.85). The following theorem makes this more precise.

Theorem 9.16. *For $L \in \mathfrak{L}(X)$, the equality*

$$\sigma_{\mathrm{IW}}(L) = \sigma(L) \cup \pi_A(L) \tag{9.108}$$

is true. In particular, if X is infinite dimensional and $L \in \mathfrak{L}(X)$ is compact, then $\sigma_{\mathrm{IW}}(L) = \sigma(L)$.

Proof. In view of Proposition 9.6 we have to prove only the inclusion $\sigma_{\mathrm{IW}}(L) \subseteq \sigma(L) \cup \pi_A(L)$. For this it suffices to show that every A-proper linear isomorphism is IW-regular.

So let $L: X \to X$ be an A-proper isomorphism. We show first that L is A-proper stable. In fact, since the resolvent set $\rho(L)$ is open, we may find $\delta > 0$ such that $\mu \in \rho(L)$ for $|\mu| < \delta$. Moreover, from Proposition 9.5 it follows that L is A-stable, and so

$$\|P_n Lx\| \geq c\|x\| \quad (x \in X_n,\ n \geq n_0)$$

for some $c > 0$ and $n_0 \in \mathbb{N}$. So for $|\mu| < \min\{\delta, c\}$ we get

$$\|P_n Lx - \mu x\| \geq \|P_n Lx\| - |\mu|\,\|x\| \geq (c - |\mu|)\|x\| \quad (x \in X_n,\ n \geq n_0).$$

This proves that $L - \mu I$ is an A-stable isomorphism, and so $L - \mu I$ is A-proper, again by Proposition 9.5.

The fact that L is A-stable also implies that $d(R) > 0$. Finally, applying once more Proposition 9.5 we see that the equation $Lx = y$ is A-solvable for each $y \in X$, and therefore $P_n L$ is FMV-regular (see Section 6.2) for sufficiently large $n \in \mathbb{N}$. Altogether we have shown that L is IW-regular, and the proof of the first assertion is complete.

The second assertion follows from the fact that $\lambda I - L$ is A-proper for $\lambda \neq 0$ and from $0 \in \sigma_{\mathrm{IW}}(L)$ if L is compact. □

9.8 Notes, remarks and references

The spectrum $\sigma_F(L, F)$ which we called semilinear Feng spectrum has been actually introduced by Feng and Webb in [110]. The semilinear FMV-spectrum we discussed in Section 9.2 imitates the Feng-Webb construction for the Furi–Martelli–Vignoli spectrum and is taken from [14]. Similarly, one could define some kind of semilinear AGV-spectrum for L and F by adding the spectral set

$$\sigma_\mu(L, F) := \{\lambda \in \mathbb{K} : \mu(\Phi_\lambda(L, F)) = 0\}, \tag{9.109}$$

where $\mu(F)$ is the operator characteristic (6.4), to the spectral set $\sigma_q(L, F)$. This subspectrum has not been studied in the literature.

As we indicated in Section 9.1 and 9.2, the spectra (9.10) and (9.17) have applications to periodic problems for semilinear ordinary differential equations. Such applications are related to the so-called *coincidence degree* of Mawhin which is described in detail, for example, in [126] and [187], [188].

Subsequently, this coincidence degree has been used by Basile and Mininni [32], [33], [193], [194] to study solvability of the operator equation (9.6), where as before L is a linear Fredholm operator of index zero, and F is nonlinear. For instance, it is proved in [194] that $L - F$ is onto if $F \in \mathfrak{C}^1(X, Y)$ with $Lx \neq F'(y)x$ for all $y \in X$ and $x \in X \setminus \{\theta\}$, provided that the coercivity condition

$$\lim_{\|x\|\to\infty} \|Lx - F(x)\| = \infty$$

holds true. Moreover, in [32], [33] the authors study the solvability of (9.6) in the "resonance case", i.e. under so-called *Landesman–Lazer conditions*. Here it is supposed, in addition, that the τ-quasinorm $[F]_Q^\tau$ of F is finite, see (9.44).

The following discreteness result for the spectra (9.10) and (9.17) which we state for the sake of completeness, generalizes the Theorems 9.2 and 9.4. The proof is analogous to that of Theorems 6.12 and 7.9.

Theorem 9.17. *Let $F \in \mathfrak{A}_L(X, Y)$ be odd. Then every $\lambda \in \sigma_{FMV}(L, F)$ with $|\lambda| > [K_{PQ}F]_A$ is an asymptotic eigenvalue of the pair (L, F). Moreover, if F is in addition 1-homogeneous, then every $\lambda \in \sigma_F(L, F)$ with $|\lambda| > [K_{PQ}F]_A$ is an eigenvalue of the pair (L, F).*

The definitions and results in Section 9.3 are all taken from the paper [9]. We remark that there exist several definitions of adjoints of nonlinear operators in the literature. Apart from that given in [253], all these definitions are essentially different from ours given in (9.22). The following quite natural definition is due to Nashed ([197], see also [52], [53], [281]): given some C^1 operator $F : H \to H$ in a Hilbert space H with $F(\theta) = \theta$, call $F^* : H \to H$ the *adjoint* of F if

$$[F^*]'(x) = [F'(x)]^* \quad (x \in H), \tag{9.110}$$

where the asterisk on the right hand side of (9.110) denotes the usual adjoint (9.20) of a linear operator. The adjoint F^* exists if and only if the difference $F'(x) - [F'(x)]^*$ does not depend on x; in this case one has the natural representation

$$F^*(x) = \int_0^1 [F'(tx)]^* x \, dt.$$

In particular, F is *selfadjoint* (i.e., $F^* = F$) if F is a *potential operator*; this provides an interesting connection between calculus of variation and nonlinear adjoints. We point out that the apparently most natural condition on a selfadjoint nonlinear operator F to satisfy

$$\langle F(x), y \rangle = \langle x, F(y) \rangle \quad (x, y \in H) \tag{9.111}$$

in a Hilbert space H does not make sense. In fact, it is not hard to see that condition (9.111) implies that F is linear.

Similarly, Cacuci, Perez and Protopopescu [58] define – obviously unaware of the articles [197] and [281] – an adjoint $F^+ \colon H \to \mathfrak{L}(H)$ for some Gâteaux differentiable nonlinear operator F in a Hilbert space H by putting

$$F^+(u) := \int_0^1 F'(tu)^* \, dt.$$

With this definition, the pair (F, F^+) satisfies the "duality relation"

$$\langle F(u), v \rangle = \langle u, F^+(u)v \rangle.$$

Of course, the differentiability requirement on F is a drawback of these constructions. A more special construction due to Shutjaev [237] associates with each continuous operator $F \colon L_2(\Omega) \times [0, 1] \to W_2^{-1}(\Omega)$ some adjoint $F^* \colon W_2^1(\Omega) \times [0, 1] \to L_2(\Omega)$ by means of the duality relation

$$\langle F(u, \varepsilon)v, w \rangle = \langle v, F^*(u, \varepsilon)w \rangle \quad (0 \le \varepsilon \le 1).$$

This definition is modelled in view of applications to quasilinear hyperbolic equations of first order [237], but has the flaw of containing a parameter. Another definition of an adjoint with interesting applications to nonlinear semigroups with accretive generators may be found in [61]. Finally, a unified approach to nonlinear adjoints which covers many of the above mentioned notions, is contained in the recent paper [236].

Let us make an observation on our definition of the pseudo-adjoint (9.22). Lemma 9.3 shows that $F^\#$ is always linear, even if F itself is nonlinear. However, we have a price to pay for this: the spaces $X^\#$ and $Y^\#$ are in general much bigger than the dual spaces X^* and Y^*. So, although one has a good explicit description for the dual X^* of many classical Banach spaces X, it is impossible to describe even the space $\mathbb{R}^\# = \mathfrak{Lip}_0(\mathbb{R}, \mathbb{R})$, say. How much bigger $X^\#$ is than X^* may be seen by means of the following observation. Let us say that a sequence $(x_n)_n$ in X is *weakly*$^\#$ *convergent* to

$x_0 \in X$ if $f(x_n) \to f(x_0)$, for all $f \in X^\#$, as $n \to \infty$. Then this is equivalent to the *strong convergence* $x_n \to x_0$! In fact, the special choice $f(x) := \|x - x_0\| - \|x_0\|$ yields $\|x_n - x_0\| = f(x_n) - f(x_0) \to 0$.

In view of the disappointing Example 9.3 one might think that a better definition than (9.23) could be

$$\sigma^\#(F) := \{\lambda \in \mathbb{K} : (\lambda I - F)^\# \text{ is not a bijection}\}.$$

However, Proposition 9.3 shows that this is nothing else but the Kachurovskij spectrum $\sigma_K(F)$ of F which we already studied in detail in Chapter 5.

The "asymmetric" spectrum (9.27) was introduced and studied by Singhof [238], [239] in the singlevalued and by Weyer [274], [275] in the multivalued case. The interested reader may find a good presentation of the theory and applications of multivalued maps, including monotone and maximal monotone operators, in the monographs [29], [47], [287]. In particular, we have taken Lemma 9.5 from the book [29]. The notion of λ-polytone operators is due to Weyer [275].

One may deduce directly from Theorem 9.7 that the Singhof–Weyer resolvent set (9.26) is open. In the thesis [238] one can find another stability result on this set with respect to perturbations by operators which reads as follows.

Theorem 9.18. *Suppose that λ belongs to the Singhof–Weyer resolvent set $\rho_{SW}(F)$ of some multivalued operator $F: H \to 2^H$, and $G: H \to H$ is singlevalued and Lipschitz continuous with*

$$[G]_{\text{Lip}} < \frac{1}{[R(\lambda; F)]_{\text{Lip}}}.$$

Then $\lambda \in \rho_{SW}(F + G)$.

Observe that the Singhof–Weyer spectrum of the operator in Example 9.4 coincides with its Neuberger spectrum (4.17). Indeed, the following result is proved in [239]; compare this with Theorem 4.2.

Theorem 9.19. *For $F \in \mathfrak{C}^1(X)$ with $F(\theta) = \theta$, the inclusion*

$$\sigma_{SW}(F) \supseteq \bigcup_{x \in X} \sigma(F'(x)) \tag{9.112}$$

holds, where $\sigma(L)$ on the right-hand side of (9.112) *denotes the usual spectrum* (1.5) *of a bounded linear operator L. In particular, $\sigma_{SW}(F) \neq \emptyset$ in case $\mathbb{K} = \mathbb{C}$.*

We point out that Singhof also develops a parallel eigenvalue theory in the thesis [238]. He calls a scalar λ a *quasi-eigenvalue* of $F: H \to 2^H$ if the operator $\lambda I - F$ is not injective. In other words, λ is a quasi-eigenvalue of F if and only if one can find $x, y \in H$ with $x \neq y$, $u \in F(x)$, and $v \in F(y)$ such that $u - \lambda x = v - \lambda y$. This notion of eigenvalue is appropriate for the Singhof–Weyer spectrum and coincides with the classical notion in the singlevalued linear case.

The entire Section 9.5 is taken from Weber's survey article [273] (see also [272]). Apart from the characteristics (9.42)–(9.45), Weber also introduces (in another notation) the characteristics

$$[F]_{\mathrm{B}}^{\varphi} := \sup_{x\neq\theta} \frac{\|F(x)\|}{\varphi(\|x\|)} \tag{9.113}$$

and

$$[F]_{\mathrm{b}}^{\varphi} := \inf_{x\neq\theta} \frac{\|F(x)\|}{\varphi(\|x\|)}, \tag{9.114}$$

which in the special case $\varphi(t) = t^\tau$ reduce to our characteristics (9.53) and (9.54). We remark that the spectrum $\sigma_{\mathrm{W}}^{\varphi}(J, F)$ was introduced independently in the same year in [39] for $\varphi(t) = t^\tau$ and $J: X \to Y$ being a homeomorphism satisfying the bilateral estimate

$$c\|x\|^\tau \leq \|J(x)\| \leq C\|x\|^\tau \quad (x \in X)$$

for some constants $C, c > 0$. This spectrum is studied in [39] by means of (a variant of) the Leray–Schauder degree to obtain surjectivity results for the operator $\lambda J - F$, in the same way as Furi and Vignoli obtain surjectivity results for the operator $\lambda I - F$ in [124].

We point out that the idea of considering surjectivity results for operators of the form $\lambda J - F$, where J may be different from the identity, is not new. In fact, a classical monograph concerned with this problem is [116]; this may be regarded as one of the starting points of nonlinear eigenvalue theory. A particular emphasis in [116] is put on so-called *nonlinear Fredholm alternatives* which we will consider in more detail in Chapter 12.

The paper [211] is also concerned with surjectivity results for the operator $\lambda J - F$ (in particular, with the solvability of equation (9.41)). Here F and J are supposed to map a Banach space X into its dual X^* and to satisfy $[F]_{\mathrm{A}} = [F]_{\mathrm{Q}}^{\tau} = 0$ and $[J]_{\mathrm{b}}^{\tau} > 0$ (see (9.44) and (9.54)). More precise results may be obtained for τ-homogeneous operators; however, the author of [211] does not assume that J and F are odd operators.

The discreteness result given in Theorem 9.9 (b) is quite similar to Lemma 4.5 from [273], where it is assumed that J is "asymptotically equivalent" to the identity in the sense that $[J - I]_{\mathrm{Q}} = 0$. In [273] the author proves another discreteness result which does not exclude the zero point in the inclusion (9.51) and reads as follows. Recall that a continuous operator $A: X \to X^*$ is said to *satisfy condition* (S) if the relations $x_n \rightharpoonup x$ (weak convergence) and $\langle A(x_n), x - x_n\rangle \to 0$ imply that $x_n \to x$.

Theorem 9.20. *Let X be a reflexive Banach space and $F, J \in \mathfrak{Q}_\tau(X, X^*)$. Denote by $s(J, F)$ the set of all $\lambda \in \mathbb{K}$ such that the operator $\lambda J - F$ satisfies condition (S). Then the inclusion*

$$\sigma_{\mathrm{W}}^{\tau}(J, F) \cap s(J, F) \subseteq \sigma_{\mathrm{p}}(J, F)$$

holds true.

The proof of this theorem is similar to that of our Theorem 9.9 (b) and uses the weak compactness of the closed unit ball $B(X)$ in a reflexive space X.

The whole material of Section 9.6 is taken from the recent paper [17]. The idea of calculating the characteristics $[F_p]_{\mathrm{A}}^p$ and $[F_p]_{\mathrm{a}}^p$ in Example 9.10 was suggested to the authors by Väth. In the paper [17] one may also find an explicit calculation of these characteristics, when the Hausdorff measure of noncompactness (1.22) is replaced by the *Kuratowski measure of noncompactness*

$$\hat{\alpha}(M) := \inf\{\delta > 0 : M \text{ may be covered by finitely many sets of diameter } \leq \delta\},$$

see [1]. We point out that the paper [17] also contains an application to an eigenvalue problem for the so-called p-Laplacian. These applications may be obtained as a consequence of some kind of *nonlinear Fredholm alternative* which we will discuss in Chapter 12. Several types of Fredholm alternatives for special nonlinear operators have been given in the monograph [116]. The alternative given in [218], together with applications to nonlinear elliptic boundary value problems, is of completely different nature.

The whole material presented in Section 9.7 is due to Infante and Webb [153]. A good reference on various classes of A-proper operators is the book [219]. In that book the case of operators $F : X \to Y$ is considered throughout, while we restricted ourselves to the case $Y = X$ which suffices for introducing the spectrum (9.93). The proof of Proposition 9.5 (for $L \in \mathfrak{L}(X, Y)$) may also be found in [219].

As may be seen in Section 9.7, the characteristic $\upsilon(F)$ defined in (9.88) plays the same role for the IW-spectrum as the measure of noncompactness $[F]_{\mathrm{a}}$ for the Feng or FMV-spectrum. The article [153] also contains some applications of the spectrum $\sigma_{\mathrm{IW}}(F)$; for instance, the following Birkhoff–Kellogg type theorem is proved there.

Theorem 9.21. *Let X be an infinite dimensional real Banach space with approximation scheme $(X_n, P_n)_n$. Suppose that $F : S(X) \to X$ is finitely continuous and bounded and satisfies*

$$\liminf_{n\to\infty} \inf_{x\in S(X_n)} \|P_n F(x)\| > \upsilon(F).$$

Then F has an eigenvalue λ with $|\lambda| > \upsilon(F)$.

A large amount of material may also be found in the thesis [152]. For example, Proposition 9.6 and the following Example 9.14 are taken from [152].

Theorem 9.16 shows that, given a bounded linear operator $L : X \to X$ and an approximation scheme $(X_n, P_n)_n$ on X, one has $\sigma(L) \subseteq \sigma_{\mathrm{IW}}(L)$, where strict inclusion may occur. Of course, the spectrum $\sigma_{\mathrm{IW}}(L)$ depends on the choice of $(X_n, P_n)_n$. It is an open problem whether or not one may always find an approximation scheme such that $\sigma(L) = \sigma_{\mathrm{IW}}(L)$.

Chapter 10

Nonlinear Eigenvalue Problems

In this chapter we consider the eigenvalue equations $F(x) = \lambda x$ and, more generally, $F(x) = \lambda J(x)$. In the first section we discuss existence of nontrivial solutions of the first equation; this may be viewed as one of the historical roots of spectral theory for nonlinear operators. Afterwards we study nonlinear operators in spaces with cones and give a certain nonlinear analogue of the classical Krejn–Rutman theorem. As we have seen in the previous chapters, however, some of the recently defined spectra require other notions of eigenvalue; this will be illustrated in the following section. Finally, the last part is dedicated to so-called connected eigenvalues (point phantoms) which we introduced in Chapter 8 and which seem to be most natural in connection with nonlinear spectral theory.

10.1 Classical eigenvalues

Let X and Y be two Banach spaces over $\mathbb{K}$ and $F, J \in \mathfrak{C}(X, Y)$. As before (see (9.49)), we call a scalar $\lambda \in \mathbb{K}$ a *classical eigenvalue* of the pair (J, F) if the nullset

$$N(\lambda J - F) = \{x \in X : F(x) = \lambda J(x)\} \tag{10.1}$$

contains an element $x \neq \theta$; any such element will be called an *eigenvector* of (J, F) corresponding to λ. The most important special case is of course $X = Y$ and $J = I$, i.e.,

$$N(\lambda I - F) = \{x \in X : F(x) = \lambda x\}. \tag{10.2}$$

As in the preceding chapter, we write $\sigma_{\mathrm{p}}(J, F)$ for the set of all classical eigenvalues of (J, F) and, in particular, $\sigma_{\mathrm{p}}(I, F) =: \sigma_{\mathrm{p}}(F)$.

Sometimes one is interested in eigenvalues with corresponding eigenvectors of prescribed norm. Therefore we introduce for $r > 0$ the sets

$$\Lambda_r(J, F) := \{\lambda \in \mathbb{K} : F(x) = \lambda J(x) \text{ for some } x \in S_r(X)\} \tag{10.3}$$

and, in particular,

$$\Lambda_r(F) := \Lambda_r(I, F). \tag{10.4}$$

Of course, for a *linear* operator L the set $\Lambda_r(L)$ is independent of r. More generally, if F is τ-homogeneous and J is τ'-homogeneous, then

$$\Lambda_r(J, F) = \{\lambda r^{\tau - \tau'} : \lambda \in \Lambda_1(J, F)\}, \tag{10.5}$$

and so one knows all eigenvectors if one only knows those in the unit sphere.

We start with two important theorems on the structure of $\Lambda_r(F)$ in case of a *compact* nonlinear operator $F\colon B_r(X) \to X$. Interestingly, the first theorem holds only in finite dimensional spaces, the second one only in infinite dimensional spaces.

Theorem 10.1. *If* $F\colon B_r(\mathbb{R}^n) \to \mathbb{R}^n \setminus \{\theta\}$ *is continuous, then the set* (10.4) *contains some* $\lambda_+ > 0$ *and some* $\lambda_- < 0$.

Proof. We prove the statement by means of Brouwer's fixed point principle. Define $G\colon B_r(\mathbb{R}^n) \to \mathbb{R}^n$ by

$$G(x) := \begin{cases} -F\left(\frac{4}{r}\|x\|x\right) & \text{if } \|x\| \le \frac{r}{2}, \\ \left(\frac{2}{r}\|x\| - 1\right)x - \left(2 - \frac{2}{r}\|x\|\right)F\left(\frac{r}{\|x\|}x\right) & \text{if } \|x\| \ge \frac{r}{2}. \end{cases}$$

Then G is continuous with $G(x) = -F(2x)$ for $x \in S_{r/2}(\mathbb{R}^n)$ and $G(x) = x$ for $x \in S_r(\mathbb{R}^n)$.

We claim that G has a zero $z \in B_r(\mathbb{R}^n)$. In fact, suppose that $G(x) \ne \theta$ for all $x \in B_r(\mathbb{R}^n)$. Then the operator $H\colon B_r(\mathbb{R}^n) \to B_r(\mathbb{R}^n)$ defined by

$$H(x) := -r\frac{G(x)}{\|G(x)\|}$$

is well-defined and continuous, and so there exists $\hat{x} \in B_r(\mathbb{R}^n)$ such that $\hat{x} = H(\hat{x})$, by Brouwer's fixed point principle. Since $H(B_r(\mathbb{R}^n)) \subseteq S_r(\mathbb{R}^n)$, we actually have $\|\hat{x}\| = r$, and so $G(\hat{x}) = \hat{x}$. But this implies that

$$\hat{x} = H(\hat{x}) = -G(\hat{x}) = -\hat{x}$$

which is absurd. So we conclude that there exists some $z \in B_r(\mathbb{R}^n)$ with $G(z) = \theta$. Since the operator F has no zero in $B_r(\mathbb{R}^n)$, by assumption, we necessarily have $\|z\| > \frac{r}{2}$, hence

$$\theta = \left(\frac{2}{r}\|z\| - 1\right)z - \left(2 - \frac{2}{r}\|z\|\right)F\left(\frac{r}{\|z\|}z\right).$$

Moreover, it is evident that we cannot have $\|z\| = r$. Consequently, putting

$$x_+ := \frac{r}{\|z\|}z, \quad \lambda_+ := \frac{(2\|z\| - r)\|z\|}{2r(r - \|z\|)}$$

we have $x_+ \in S_r(\mathbb{R}^n)$, $\lambda_+ > 0$, and $F(x_+) = \lambda_+ x_+$ as claimed. Applying the same reasoning to the operator $-F$ yields the existence of $\lambda_- < 0$ and $x_- \in S_r(\mathbb{R}^n)$ such that $F(x_-) = \lambda_- x_-$. □

We have proved Theorem 10.1 as a consequence of Brouwer's fixed point principle. Vice versa, one may also prove Brouwer's principle as a consequence of Theorem 10.1. To see this, suppose that there exists a fixed point free operator $G\colon B_r(\mathbb{R}^n) \to B_r(\mathbb{R}^n)$. Then $F\colon B_r(\mathbb{R}^n) \to \mathbb{R}^n$ defined by $F(x) := G(x) - x$ has no zeros in $B_r(\mathbb{R}^n)$, and so by Theorem 10.1 we find $\lambda > 0$ and $z \in S_r(\mathbb{R}^n)$ such that $F(z) = \lambda z$, hence $(\lambda + 1)z = G(z)$. Taking norms in this equality we obtain

$$|\lambda + 1|r = \|(\lambda + 1)z\| = \|G(z)\| \le r,$$

hence $-2 \le \lambda \le 0$, a contradiction.

Since Theorem 10.1 is equivalent to Brouwer's fixed point principle, it is not too surprising that it becomes false in infinite dimensions. We illustrate this by means of an example which comes from a classical fixed point free noncompact operator in the sequence space l_2.

Example 10.1. In $X = l_2$, consider the operator $F\colon B(X) \to X \setminus \{\theta\}$ defined by

$$F(x_1, x_2, x_3, \dots) := (1 - \|x\|^2 - x_1, x_1 - x_2, x_2 - x_3, \dots). \tag{10.6}$$

Suppose that $F(z) = \lambda z$ for some $\lambda > 0$ and $z = (z_1, z_2, z_3, \dots) \in B(X)$. Writing this equality in coordinates yields

$$z_n = \frac{1 - \|z\|^2}{(1 + \lambda)^n} \quad (n = 1, 2, 3, \dots),$$

which is impossible for $z \in B(X)$. ♡

The next theorem shows that a compact operator $F\colon B_r(X) \to X$ in an infinite dimensional space has even *two* eigenvalues (of different sign) if F is bounded away from zero on the sphere $S_r(X)$.

Theorem 10.2. *Let X be an infinite dimensional real Banach space, and let $F\colon B_r(X) \to X$ be a compact operator such that*

$$\inf_{\|x\|=r} \|F(x)\| > 0. \tag{10.7}$$

Then the set (10.4) *contains some $\lambda_+ > 0$ and some $\lambda_- < 0$.*

Since Theorem 10.2 is a special case of Theorem 10.3 below, we drop the proof. Instead, we make some remarks on its hypotheses.

First of all, the trivial example $F(x_1, x_2) = (x_2, -x_1)$ shows that Theorem 10.2 is not true in finite dimensional spaces. Moreover, we point out that the requirement (10.7) is so restrictive that it excludes linear operators! In fact, (10.7) for a compact operator $L \in \mathfrak{L}(X)$ would imply that L^{-1} exists and is bounded. On the other hand, since X is infinite dimensional, we may find a sequence $(e_n)_n$ in $S(X)$ such that $\|e_m - e_n\| \ge 1/2$ for $m \ne n$ and (without loss of generality) $(Le_n)_n$ being Cauchy, a contradiction.

Finally, the operator F_p from Example 9.10 shows that one cannot drop the compactness assumption in Theorem 10.2. Indeed, here we have

$$\inf_{\|x\|=r} \|F_p(x)\| = r^p > 0,$$

but the set $\Lambda_r(F_p) = \{r^{p-1}\}$ is a singleton.

We remark that it is not really important that the operator F in Theorem 10.2 is defined on a closed ball; the same is true for operators on the closure $\overline{\Omega}$ of some set $\Omega \in \mathfrak{OBC}(X)$. Of course, the eigenvectors corresponding to the eigenvalues λ_+ and λ_- belong then to the boundary $\partial\Omega$ of Ω.

We illustrate Theorem 10.2 by two examples.

Example 10.2. Let X be an infinite dimensional real Banach space, $e \in S(X)$ fixed, and $\varphi\colon [0,\infty) \to [0,\infty)$ a strictly increasing continuous function with $\varphi(0) = 0$. Then the compact operator $F\colon X \to X$ defined by

$$F(x) := \varphi(\|x\|)e$$

satisfies

$$\inf_{\|x\|=r} \|F(x)\| = \varphi(r) > 0,$$
$$\Lambda_r(F) = \left\{\pm\frac{\varphi(r)}{r}\right\},$$

and so

$$\sigma_{\mathrm{p}}(F) = \left\{\pm\frac{\varphi(r)}{r} : 0 < r < \infty\right\}.$$

In case $\varphi(t) = t$ we get the operator F from Example 2.46 which clearly satisfies $\Lambda_r(F) = \sigma_{\mathrm{p}}(F) = \{\pm 1\}$. ♡

Example 10.3. Let $X = L_2[0,1]$ and $F\colon B_r(X) \to X$ be given by

$$F(x)(s) = \int_0^1 (s^2 + t^2)x(t)^2\,dt. \tag{10.8}$$

The compactness of F is an obvious consequence of the fact that the range $R(F)$ of F is two-dimensional. Now, for $x \in S_r(X)$ we have

$$|F(x)(s)| = \left|s^2\int_0^1 x(t)^2\,dt + \int_0^1 t^2x(t)^2\,dt\right| \geq s^2r^2,$$

hence $\|F(x)\| \geq r^2/\sqrt{5}$. By Theorem 10.2, there exist numbers $\lambda_+ > 0$ and $\lambda_- < 0$ such that

$$\lambda_\pm x_\pm(s) = \int_0^1 (s^2 + t^2)x_\pm(t)^2\,dt$$

for some functions $x_+, x_- \in X$ with $\|x_\pm\| = r$, and so $\lambda_\pm \in \Lambda_r(F)$.

This can be made more explicit. To begin with, let us first take $r = 1$. It is clear that 0 is not an eigenvalue of the operator (10.8). Every eigenfunction x which corresponds to an eigenvalue $\lambda \neq 0$ is necessarily of the form

$$x(s) = \frac{s^2}{\lambda} + \frac{\xi}{\lambda}, \quad \xi := \int_0^1 t^2 x(t)^2 \, dt.$$

Evidently, $\xi > 0$, since $x(t) \equiv 0$ is not an eigenfunction. Integrating $\lambda^2 x(t)^2$ over $[0, 1]$ and using the fact that $\|x\| = 1$ yields

$$\lambda^2 = \frac{1}{5} + \frac{2}{3}\xi + \xi^2, \tag{10.9}$$

while integrating $\lambda^2 t^2 x(t)^2$ over $[0, 1]$ and using the definition of ξ gives

$$\lambda^2 = \frac{1}{7\xi} + \frac{2}{5} + \frac{1}{3}\xi. \tag{10.10}$$

A straightforward calculation shows that there is precisely one $\hat{\xi} > 0$ which satisfies both equations (10.9) and (10.10). If $\hat{\lambda}^2$ is the corresponding value, then $\lambda_+ = |\hat{\lambda}|$ and $\lambda_- = -|\hat{\lambda}|$ are precisely the eigenvalues whose existence is claimed in Theorem 10.2, and there are no other eigenvalues with eigenvectors of norm 1. In other words, we have shown that

$$\Lambda_1(F) = \{\lambda_+, \lambda_-\}. \tag{10.11}$$

Now, since the operator (10.8) is 2-homogeneous, from (10.5) and (10.11) we conclude that

$$\Lambda_r(F) = \{\lambda r : \lambda \in \Lambda_1(F)\} = \{r\lambda_+, r\lambda_-\}. \tag{10.12}$$

Consequently, $\sigma_{\mathrm{p}}(F) = \mathbb{R} \setminus \{0\}$. ♡

Theorem 10.3. *Let X be an infinite dimensional real Banach space, and let $F : B_r(X) \to X$ be an operator such that $[F]_{\mathrm{A}} < \infty$ and*

$$\inf_{\|x\|=r} \|F(x)\| > r\,[F]_{\mathrm{A}}. \tag{10.13}$$

Then the set (10.4) *contains some $\lambda_+ > 0$ and some $\lambda_- < 0$.*

Proof. As in (3.8), define $F_r : X \to X$ by

$$F_r(x) := \begin{cases} \frac{\|x\|}{r} F\left(\frac{r}{\|x\|}x\right) & \text{if } x \neq \theta, \\ \theta & \text{if } x = \theta. \end{cases} \tag{10.14}$$

Then F_r is 1-homogeneous, F_r and F coincide on $S_r(X)$, and $[F_r]_{\mathrm{A}} = [F]_{\mathrm{A}}$ (see Proposition 3.3). Moreover, (10.13) implies that

$$[F_r]_{\mathrm{q}} = \frac{1}{r} \liminf_{\|x\|\to\infty} \|F(\tfrac{r}{\|x\|}x)\| = \frac{1}{r} \inf_{\|x\|=r} \|F(x)\| > [F]_{\mathrm{A}} = [F_r]_{\mathrm{A}}. \tag{10.15}$$

By Theorem 6.8 we find an element $\lambda \in \sigma_q(F_r)$, and so $|\lambda| \geq [F_r]_q > [F_r]_A$, by (2.33) and (10.15). Moreover,

$$[\lambda I - F_r]_a \geq |\lambda| - [F_r]_A > 0, \tag{10.16}$$

by Proposition 2.4 (d).

Now we apply the homotopy property of k-epi operators (see Property 7.4 in Section 7.1) to the identity $F_0 = I$ on $\Omega = B_r^o(X)$ and the homotopy $H(x,t) := -tF_r(x)/\lambda$. So consider the set

$$S := \{x \in B_r(X) : \lambda x = tF_r(x) \text{ for some } t \in [0,1]\},$$

and assume that $S \cap S_r(X) = \emptyset$. Then the operator $F_1 := F_0 + H(\cdot, 1) = I - F_r/\lambda$ is k-epi on $B_r(X)$ for any $k < 1 - [F_r]_A/|\lambda|$, and so $\lambda I - F_r$ is $|\lambda|k$-epi on $B_r(X)$. But this, together with (10.16), implies that $\lambda \in \rho_F(F_r) \subseteq \rho_{FMV}(F_r)$, contradicting the fact that $\lambda \in \sigma_q(F_r)$.

So we know that $\lambda x_+ = tF_r(x_+)$ for some $x_+ \in S_r(X)$ and $t \in (0,1]$. Putting this into (10.14) yields $F(x_+) = \lambda_+ x_+$ with $\lambda_+ := \lambda/t$. We may assume without loss of generality (otherwise we change the notation) that $\lambda > 0$, hence $\lambda_+ > 0$ as well. On the other hand, the operator $-F$ satisfies the same hypotheses as F, and so by the same reasoning we get another eigenvalue $\lambda_- < 0$ such that $F(x_-) = \lambda_- x_-$ for some $x_- \in S_r(X)$. □

Since the right-hand side of (10.13) is zero for compact F, Theorem 10.3 contains Theorem 10.2 as a special case. We give a simple example where Theorem 10.2 does not apply, but Theorem 10.3 does.

Example 10.4. Let X be an infinite dimensional real Banach space. Fix $y \in X$ with $\|y\| > r$, and consider the operator $F\colon B_r(X) \to X$ defined by $F(x) := x + y$. Clearly, $[F]_A = 1$, and so F is not compact.

Now, it is easy to see that the eigenvalue equation $F(x) = \lambda x$ has precisely two solutions $(\lambda, x) \in \mathbb{R} \times S_r(X)$, namely

$$\lambda_+ = 1 + \frac{\|y\|}{r} > 0, \quad x_+ = \frac{r}{\|y\|}y$$

and

$$\lambda_- = 1 - \frac{\|y\|}{r} < 0, \quad x_- = -\frac{r}{\|y\|}y.$$

Consequently, we have $\Lambda_r(F) = \left\{1 \pm \frac{\|y\|}{r}\right\}$ and $\sigma_p(F) = \mathbb{R} \setminus \{1\}$. ♡

The operator F in Example 10.4 illustrates again the importance of the condition (10.13). Indeed, if we choose $\|y\| = r$ instead of $\|y\| > r$, then the left-hand side of (10.13) is zero and $\Lambda_r(F) = \{0, 2\}$.

We use the eigenvalue set (10.4) to introduce a special class of operators. For $F \in \mathfrak{C}(X)$ and $r > 0$ we put

$$\delta_r(F) := \sup\{|\lambda| : \lambda \in \Lambda_r(F)\} \tag{10.17}$$

and

$$[F]_{SQ} := \sup_{z\in X} \inf_{r>0} \delta_r(F_z), \tag{10.18}$$

where $F_z(x) := F(x) + z$ as in (2.24). (If the set on the right-hand side of (10.17) is empty we put $\delta_r(F) := 0$.) The number (10.18) will be called the *s-quasinorm of* F in what follows, and we say that F is *s-quasibounded* if $[F]_{SQ} < \infty$. For example, every quasibounded operator is also s-quasibounded with $[F]_{SQ} \le [F]_{\mathrm{Q}}$. In fact, from the estimate

$$\delta_r(F) \le \sup_{\|x\|=r} \frac{\|F(x)\|}{\|x\|} \le \sup_{\|x\|\ge r} \frac{\|F(x)\|}{\|x\|}$$

it follows that

$$\inf_{r>0} \delta_r(F_z) \le \inf_{r>0} \sup_{\|x\|\ge r} \frac{\|F(x)+z\|}{\|x\|} = \limsup_{\|x\|\to\infty} \frac{\|F(x)+z\|}{\|x\|} = [F_z]_{\mathrm{Q}} = [F]_{\mathrm{Q}}$$

for each $z \in X$, and this implies that also $[F]_{SQ} \le [F]_{\mathrm{Q}}$. In the next proposition we show that the s-quasinorm of F may be regarded as a certain "measure of surjectivity" of the operator $I - F$.

Proposition 10.1. *Let* $F\colon X \to X$ *be* α*-contractive, i.e.,* $[F]_{\mathrm{A}} < 1$. *Then the following is true.*

(a) *If* $\delta_r(F) \le 1$ *for some* $r > 0$, *then* F *has a fixed point in* $B_r(X)$.

(b) *If* $\delta_r(F_z) \le 1$ *for some* $r > 0$, *then* $z \in (I - F)(B_r(X))$.

(c) *If* $[F]_{SQ} < 1$, *then* $I - F$ *is onto.*

Proof. Fix $r > 0$ such that $\delta_r(F) \le 1$, and consider the radial retraction $\rho\colon X \to B_r(X)$ given by (2.20). Since $\rho F\colon B_r(X) \to B_r(X)$ satisfies $[\rho F]_{\mathrm{A}} \le [\rho]_{\mathrm{A}}[F]_{\mathrm{A}} \le [F]_{\mathrm{A}} < 1$, Theorem 2.1 implies that there exists $\hat{x} \in B_r(X)$ with $\hat{x} = \rho(F(\hat{x}))$. The assumption $\|F(\hat{x})\| > r$ leads to the equality

$$F(\hat{x}) = \frac{\|F(\hat{x})\|}{r}\hat{x} = \lambda\hat{x}$$

with $\lambda := \|F(\hat{x})\|/r > 1$ and $\|\hat{x}\| = r$, hence $\lambda \in \Lambda_r(F)$, contradicting the hypothesis $\delta_r(F) \le 1$. So we necessarily have $\|F(\hat{x})\| \le r$. But this implies that $\rho(F(\hat{x})) = F(\hat{x})$, and so $\hat{x} \in B_r(X)$ is a fixed point of F, which proves (a).

From (a) it follows that the operator (2.24) has a fixed point $\hat{x} \in B_r(X)$, and so $(I - F)(\hat{x}) = z$ which proves (b). Finally, the assertion (c) follows immediately from the definition of $[F]_{SQ}$ and from (b). □

The estimate $[F]_{SQ} \leq [F]_{\mathrm{Q}}$ shows that Proposition 10.1 (c) extends Theorem 2.2. The simple example $L(x_1, x_2) = (-x_2, x_1)$ on $X = \mathbb{R}^2$ shows that this is a proper extension even for linear operators. Indeed, $[L]_{SQ} < 1 = [L]_{\mathrm{Q}}$ in this example. In the following Example 10.5 we show how the hypotheses of Proposition 10.1 may be easily checked in Hilbert spaces.

Example 10.5. Let H be a real Hilbert space with scalar product $\langle \cdot, \cdot \rangle$, and let $F\colon H \to H$ be an operator satisfying

$$\lim_{\|x\|\to\infty} \frac{1}{\|x\|} \langle x - F(x), x \rangle = \infty. \tag{10.19}$$

We claim that there exists $R > 0$ such that $\delta_r(F) \leq 1$ for all $r > R$. In fact, otherwise there exist sequences $(\lambda_n)_n$ with $\lambda_n > 1$ and $(x_n)_n$ with $\|x_n\| = n$ such that $F(x_n) = \lambda_n x_n$. Then

$$\frac{1}{\|x_n\|} \langle x_n - F(x_n), x_n \rangle = n(1 - \lambda_n) < 0,$$

contradicting (10.19). So we find $R > 0$ such that $\delta_r(F) \leq 1$ for $r \geq R$. But from (10.19) it follows that also

$$\lim_{\|x\|\to\infty} \frac{1}{\|x\|} \langle x - F_z(x), x \rangle = \infty$$

for each $z \in X$, and so Proposition 10.1 implies that $I - F$ is surjective. ♡

10.2 Eigenvalue problems in cones

Sometimes one is interested in solutions of eigenvalue problems with additional properties. One of the most important examples is that of *positive* solutions; in this case one usually works in spaces with cones. In this section we consider some existence results for solutions of eigenvalue problems in Banach spaces with cones.

Let X be a real Banach space. A *cone* in X is a convex set K with the property that $tx \in K$ for every $x \in K$ and $t \geq 0$, and $K \cap (-K) = \{\theta\}$, i.e., $x \in K$ and $-x \in K$ is possible only for $x = \theta$. In every cone K one may find a point $x_0 \in K \cap S(X)$ and a number $\gamma > 0$ such that

$$\|x + \lambda x_0\| \geq \gamma \|x\| \quad (x \in K,\ \lambda \geq 0). \tag{10.20}$$

The biggest constant γ for which (10.20) holds is called the *cone constant* of K and denoted by $\gamma(K)$. To calculate the constant $\gamma(K)$ for a given cone K is in general not easy; a pleasant exception where this is possible is described in the following example.

Example 10.6. Let H be a real Hilbert space with scalar product $\langle\cdot,\ \cdot\rangle$, and let K be some cone in H. We claim that

$$\gamma(K) = 1. \tag{10.21}$$

Without loss of generality we may assume that K is not contained in a proper subspace of H. There exists a continuous linear functional $\ell \in H^*$ such that $\|\ell\| = 1$ and $\ell(x) \geq 0$ for any $x \in K$. By Riesz' representation theorem we have

$$\ell(x) = \langle y, x\rangle \quad (x \in K)$$

for some $y \in S(H)$. Now we distinguish the two cases $y \in K$ and $y \notin K$.

Suppose first that $y \in K$. Then we set $x_0 := y$ and get for every $x \in K$ and $\lambda \geq 0$, using the shortcut $\mu := \langle x, x_0\rangle$,

$$\begin{aligned}\|x + \lambda x_0\|^2 &= \|(x - \mu x_0) + (\lambda + \mu)x_0\|^2\\ &= \|x - \mu x_0\|^2 + (\lambda + \mu)^2\|x_0\|^2\\ &\geq \|x - \mu x_0\|^2 + \mu^2\|x_0\|^2 = \|x\|^2,\end{aligned} \tag{10.22}$$

i.e., (10.20) holds. On the other hand, suppose that $y \notin K$, and denote by z the point in K which is closest to y, i.e.,

$$\|y - z\| = \operatorname{dist}(y, K).$$

It is easy to see that $0 < \|z\| < 1$, since K is not contained in a proper subspace of H. By definition of z, we further have

$$\langle y - z, x - z\rangle \leq 0 \quad (x \in K)$$

and $\langle y - z, z\rangle = 0$. We therefore obtain

$$\langle y, x\rangle \leq \langle z, x\rangle + \langle y - z, z\rangle = \langle z, x\rangle.$$

So we may set $x_0 := z/\|z\|$ and repeat the above calculation. Putting $\lambda = 0$ in (10.22) we see that $\gamma(K) = 1$. ♡

Now we are going to study nonlinear operators in cones. Given a cone K in a Banach space X, we put

$$\mathfrak{OBC}_K(X) := \{\Omega \cap K : \Omega \in \mathfrak{OBC}(X)\}, \tag{10.23}$$

where $\mathfrak{OBC}(X)$ is the family of all open, bounded, connected subsets of X containing θ (see Section 7.1). In this section we will always consider operators which are defined on the closure of some set from $\mathfrak{OBC}_K(X)$ and take values in K. In order to not overburden the notation, we simple write $F\colon \overline{\Omega} \to K$, where we identify Ω with $\Omega \cap K$ and $\overline{\Omega}$ means the relative closure (in K) of $\Omega \in \mathfrak{OBC}_K(X)$. Similarly, $\partial\Omega = \overline{\Omega} \cap \overline{K \setminus \Omega}$ denotes the relative boundary (in K) of $\Omega \in \mathfrak{OBC}_K(X)$.

Before stating our first existence result for eigenvectors in cones we need a technical lemma.

Lemma 10.1. *Suppose that K is a cone in a real Banach space X, $\Omega \in \mathfrak{OBC}_K(X)$, and $G\colon \overline{\Omega} \to K$ is compact with*

$$\delta := \inf_{x\in\partial\Omega} \|G(x)\| > \sup_{x\in\overline{\Omega}} \|x\| =: r. \tag{10.24}$$

Then the operator $I - G$ is not epi on $\overline{\Omega}$.

Proof. Without loss of generality, we may assume that $\|G(x)\| \equiv \delta$ on $\partial\Omega$. Put $R := \sup\{\|x\| + \|G(x)\| : x \in \overline{\Omega}\}$, and choose $\varepsilon > 0$ such that $\delta > r + \varepsilon$. Fix $y \in K \setminus \{\theta\}$ with $\|y\| < \varepsilon$, and define an operator $T\colon \overline{\Omega} \setminus \{\theta\} \to S_R(X) \cap K$ by

$$T(x) := R\frac{G(x) + y}{\|G(x) + y\|} \quad (x \in \overline{\Omega},\ x \neq \theta).$$

This operator is well defined and continuous, since $G(x) + y \neq \theta$ on $\overline{\Omega}$, by our choice of ε.

Now we use Property 7.4 of k-epi operators with $F_0 := I - G$. Suppose that $I - G$ is epi on $\overline{\Omega}$, and define a homotopy $H\colon \overline{\Omega} \times [0, 1] \to K$ by $H(x, t) := t[G(x) - T(x)]$. Then our assumption $\delta > r + \varepsilon$ implies that $x - G(x) \neq t[T(x) - G(x)]$ for $t \in [0, 1]$ and $x \in \partial\Omega$, and so the set

$$S := \{x \in \overline{\Omega} : x - G(x) + H(x, t) = \theta \text{ for some } t \in [0, 1]\}$$

satisfies $S \cap \partial\Omega = \emptyset$. From Property 7.4 we conclude that the operator $F_1 := F_0 + H(\cdot, 1) = I - T$ is epi on $\overline{\Omega}$. So from Property 7.1 we further conclude that the equation $x - T(x) = \theta$ has a solution $\hat{x} \in \Omega$. But for this solution we get

$$R = \|T(\hat{x})\| = \|\hat{x}\| \leq r < R,$$

a contradiction. So the operator $I - G$ cannot be epi on $\overline{\Omega}$, and the lemma is proved. □

We state now a theorem which not only provides eigenvectors in cones, but also a lower estimate for the corresponding eigenvalues. First we make a simple but useful remark. Suppose that $F\colon \overline{\Omega} \to K$ satisfies

$$\delta := \inf_{x\in\partial\Omega} \|F(x)\| > 0, \tag{10.25}$$

and r is defined as in (10.24). Then F cannot have eigenvalues $\lambda < \delta/r$ with corresponding eigenvectors on the boundary of Ω. In fact, taking norms in the eigenvalue equation $F(x) = \lambda x$ with $x \in \partial\Omega$ yields

$$\delta \leq \|F(x)\| = \lambda\|x\| \leq \lambda r,$$

and so every eigenvalue λ necessarily belongs to the interval $[\delta/r, \infty)$. The next theorem shows that this interval contains in fact an eigenvalue, at least for compact operators.

Theorem 10.4. *Given $\Omega \in \mathfrak{DBC}_K(X)$, let $F: \overline{\Omega} \to K$ be a compact operator such that* (10.25) *is true, and let r be defined as in* (10.24). *Then there exist $x \in \partial\Omega$ and $\lambda \geq \delta/r$ such that $F(x) = \lambda x$.*

Proof. Assume that $F(x) \neq \lambda x$ for all $x \in \partial\Omega$ and $\lambda \geq \delta/r$. Then $I - F$ is epi on $\overline{\Omega}$, as may be seen by applying Property 7.4 to $F_0 := I$ and $H(x, t) := -tF(x)$. Indeed H is a compact homotopy, and the assumption $x - tF(x) = \theta$ for some $t \in [0, 1]$ is impossible for $x \in \partial\Omega$, by (10.24).

Define $G: \overline{\Omega} \to K$ by $G(x) := \mu F(x)$, where $\mu\delta > r$. Then

$$\inf_{x\in\partial\Omega} \|G(x)\| = \mu \inf_{x\in\partial\Omega} \|F(x)\| > \frac{r}{\delta} \inf_{x\in\partial\Omega} \|F(x)\| = r. \tag{10.26}$$

Moreover, our choice of μ implies that $x - F(x) \neq t[G(x) - F(x)]$ for $t \in [0, 1]$ and $x \in \partial\Omega$. So we may again apply Property 7.4 with $F_0 := I - F$ and $H(x, t) := t[F(x) - G(x)]$ and conclude that the operator $F_1 = I - F + H(\cdot, 1) = I - G$ is epi on $\overline{\Omega}$. On the other hand, the estimate (10.26) and Lemma 10.1 imply that $I - G$ is not epi on $\overline{\Omega}$. This contradiction shows that our assumption was false, and so the assertion is proved. □

The following example shows that the assumption (10.25) which has been also important in Theorem 10.2, cannot be dropped in Theorem 10.4.

Example 10.7. Let $X = l_2$, $\Omega = B^o(X)$, and $L: \overline{\Omega} \to X$ be the linear operator defined by

$$L(x_1, x_2, x_3, x_4, \dots) := (0, x_1, \tfrac{1}{2}x_2, \tfrac{1}{3}x_3, \dots).$$

Since $Le_k = e_{k+1}/k$, it is clear that $\inf\{\|Lx\| : \|x\| = 1\} = 0$. In fact, we have already seen in Example 1.5 (see (1.45)) that L has no eigenvalues. ♡

We give now an extension of Theorem 10.4 from compact to α-Lipschitz operators (Theorem 10.5). To this end, we need again a technical lemma .

Lemma 10.2. *Suppose that K is a cone in a real Banach space X with cone constant $\gamma(K)$, $\Omega \in \mathfrak{DBC}_K(X)$, and $G: \overline{\Omega} \to K$ satisfies $[G]_A < 1$ and*

$$\delta := \inf_{x\in\partial\Omega} \|G(x)\| > \frac{r}{\gamma(K)}, \tag{10.27}$$

with r as in (10.24). *Then the operator $I - G$ is not epi on $\overline{\Omega}$.*

Proof. Fix $x_0 \in K \cap S(X)$ such that $\|x + \lambda x_0\| \geq \gamma(K)\|x\|$ for all $x \in K$ and $\lambda \geq 0$. By the Hahn–Banach theorem we may choose $\ell \in X^*$ such that $\ell(x_0) = 1$ and $\|\ell\| = 1$. Fix a number M such that

$$M > \max\left\{\sup_{x\in\overline{\Omega}} |\ell(G(x))|, \sup_{x\in\overline{\Omega}} \|x - G(x) + \ell(G(x))x_0\|\right\}. \tag{10.28}$$

We define two operators by

$$\tilde{F}(x) := x - G(x), \quad \tilde{G}(x) := [\ell(G(x)) - M]x_0.$$

So we have to show that $\tilde{F} = I - G$ is not epi on $\overline{\Omega}$. Suppose that this is not true. Then we apply Lemma 8.1 to the pair $(\tilde{F}, \tilde{G})$. By Theorem 7.1 we have

$$\nu_\Omega(\tilde{F}) \geq [\tilde{F}|_{\overline{\Omega}}]_a > 0,$$

and so

$$[\tilde{G}|_{\overline{\Omega}}]_A = 0 < \nu_\Omega(\tilde{F}),$$

since $\tilde{G}$, having a one-dimensional range, is certainly compact. Finally,

$$\sup_{x \in \partial\Omega} \|\tilde{G}(x)\| = \sup_{x \in \partial\Omega} |\ell(G(x)) - M| > \inf_{x \in \partial\Omega} \|x - G(x)\| = \inf_{x \in \partial\Omega} \|\tilde{F}(x)\|.$$

So all hypotheses of Lemma 8.1 are satisfied, and thus the operator $\tilde{F} + \tilde{G}$ is epi on $\overline{\Omega}$. In particular, we find some $\tilde{x} \in \Omega$ such that $\tilde{F}(\tilde{x}) + \tilde{G}(\tilde{x}) = \tilde{x} - G(\tilde{x}) + \ell(G(\tilde{x}))x_0 - Mx_0 = \theta$. But this implies that

$$M = M\|x_0\| = \|\tilde{x} - G(\tilde{x}) + \ell(G(\tilde{x}))x_0\|,$$

contradicting our choice (10.28) of M. So our assumption was false, and we see that $I - G$ is not epi on $\overline{\Omega}$. □

We illustrate the importance of the estimate (10.27) by means of the following

Example 10.8. Let $X = l_2$, $\Omega = B^o(X)$, and $L\colon \overline{\Omega} \to X$ be the linear right shift operator (1.38). Since $\|Lx\| = \|x\|$, we have $[\mu L]_A = \|\mu L\| = |\mu|$ for any $\mu \in \mathbb{R}$. Moreover, since l_2 is a Hilbert space, the usual cone $K = \{(x_n)_n \in l_2 : x_n \geq 0 \ (n \in \mathbb{N})\}$ has cone constant $\gamma(K) = 1$, as we have seen in Example 10.6. So condition (10.27) holds for $G = \mu L$ precisely if $|\mu| > 1$.

In fact, for $|\mu| < 1$ the operator $I - \mu L$ is a linear isomorphism, because $\sigma(L) = [-1, 1]$ (see Example 1.4), and so $I - \mu L$ is epi on $B(X)$ for $|\mu| < 1$. More precisely, Example 7.2 shows that $\nu_{B^o(X)}(\mu L) \geq [\![\mu L]\!] = |\mu|$. ♡

Theorem 10.5. *Given $\Omega \in \mathfrak{OBC}_K(X)$, let $F\colon \overline{\Omega} \to K$ be an operator satisfying $[F]_A < \infty$ and*

$$\delta := \inf_{x \in \partial\Omega} \|F(x)\| > \frac{r}{\gamma(K)}[F]_A, \tag{10.29}$$

where r is defined as in (10.24). *Then there exist $x \in \partial\Omega$ and $\lambda \geq \delta/r$ such that $F(x) = \lambda x$.*

Proof. Suppose that $F(x) \neq \lambda x$ for $\lambda \geq \delta/r$, and so for all $\lambda \in \mathbb{R}$, by the remark preceding Theorem 10.4. Define $G\colon \overline{\Omega} \to K$ by $G(x) := \mu F(x)$, where

$$\frac{r}{\delta\gamma(K)} < \mu < \frac{1}{[F]_{\mathrm{A}}}.$$

Then

$$[G]_{\mathrm{A}} = \mu[F]_{\mathrm{A}} < 1, \tag{10.30}$$

i.e., $G\colon \overline{\Omega} \to K$ is an α-contraction. Moreover,

$$\inf_{x\in\partial\Omega} \|G(x)\| = \mu \inf_{x\in\partial\Omega} \|F(x)\| = \mu\delta > \frac{r}{\gamma(K)}. \tag{10.31}$$

By Lemma 10.2, the operator $I - G$ is not epi on $\overline{\Omega}$. On the other hand, $x - tG(x) \neq \theta$ for $t \in [0, 1]$ and $x \in \partial\Omega$, by our assumption, and so Property 7.4 (with $F_0 := I$ and $H(x, t) := -tG(x)$) implies that $I - G$ is epi on $\overline{\Omega}$. This contradiction shows that our assumption was false, and so we are done. □

10.3 A nonlinear Krejn–Rutman theorem

In this section we are going to present a theorem of Krejn–Rutman type for eigenvalues of homogeneous nonlinear operators. Recall that the classical Krejn–Rutman theorem may be stated as follows: *given a real Banach space X with cone K and a compact linear operator $L\colon X \to X$ with $L(K) \subseteq K$, the spectral radius $r(L)$ of L is a simple eigenvalue of L with eigenvector x in the interior of C.*

Let X be a real Banach space with cone K. Then $\mathcal{K} := K \times [0, \infty)$ is a cone in the space $X \times \mathbb{R}$ equipped with the norm $\|(x, \lambda)\| := \max\{\|x\|, |\lambda|\}$.

Proposition 10.2. *Let $F\colon \mathcal{K} \to K$ be compact with $F(x, 0) \equiv \theta$. Then the set*

$$\Sigma := \{(x, \lambda) \in \mathcal{K} : F(x, \lambda) = x\} \tag{10.32}$$

contains an unbounded component C containing $(\theta, 0)$.

Proof. For $n = 1, 2, 3, \ldots$ we set

$$B_n(\mathcal{K}) := \{(x, \lambda) \in \mathcal{K} : \|(x, \lambda)\| \leq n\}$$

and

$$S_n(\mathcal{K}) := ([B_n(X) \cap K] \times \{n\}) \cup ([S_n(X) \cap K] \times (0, n]) .$$

Let C be the connected component of Σ which contains $(\theta, 0)$. We claim that $C \cap S_n(\mathcal{K}) \neq \emptyset$ for all $n \in \mathbb{N}$; this will of course imply the unboundedness of C.

Assume that this is false. Then there exist a number n and an open neighbourhood $U \subseteq B_n(\mathcal{K})$ of $(\theta, 0)$ such that $\partial U \cap \Sigma = \emptyset$. Let $\pi : B_n(\mathcal{K}) \to [0, 1]$ be some Uryson function which is 1 on $\Sigma \cap U$ and 0 on $B_n(\mathcal{K}) \setminus U$. Define $G\colon B_n(\mathcal{K}) \to X \times [0, \infty)$ by

$$G(x, \lambda) := (F(x, \pi(x, \lambda)n), \pi(x, \lambda)n).$$

Then G is compact, because F is, and $G(x, \lambda) \equiv (\theta, 0)$ for $(x, \lambda) \in S_n(\mathcal{K})$ and for $(x, \lambda) \in ([B_n(X) \cap K] \times \{0\})$. Therefore G has a fixed point $(x_n, \lambda_n) \in B_n(\mathcal{K})$, i.e.,

$$(x_n, \lambda_n) = (F(x_n, \pi(x_n, \lambda_n)n), \pi(x_n, \lambda_n)n). \tag{10.33}$$

Observe that (10.33) implies $\lambda_n = \pi(x_n, \lambda_n)n \in [0, n]$. We distinguish two cases. First, suppose that $(x_n, \lambda_n) \in U$. Then $x_n = F(x_n, \lambda_n)$, hence $(x_n, \lambda_n) \in \Sigma$ and $\pi(x_n, \lambda_n) = 1$. Consequently, $\lambda_n = n$ and $x_n = F(x_n, n)$, and so $(x_n, n) \in S_n(\mathcal{K})$. But this implies $\pi(x_n, \lambda_n) = 0$, a contradiction.

On the other hand, suppose that $(x_n, \lambda_n) \notin U$. Then $\pi(x_n, \lambda_n) = 0$, and so $\lambda_n = 0$ and $x_n = F(x_n, 0) = \theta$. But $(\theta, 0) \in C$ and thus $\pi(\theta, 0) = 1$, again a contradiction. We conclude that the component C intersects $S_n(\mathcal{K})$, and so it is unbounded, since this is true for any $n \in \mathbb{N}$. □

Before stating now a Krejn–Rutman type theorem for nonlinear operators, we need two auxiliary lemmas. For the remaining part of this section we suppose that $F\colon K \to K$ is 1-homogeneous, i.e., $F(tx) = tF(x)$ for $t \geq 0$ and $x \in K$. In this case the characteristic (2.6) is simply

$$[F]_{\mathrm{B}} = \sup_{x \neq \theta} \frac{\|F(x)\|}{\|x\|} = \sup_{x \in S(X)} \|F(x)\|,$$

while the measure of noncompactness (2.13) of F may be expressed simply by

$$[F]_{\mathrm{A}} = \sup \left\{ \frac{\alpha(F(M))}{\alpha(M)} : M \subseteq S(X),\ \alpha(M) > 0 \right\}.$$

In the following two lemmas we use the abbreviation

$$r(F) := \limsup_{n \to \infty} [F^n]_{\mathrm{B}}^{1/n} \tag{10.34}$$

and call (10.34) the *spectral radius* of F. For linear operators this is of course compatible with the usual notion (1.8) of spectral radius.

Lemma 10.3. *Let $F\colon K \to K$ be 1-homogeneous. Assume that there exist $u \in S(X) \cap K$ and $\rho \leq r(F)$ such that*

$$\limsup_{n \to \infty} \frac{\|F^n(u)\|}{\rho^n} > 0. \tag{10.35}$$

Then the sequence $(\|(\mu F)^n(u)\|)_n$ is unbounded for every $\mu > 1/\rho$.

Proof. Since

$$\|(\mu F)^n(u)\| = \mu^n \|F^n(u)\| = \frac{\|F^n(u)\|}{\rho^n}(\mu\rho)^n$$

and $\mu\rho > 1$, the assertion follows. □

Lemma 10.4. *Let $F\colon K \to K$ be 1-homogeneous with*

$$\limsup_{n\to\infty} [F^n]_{\mathrm{A}}^{1/n} < 1. \tag{10.36}$$

Assume that there exists $u \in S(X) \cap K$ such that the sequence $(\|F^n(u)\|)_n$ is unbounded. Then the sequence $(v_n)_n$ defined by

$$v_n := \frac{F^n(u)}{\|F^n(u)\|} \tag{10.37}$$

admits a convergent subsequence.

Proof. It is not hard to see that we may choose a subsequence $(\|F^{n_k}(u)\|)_k$ such that

$$\|F^{n_k - j}(u)\| \leq \|F^{n_k}(u)\| \quad (j = 1, 2, \ldots, n_k - 1). \tag{10.38}$$

Fix $m \in \mathbb{N}$ such that $[F^m]_{\mathrm{A}} < 1$, which is possible by our assumption (10.36). We put

$$M := \{v_{n_k} : k \in \mathbb{N}\}, \quad M_m := \{v_{n_k} : n_k > m\}.$$

Obviously, $\alpha(M) = \alpha(M_m)$. The set

$$N_m := \left\{ \frac{F^{n_k - m}(u)}{\|F^{n_k}(u)\|} : n_k > m \right\}$$

satisfies $N_m \subseteq S(X) \cap K$ and $F^m(N_m) = M_m$. Consequently,

$$\alpha(M) = \alpha(M_m) \leq [F^m]_{\mathrm{A}}\alpha(N_m) \leq [F^m]_{\mathrm{A}}\alpha(S(X) \cap K).$$

Similarly, replacing m by m^2 we get

$$\alpha(M) \leq [F^m]_{\mathrm{A}}^2 \alpha(S(X) \cap K),$$

and, for general $p \in \mathbb{N}$,

$$\alpha(M) \leq [F^m]_{\mathrm{A}}^p \alpha(S(X) \cap K).$$

But this implies that $\alpha(M) = 0$, since $[F^m]_{\mathrm{A}} < 1$. Consequently, the set M is precompact, and so $(v_{n_k})_k$ contains a convergent subsequence. □

Recall that every cone K in a linear space X induces a partial ordering $\preceq$ by defining $x \preceq y$ if $y - x \in K$. Conversely, given a partial ordering $\preceq$ on X, the set $K := \{x \in X : \theta \preceq x\}$ is a cone in X. An operator $F\colon K \to K$ is called *order preserving* if $x \preceq y$ implies $F(x) \preceq F(y)$. A *linear* operator $L\colon X \to X$ with $L(K) \subseteq K$ is always order preserving, since $y - x \in K$ implies $Ly - Lx = L(y - x) \in K$. For nonlinear operators $F\colon K \to K$, however, this is an additional requirement.

Theorem 10.6. *Let $F\colon K \to K$ be* 1*-homogeneous and order preserving with*

$$[F]_{\mathrm{A}} < r(F), \tag{10.39}$$

where $r(F)$ is given by (10.34)*. Assume that there exist $u \in S(X) \cap K$ and $\rho > 0$ satisfying $[F]_{\mathrm{A}} < \rho \le r(F)$ and* (10.35)*. Then there exist $\hat{\lambda} \in [\rho, r(F)]$ and $\hat{x} \in S(X) \cap K$ such that $F(\hat{x}) = \hat{\lambda}\hat{x}$.*

Proof. Define $G\colon K \to K$ by $G(x) := \mu F(x)$, where

$$\frac{1}{\rho} < \mu < \frac{1}{[F]_{\mathrm{A}}}. \tag{10.40}$$

Then the sequence $(\|G^n(u)\|)_n$ is unbounded, by Lemma 10.3. Moreover, there exists a subsequence $(\|F^{n_k}(u)\|)_k$ of $(\|F^n(u)\|)_n$ such that

$$\lim_{k\to\infty} \frac{\|F^{n_k}(u)\|}{\rho^{n_k}} = \limsup_{n\to\infty} \frac{\|F^n(u)\|}{\rho^n} > 0.$$

So by Lemma 10.4 we may assume, without loss of generality, that the sequence $(v_k)_k$ defined by

$$v_k := \frac{F^{n_k}(u)}{\|F^{n_k}(u)\|} \tag{10.41}$$

converges to some $v \in S(X)$. For $\varepsilon > 0$, consider the equation

$$x = \lambda[G(x) + \varepsilon G(u)]. \tag{10.42}$$

Since $[G]_{\mathrm{A}} = \mu[F]_{\mathrm{A}} < 1$, by Proposition 10.2 we find a connected branch of solutions of (10.42) joining $(\theta, 0) \in \mathcal{K} = K \times [0, \infty)$ with $S_{1/\mu}(\mathcal{K})$. From (10.42) we obtain $\lambda G(x) \preceq x$ and $\lambda\varepsilon G(u) \preceq x$. Similarly, by induction we find $\lambda^n \varepsilon G^n(u) \preceq x$, hence

$$\lambda^n \varepsilon \frac{G^n(u)}{\|G^n(u)\|} \preceq \frac{x}{\|G^n(u)\|}.$$

Passing to the convergent subsequence (10.41) and observing that $G^n(u) = \mu^n F^n(u)$ we obtain

$$\varepsilon v_k \preceq \frac{x}{\mu^{n_k}\lambda^{n_k}\|F^{n_k}(u)\|} = \frac{1}{(\mu\rho\lambda)^{n_k}} \frac{\rho^{n_k} x}{\|F^{n_k}(u)\|}.$$

We cannot have $(\mu\rho\lambda)^k \to \infty$, since this would imply the contradiction $\varepsilon v \preceq \theta$. So $\mu\rho\lambda \leq 1$, i.e., $\lambda \leq \frac{1}{\mu\rho} < 1$, by (10.40). We conclude that equation (10.42) has a solution $x_\varepsilon \in S(X)$. Applying this reasoning to $\varepsilon = 1/n$ yields sequences $(x_n)_n$ in $S(X)$ and $(\lambda_n)_n$ in $[0, 1]$ such that

$$x_n = \lambda_n \left[G(x_n) + \tfrac{1}{n} G(u)\right].$$

The set $\{x_1, x_2, x_3, \dots\}$ is obviously precompact, and so we find $\hat{x} \in S(X)$ and $\hat{\lambda} \in [0, 1]$ such that $x_n \to \hat{x}$, $\lambda_n \to \hat{\lambda}$, and

$$\hat{x} = \hat{\lambda} G(\hat{x}) = \mu\hat{\lambda} F(\hat{x}). \tag{10.43}$$

But $\lambda_n \rho\mu \leq 1$, and so also $\hat{\lambda}\rho\mu \leq 1$. Finally,

$$\frac{1}{\mu\hat{\lambda}} = \lim_{n\to\infty} \|F^n(\hat{x})\|^{1/n} \leq \limsup_{n\to\infty} [F^n]_{\mathrm{B}}^{1/n} = r(F),$$

and so the proof is complete. □

The following example shows how crucial is the assumption (10.35) occurring in Lemma 10.3 and Theorem 10.6.

Example 10.9. Consider the Cauchy problem for the functional-differential equation

$$\begin{cases} x'(t) = \mu\sqrt{x(t)^2 + x(1-t)^2}, \\ x(0) = 0, \end{cases} \tag{10.44}$$

where $\mu \neq 0$ is a real parameter. We study this problem in the space $X = C[0, 1]$ of continuous functions with the usual norm and the cone K of nonnegative functions. Defining $F\colon X \to X$ as usual by

$$F(x)(s) := \int_0^s \sqrt{x(t)^2 + x(1-t)^2}\, dt \quad (0 \leq s \leq 1), \tag{10.45}$$

we see that F is 1-homogeneous, compact, order preserving, and satisfies $\|F(x)\| \leq \sqrt{2}\|x\|$, so $[F]_{\mathrm{B}} \leq \sqrt{2}$. Moreover, every nonnegative solution x of the problem (10.44) solves the eigenvalue problem $F(x) = \lambda x$ with $\lambda = 1/\mu$ and $x \in K$.

We consider the operator (10.45) on the intersection $B(X) \cap K$ of the unit ball and the cone in X. It is not hard to see that $\inf\{\|F(x)\| : x \in S(X)\} = 0$, so Theorem 10.2 does not apply. On the other hand, a cumbersome but straightforward calculation shows that

$$r(F) = \limsup_{n\to\infty} [F^n]_{\mathrm{B}}^{1/n} = \frac{1}{\sqrt{2}\log(1+\sqrt{2})} > 0 \tag{10.46}$$

and

$$\lim_{n\to\infty} \frac{\|F^n(u)\|}{\rho^n} > 0 \tag{10.47}$$

for $u(t) \equiv 1$ and $\rho = 1/\sqrt{2}$. So from Theorem 10.6 we conclude that there exist $\hat{\lambda} \in [1/\sqrt{2}, r(F)]$ and $\hat{x} \in S(X) \cap K$ such that $F(\hat{x}) = \hat{\lambda}\hat{x}$.

A scrutiny of problem (10.44) leads to the following alternative. On the one hand, if $\mu \neq \sqrt{2}\log(1+\sqrt{2})$, the only solution of (10.44) is $x(t) \equiv 0$, and so $\lambda = 1/\mu$ is not an eigenvalue of the operator (10.45). On the other hand, for $\hat{\mu} = \sqrt{2}\log(1+\sqrt{2})$ we get the nontrivial solution

$$\hat{x}(t) = \frac{1}{2}\sinh\left[2\log\left(1+\sqrt{2}\right)t + \log\left(-1+\sqrt{2}\right)\right] + \frac{1}{2}.$$

It is easy to see that this solution belongs to $S(X) \cap K$. Since $\sqrt{2}\log(1+\sqrt{2}) < \sqrt{2}$, the eigenvalue $\hat{\lambda} = 1/\hat{\mu}$ belongs to the interval $(1/\sqrt{2}, r(F)]$. ♡

We will come back to Example 10.9 in Section 12.1 in connection with general solvability results for nonlinear operator equations (see Example 12.1).

10.4 Other notions of eigenvalue

We already noticed several times why the notion of eigenvalue in the classical sense of (10.1) or (10.2) is not appropriate. For this reason we considered other notions of eigenvalues, namely the *asymptotic point spectrum*

$$\sigma_q(J,F) = \Big\{\lambda \in \mathbb{K} : \frac{\|\lambda J(x_n) - F(x_n)\|}{\|x_n\|} \to 0 \quad \text{for some unbounded sequence } (x_n)_n\Big\}, \tag{10.48}$$

the *increasing point spectrum*

$$\sigma_q^0(J,F) = \{\lambda \in \mathbb{K} : \|\lambda J(x_n) - F(x_n)\| \to 0 \quad \text{for some unbounded sequence } (x_n)_n\}, \tag{10.49}$$

the *unbounded point spectrum*

$$\sigma_p^0(J,F) = \{\lambda \in \mathbb{K} : F(x_n) = \lambda J(x_n) \quad \text{for some unbounded sequence } (x_n)_n\}, \tag{10.50}$$

and the *point phantom*

$$\phi_p(J,F) = \{\lambda \in \mathbb{K} : N(\lambda J - F) \text{ contains an unbounded connected set } C \text{ with } \theta \in C\} \tag{10.51}$$

which we already considered in (9.74). For $J = I$, the set (10.48) was introduced in (2.29), the set (10.50) in (6.36), and the set (10.51) in (8.18). Since the set (10.49) is new, let us see what it becomes in the linear case. To this end, we first need a lemma. In analogy to (2.4) and (9.45), for $F \in \mathfrak{C}(X, Y)$ let us put

$$[F]_q^0 := \liminf_{\|x\|\to\infty} \|F(x)\|. \tag{10.52}$$

Obviously, $[F]_q^0 \geq [F]_q$ for general F, and it is very easy to find examples of operators F for which $[F]_q^0 > 0$ but $[F]_q = 0$. For linear operators, however, this is impossible.

Lemma 10.5. *For $L \in \mathfrak{L}(X, Y)$, the following three statements are equivalent:*

(a) $[L]_q > 0$.

(b) $[L]_q^0 > 0$.

(c) *L is injective with $R(L)$ closed.*

Proof. The implication (a) ⇒ (b) is a trivial consequence of the estimate $[L]_q^0 \geq [L]_q$. To prove (b) ⇒ (c), suppose that L is not injective. Then we find an unbounded sequence $(x_n)_n$ such that $Lx_n \equiv \theta$, and so $[L]_q^0 = 0$. So L must be injective. Now, if $R(L)$ is not closed, the inverse operator $L^{-1}\colon R(L) \to X$ cannot be continuous. This means that we find a sequence $(x_n)_n$ in X and $\varepsilon > 0$ such that $\|x_n\| \geq \varepsilon$ and, without loss of generality, $\|Lx_n\| \leq 1/n^2$. But then the sequence $(nx_n)_n$ satisfies $\|nx_n\| \geq n\varepsilon \to \infty$ and $\|L(nx_n)\| \to 0$, and so again $[L]_q^0 = 0$.

It remains to show that (c) ⇒ (a). If (c) is true, then $L\colon X \to R(L)$ is a linear isomorphism, and so $[L^{-1}|_{R(L)}]_Q = \|L^{-1}\| < \infty$. But then $[L]_q = \| L \| = \|L^{-1}\|^{-1} > 0$, by Proposition 2.2 (e), and so we are done. □

The equivalence of (a) and (c) in Lemma 10.5 has been already used implicitly in Chapter 1, e.g., in the decomposition (1.53) or (1.55). Moreover, linearity is such a rigid structure that a linear operator L can only satisfy either $[L]_q^0 = 0$ or $[L]_q^0 = \infty$.

Now we introduce yet three other notions of eigenvalue and compare them. We set

$$\phi_{pp}(J, F) = \{\lambda \in \mathbb{K} : N(\lambda J - F) \text{ contains an unbounded pathwise connected set } C \text{ with } \theta \in C\}, \tag{10.53}$$

$$\phi_{pr}(J, F) = \{\lambda \in \mathbb{K} : N(\lambda J - F) \text{ contains a ray } \{tx : t \geq 0\}\}, \tag{10.54}$$

and

$$\phi_{ps}(J, F) = \{\lambda \in \mathbb{K} : N(\lambda J - F) \text{ contains a nontrivial subspace of } X\}. \tag{10.55}$$

We call each $\lambda \in \phi_{pp}(J, F)$ a *pathwise connected eigenvalue*, each $\lambda \in \phi_{pr}(J, F)$ a *ray eigenvalue*, and each $\lambda \in \phi_{ps}(J, F)$ a *subspace eigenvalue* of (J, F). As usual, we simply write $\phi_{pp}(F)$, $\phi_{pr}(F)$, and $\phi_{ps}(F)$ in case $X = Y$ and $J = I$.

For further use we collect some relations among all these eigenvalue sets in the following proposition.

Proposition 10.3. *The inclusions*

$$\begin{aligned}\phi_{\mathrm{ps}}(J, F) &\subseteq \phi_{\mathrm{pr}}(J, F) \subseteq \phi_{\mathrm{pp}}(J, F)\\ &\subseteq \phi_{\mathrm{p}}(J, F) \subseteq \sigma_{\mathrm{p}}^0(J, F)\\ &\subseteq \sigma_{\mathrm{q}}^0(J, F) \subseteq \sigma_{\mathrm{q}}(J, F)\end{aligned} \tag{10.56}$$

are true. Moreover,

$$\begin{aligned}\phi_{\mathrm{ps}}(J, F) &\subseteq \phi_{\mathrm{pr}}(J, F) = \phi_{\mathrm{pp}}(J, F)\\ &= \phi_{\mathrm{p}}(J, F) = \sigma_{\mathrm{p}}^0(J, F)\\ &\subseteq \sigma_{\mathrm{q}}^0(J, F) \subseteq \sigma_{\mathrm{q}}(J, F)\end{aligned} \tag{10.57}$$

if F and J are τ-homogeneous, and

$$\phi_{\mathrm{ps}}(L) = \phi_{\mathrm{pr}}(L) = \phi_{\mathrm{pp}}(L) = \phi_{\mathrm{p}}(L) = \sigma_{\mathrm{p}}^0(L) \subseteq \sigma_{\mathrm{q}}^0(L) = \sigma_{\mathrm{q}}(L) \tag{10.58}$$

if L is linear. Finally, $\sigma_{\mathrm{p}}^0(J, F) \subseteq \sigma_{\mathrm{p}}(J, F)$ in general, and $\sigma_{\mathrm{p}}^0(J, F) = \sigma_{\mathrm{p}}(J, F)$ if F and J are τ-homogeneous.

Proof. The inclusions (10.56) and $\sigma_{\mathrm{p}}^0(J, F) \subseteq \sigma_{\mathrm{p}}(J, F)$ are trivial. Let both F and J be τ-homogeneous, and fix $\lambda \in \sigma_{\mathrm{p}}(J, F)$ and $x \in X \setminus \{\theta\}$ such that $F(x) = \lambda J(x)$. Then

$$F(tx) = t^\tau F(x) = \lambda t^\tau J(x) = \lambda J(tx),$$

and so $tx \in N(\lambda J - F)$ for all $t > 0$. This shows that $\lambda \in \phi_{\mathrm{pr}}(J, F)$, and so (10.57) is true. Moreover, for $L \in \mathfrak{L}(X)$ the equality $\phi_{\mathrm{ps}}(L) = \phi_{\mathrm{pr}}(L)$ is again trivial, while the equality $\sigma_{\mathrm{q}}^0(L) = \sigma_{\mathrm{q}}(L)$ follows from Lemma 10.5. □

We summarize the statement of Proposition 10.3 in the following table for the point spectra $\sigma_{\mathrm{p}}(J, F)$, $\sigma_{\mathrm{p}}^0(J, F)$, $\sigma_{\mathrm{q}}(J, F)$, and $\sigma_{\mathrm{q}}^0(J, F)$. Here the general case is represented on the left, the case of τ-homogeneous operators F and J in the middle, and the linear case (for $J = I$) on the right.

Table 10.1

$$\begin{array}{ccc} \sigma_{\mathrm{p}}^0(J, F) & \subseteq & \sigma_{\mathrm{q}}^0(J, F)\\ \rotatebox{-90}{$\subseteq$} & & \rotatebox{-90}{$\subseteq$}\\ \sigma_{\mathrm{p}}(J, F) & & \sigma_{\mathrm{q}}(J, F)\end{array} \qquad \begin{array}{ccc} \sigma_{\mathrm{p}}^0(J, F) & \subseteq & \sigma_{\mathrm{q}}^0(J, F)\\ \| & & \rotatebox{-90}{$\subseteq$}\\ \sigma_{\mathrm{p}}(J, F) & \subseteq & \sigma_{\mathrm{q}}(J, F)\end{array} \qquad \begin{array}{ccc} \sigma_{\mathrm{p}}^0(L) & \subseteq & \sigma_{\mathrm{q}}^0(L)\\ \| & & \|\\ \sigma_{\mathrm{p}}(L) & \subseteq & \sigma_{\mathrm{q}}(L)\end{array}$$

We have seen in Chapter 6 that the inclusion $\sigma_{\mathrm{p}}^0(L) \subseteq \sigma_{\mathrm{q}}^0(L)$ in (10.58) (equivalently, the inclusion $\sigma_{\mathrm{p}}(L) \subseteq \sigma_{\mathrm{q}}(L)$ in (6.34)) may be strict. We illustrate the other inclusions by a series of examples.

Example 10.10. Let $F : \mathbb{R} \to \mathbb{R}$ be defined by

$$F(x) := \sqrt{|x|} \tag{10.59}$$

(see Example 3.16). A straightforward calculation shows that

$$\phi_{\mathrm{ps}}(F) = \phi_{\mathrm{pr}}(F) = \phi_{\mathrm{pp}}(F) = \phi_{\mathrm{p}}(F) = \sigma_{\mathrm{p}}^0(F) = \sigma_{\mathrm{q}}^0(F) = \emptyset,$$
$$\sigma_{\mathrm{q}}(F) = \{0\}, \quad \sigma_{\mathrm{p}}(F) = \mathbb{R} \setminus \{0\},$$

in this example. ♡

Example 10.11. Let $F : \mathbb{R} \to \mathbb{R}$ be defined by

$$F(x) := \begin{cases} -x & \text{if } x \leq 0, \\ 2x & \text{if } 0 \leq x \leq 1, \\ x + \frac{1}{x} & \text{if } x \geq 1. \end{cases} \tag{10.60}$$

Again, an easy calculation shows that

$$\phi_{\mathrm{ps}}(F) = \emptyset, \quad \phi_{\mathrm{pr}}(F) = \phi_{\mathrm{pp}}(F) = \phi_{\mathrm{p}}(F) = \sigma_{\mathrm{p}}^0(F) = \{-1\},$$
$$\sigma_{\mathrm{q}}^0(F) = \sigma_{\mathrm{q}}(F) = \{\pm 1\}, \quad \sigma_{\mathrm{p}}(F) = \{-1\} \cup (1, 2].$$

in this example. ♡

Example 10.12. In $X = l_2$ over $\mathbb{K} = \mathbb{C}$, let $F : X \to X$ be defined by

$$F(x_1, x_2, x_3, \ldots) := (\|x\|, x_1, x_2, \ldots) \tag{10.61}$$

and $J = I$. Obviously, F is 1-homogeneous. We already know from Example 9.11 that

$$\sigma_{\mathrm{p}}(F) = \sigma_{\mathrm{p}}^0(F) = \phi_{\mathrm{p}}(F) = \sigma_{\mathrm{q}}(F) = \sigma_{\mathrm{q}}^0(F) = \phi_{\mathrm{q}}(F) = \mathbb{S}_{\sqrt{2}},$$

and so

$$\phi_{\mathrm{pr}}(F) = \phi_{\mathrm{pp}}(F) = \mathbb{S}_{\sqrt{2}}$$

as well, by (10.57). Moreover, we have seen that $x_\lambda := (\lambda^{-1}, \lambda^{-2}, \lambda^{-3}, \ldots)$ is an eigenvector for F with respect to $\lambda \in \sigma_{\mathrm{p}}(F)$, and the same is true for every *positive* multiple of x_λ. However, this is not true for *negative* (let alone complex) multiples of x_λ, and so $\phi_{\mathrm{ps}}(F) = \emptyset$. ♡

In case $X = \mathbb{R}$ it is possible to describe the topological structure of some of the above eigenvalue sets. For example, $\sigma_{\mathrm{p}}(F)$ is always an interval or a union of two intervals, while $\sigma_{\mathrm{p}}^0(F)$ may be empty, finite, an interval, or a union of two intervals. In case $\dim X \geq 2$ such a simple description is not possible. The following example shows that all point spectra and phantoms may be uncountable, even if F is compact.

Example 10.13. Let $\varphi\colon \mathbb{R}^2 \to \mathbb{R}$ be defined in polar coordinates by

$$\varphi(r\cos t, r\sin t) := |\sin t| \quad (r \geq 0,\ 0 \leq t < 2\pi),$$

and define $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ by

$$F(x) := \varphi(x)x. \tag{10.62}$$

We have seen in Example 8.4 that $\phi_{\mathrm{p}}(F) = [0, 1]$. But it is easy to see that

$$\phi_{\mathrm{ps}}(F) = \phi_{\mathrm{pr}}(F) = \phi_{\mathrm{pp}}(F) = \phi_{\mathrm{p}}(F) = \sigma_{\mathrm{p}}^0(F) = \sigma_{\mathrm{q}}^0(F) = \sigma_{\mathrm{q}}(F) = \sigma_{\mathrm{p}}(F) = [0, 1]$$

for *all* point spectra and phantoms, because $N(\lambda I - F) = \{x \in \mathbb{R}^2 : \varphi(x) = \lambda\} \cup \{0\}$. ♡

Example 10.14. Let $\varphi\colon [0, 1) \times \mathbb{R} \to \mathbb{R}$ be defined by

$$\varphi(x_1, x_2) := \begin{cases} 0 & \text{if } x_1 = 0 \text{ and } |x_2| \leq 1, \\ |\sin\frac{1}{x_1} - x_2|\, x_1(1 - x_1) & \text{if } x_1 > 0 \text{ and } |x_2| \leq 1, \\ |x_2| - 1 & \text{if } x_1 = 0 \text{ and } |x_2| > 1, \\ |x_2| - 1 + |\sin\frac{1}{x_1} - x_2|\, x_1(1 - x_1) & \text{if } x_1 > 0 \text{ and } |x_2| > 1, \end{cases}$$

and extend φ periodically in the first argument to the whole real line. Define $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ as in (10.62). Then we have

$$N(F) = \bigcup_{k \in \mathbb{Z}} [(\{k\} \times [-1, 1]) \cup \{(t + k, \sin\tfrac{1}{t}) : 0 < t < 1\}],$$

and this set is connected, but contains no unbounded pathwise connected subset. This means that $\lambda = 0$ is a connected eigenvalue of F, but not a pathwise connected eigenvalue, and so the inclusion $\phi_{\mathrm{pp}}(F) \subseteq \phi_{\mathrm{p}}(F)$ may be strict. ♡

Example 10.15. Let $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$F(x_1, x_2) := (x_1 \cos x_1, |x_2| \cos x_1). \tag{10.63}$$

For $\lambda \in \mathbb{R}$ fixed, the function $f(t) := \lambda\pi + (2t + \pi)\cos t$ has a zero t_0, since f is continuous and assumes positive and negative values. Hence, $(\lambda J - F)(t_0, 0) = (\lambda J - F)(t_0 + \pi, 0)$. Using the shortcut

$$N_\lambda := \{(x_1, x_2) : \cos x_1 = \lambda,\ x_2 \geq 0\} \cup \{(0, 0)\},$$

we have

$$N(\lambda J - F) = \begin{cases} N_\lambda \cup \{(x_1, x_2) : x_1 = 0,\ x_2 \leq 0\} & \text{if } \lambda = -1, \\ N_\lambda \cup \{(x_1, x_2) : \cos x_1 = 0\} & \text{if } \lambda = 0, \\ N_\lambda & \text{otherwise.} \end{cases}$$

Consequently, we obtain the relations

$$\phi_{\rm ps}(F) = \emptyset, \quad \phi_{\rm pr}(F) = \{\pm 1\},$$
$$\phi_{\rm pp}(F) = \phi_{\rm p}(F) = \sigma_{\rm p}^0(F) = \sigma_{\rm q}^0(F) = \sigma_{\rm q}(F) = \sigma_{\rm p}(F) = [-1, 1]$$

in this example. ♡

Example 10.16. Let $X = \mathbb{R}^2$, and let $x_k = (1, k)$ $(k \in \mathbb{Z})$. Denote by C_k that part of the line through θ and x_k which starts at x_k and lies opposite of θ. Put $\lambda_k := k + 1$ for $k = 0, 1, 2, \ldots$ and $\lambda_k := -1/k$ for $k = -1, -2, -3, \ldots$. We define a continuous function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ in the following way. On C_k, we put $\varphi(x) = \lambda_k$. Since the union of all sets C_k is closed and $\lambda_k > 0$ for all $k \in \mathbb{Z}$, we may extend φ to a continuous and strictly positive function on $\mathbb{R}^2$ which we still denote by φ. (One may use an abstract extension theorem to see this, but one may also give an explicit formula for such an extension in our particular situation.) Now we define $F : \mathbb{R}^2 \to \mathbb{R}^2$ again as in (10.62). Then $F(\theta) = \theta$ and $F(x) = \lambda_k x$ on C_k. In particular, each scalar λ_k is a ray eigenvalue of F and thus all point spectra and phantoms in (10.56) are unbounded. Moreover, they all have 0 as an accumulation point, but do not contain 0, because $\varphi(x) \neq 0$ on $\mathbb{R}^2$. Thus, neither of the point spectra or phantoms introduced above is closed here. ♡

The most important and appropriate notion of point spectrum seems to be the point phantom (10.51). We will study this set more systematically in Section 10.5 below.

Let us return now to the problem of finding eigenvalues with corresponding eigenvectors in a given subset $M \subset X$. Consider the "constrained" eigenvalue sets

$$\sigma_{\rm p}(F, M) := \{\lambda \in \mathbb{K} : F(x) = \lambda x \text{ for some } x \in M \setminus \{\theta\}\}, \tag{10.64}$$

$$\sigma_{\rm q}(F, M) := \Big\{\lambda \in \mathbb{K} : \tfrac{\|\lambda x_n - F(x_n)\|}{\|x_n\|} \to 0 \text{ for some sequence } (x_n)_n \text{ in } M \setminus \{\theta\}\Big\}, \tag{10.65}$$

and

$$\sigma_{\rm q}^0(F, M) := \{\lambda \in \mathbb{K} : \|\lambda x_n - F(x_n)\| \to 0 \text{ for some sequence } (x_n)_n \text{ in } M \setminus \{\theta\}\}. \tag{10.66}$$

Obviously, $\sigma_{\rm p}(F, X) = \sigma_{\rm p}(F)$, $\sigma_{\rm q}(F, X) \supseteq \sigma_{\rm q}(F)$, and $\sigma_{\rm q}^0(F, X) \supseteq \sigma_{\rm q}^0(F)$. The set $\sigma_{\rm p}(F, S_r(X))$ is of course nothing else but the eigenvalue set $\Lambda_r(F)$ we introduced in (10.4).

It is easy to establish some simple relations for the sets (10.64)–(10.66). For example, one has

$$\sigma_{\rm p}(F, M \cup N) = \sigma_{\rm p}(F, M) \cup \sigma_{\rm p}(F, N)$$

and

$$\sigma_{\rm p}(F, M \cap N) \subseteq \sigma_{\rm p}(F, M) \cap \sigma_{\rm p}(F, N),$$

and similarly for $\sigma_{\rm q}(F, M)$ and $\sigma_{\rm q}^0(F, M)$. Moreover,

$$\sigma_{\rm p}(F, M) \subseteq \sigma_{\rm q}^0(F, M). \tag{10.67}$$

Example 10.10 with $M = [-1, 1]$ shows that the inclusion $\sigma_{\rm q}^0(F, M) \subseteq \sigma_{\rm q}(F, M)$ is not always true. The next proposition shows how the properties of the set M determine the structure of the eigenvalue sets $\sigma_{\rm p}(F, M)$, $\sigma_{\rm q}(F, M)$, and $\sigma_{\rm q}^0(F, M)$.

Proposition 10.4. *The sets* (10.64)–(10.66) *have the following properties.*

(a) *The set $\sigma_{\rm q}(F, M)$ is always closed.*

(b) *If M is bounded then $\sigma_{\rm q}^0(F, M)$ is closed.*

(c) *If M is bounded away from zero then $\sigma_{\rm q}^0(F, M) \subseteq \sigma_{\rm q}(F, M)$.*

(d) *If M is bounded then $\sigma_{\rm q}(F, M) \subseteq \sigma_{\rm q}^0(F, M)$.*

(e) *If $F(\theta) = \theta$ and θ is an accumulation point of M then $\sigma_{\rm q}^0(F, M) = \mathbb{K}$.*

(f) *If M is compact with $\theta \notin M$ then*

$$\sigma_{\rm p}(F, M) = \sigma_{\rm q}^0(F, M) = \sigma_{\rm q}(F, M). \tag{10.68}$$

Proof. (a) Let $(\lambda_k)_k$ be a sequence in $\sigma_{\rm q}(F, M)$ with $\lambda_k \to \lambda$ $(k \to \infty)$. For $\varepsilon > 0$ and each k we choose $(x_{n,k})_n$ in $M \setminus \{\theta\}$ such that $\|F(x_{n,k}) - \lambda_k x_{n,k}\| \leq \varepsilon \|x_{n,k}\|$ for $n \geq n(\varepsilon, k)$. Then the "diagonal sequence" $x_n := x_{n,n(\varepsilon,n)}$ satisfies

$$\|F(x_n) - \lambda x_n\| \leq \|F(x_n) - \lambda_n x_n\| + |\lambda_n - \lambda|\, \|x_n\| \leq (\varepsilon + |\lambda_n - \lambda|)\|x_n\|,$$

and so $\lambda \in \sigma_{\rm q}(F, M)$.

(b) Choosing $(x_{n,k})_n$ with $\|F(x_{n,k}) - \lambda_k x_{n,k}\| \leq \varepsilon$ and $x_n = x_{n,n}$ as above we obtain

$$\|F(x_n) - \lambda x_n\| \leq \|F(x_n) - \lambda_n x_n\| + |\lambda_n - \lambda|\, \|x_n\| \leq \varepsilon + |\lambda_n - \lambda| \sup_{x \in M} \|x\|.$$

(c) Given $\lambda \in \sigma_{\rm q}^0(F, M)$ and $\varepsilon > 0$, let $(x_n)_n$ be a sequence in $M \setminus \{\theta\}$ such that $\|F(x_n) - \lambda x_n\| \leq \varepsilon$ for $n \geq n(\varepsilon)$. Then

$$\frac{1}{\|x_n\|}\|F(x_n) - \lambda x_n\| \leq \frac{\varepsilon}{\inf_{x \in M} \|x\|} \quad (n \geq n(\varepsilon)),$$

hence $\lambda \in \sigma_{\rm q}(F, M)$.

(d) Given $\lambda \in \sigma_{\rm q}(F, M)$ and $\varepsilon > 0$, let $(x_n)_n$ be a sequence in $M \setminus \{\theta\}$ such that $\|F(x_n) - \lambda x_n\| \leq \varepsilon \|x_n\|$ for per $n \geq n(\varepsilon)$. Then

$$\|F(x_n) - \lambda x_n\| \leq \sup_{x \in M} \|x\| \varepsilon \quad (n \geq n(\varepsilon)),$$

hence $\lambda \in \sigma_{\mathrm{q}}^0(F, M)$.

(e) Given $x_n \in M \setminus \{\theta\}$ with $x_n \to \theta$, we have $F(x_n) \to F(\theta) = \theta$, by continuity. Consequently,

$$\|F(x_n) - \lambda x_n\| \leq \|F(x_n)\| + |\lambda| \, \|x_n\| \to 0$$

for any $\lambda \in \mathbb{K}$.

(f) The relation (10.67) and properties (c) and (d) imply that

$$\sigma_{\mathrm{p}}(F, M) \subseteq \sigma_{\mathrm{q}}^0(F, M) = \sigma_{\mathrm{q}}(F, M)$$

for compact M with $\theta \notin M$. Now, for $\lambda \in \sigma_{\mathrm{q}}^0(F, M)$ we may find $(x_n)_n$ in M with $\|F(x_n) - \lambda x_n\| \to 0$ as $n \to \infty$. Since M is compact, there exists a subsequence $(x_{n_k})_k$ such that $x_{n_k} \to x \in M$, and so

$$\|F(x) - \lambda x\| \leq \|F(x) - F(x_{n_k})\| + \|F(x_{n_k}) - \lambda x_{n_k}\| + |\lambda| \, \|x_{n_k} - x\| \to 0 \quad (k \to \infty),$$

which shows that $\lambda \in \sigma_{\mathrm{p}}(F, M)$. □

The next proposition shows how the properties of the operator F determine the structure of the eigenvalue sets $\sigma_{\mathrm{p}}(F, M)$, $\sigma_{\mathrm{q}}(F, M)$, and $\sigma_{\mathrm{q}}^0(F, M)$.

Proposition 10.5. *The following is true.*

(a) *If F is* 1*-homogeneous then $\sigma_{\mathrm{p}}(F, tM) \equiv \sigma_{\mathrm{p}}(F, M)$ for every $t > 0$, and similarly for $\sigma_{\mathrm{q}}(F, M)$ and $\sigma_{\mathrm{q}}^0(F, M)$.*

(b) *If M is closed and bounded with $\theta \notin M$, and F is compact then* (10.68) *holds.*

(c) *If M is bounded and bounded away from zero, and F is bounded then $\sigma_{\mathrm{q}}(F, M)$ and $\sigma_{\mathrm{q}}^0(F, M)$ are compact.*

Proof. (a) Fix $t > 0$, $\lambda \in \sigma_{\mathrm{p}}(F, M)$ and $x \in M \setminus \{\theta\}$ with $F(x) = \lambda x$. Then $F(tx) = tF(x) = t\lambda x$, hence $\lambda \in \sigma_{\mathrm{p}}(F, tM)$. The proof for $\sigma_{\mathrm{q}}(F, M)$ and $\sigma_{\mathrm{q}}^0(F, M)$ is analogous.

(b) As before it suffices to show that $\sigma_{\mathrm{q}}^0(F, M) \subseteq \sigma_{\mathrm{p}}(F, M)$. Given $\lambda \in \sigma_{\mathrm{q}}^0(F, M) \setminus \{0\}$, choose a sequence $(x_n)_n$ in M such that $\|F(x_n) - \lambda x_n\| \to 0$ as $n \to \infty$. Since F is compact we may suppose, without loss of generality that the sequence $(F(x_n))_n$ converges. The estimate

$$|\lambda| \, \|x_m - x_n\| \leq \|\lambda x_m - F(x_m)\| + \|F(x_m) - F(x_n)\| + \|F(x_n) - \lambda x_n\| \to 0,$$

as $m, n \to \infty$, implies that $(x_n)_n$ is a Cauchy sequence, and so $x_n \to x$ for some $x \in M \setminus \{\theta\}$. Consequently, $F(x) = \lambda x$, i.e., $\lambda \in \sigma_{\mathrm{p}}(F, M)$.

(c) We already know that $\sigma_{\mathrm{q}}^0(F, M) = \sigma_{\mathrm{q}}(F, M)$ is closed. Suppose that $\sigma_{\mathrm{q}}^0(F, M)$ is unbounded. Then we find $\lambda \in \sigma_{\mathrm{q}}^0(F, M)$ such that

$$|\lambda| \inf_{x \in M} \|x\| > \sup_{x \in M} \|F(x)\|.$$

Moreover, we may choose a sequence $(x_n)_n$ in $M \setminus \{\theta\}$ such that $\|F(x_n) - \lambda x_n\| \to 0$ as $n \to \infty$. On the other hand,

$$\|\lambda x_n - F(x_n)\| \geq |\lambda| \inf_{x \in M} \|x\| - \sup_{x \in M} \|F(x)\| > 0,$$

a contradiction. Consequently, $\sigma_q^0(F, M)$ is bounded as claimed. □

To illustrate Proposition 10.5, we collect the spectral sets $\sigma_p(F, M)$, $\sigma_q(F, M)$, and $\sigma_q^0(F, M)$, for $M = B(X)$ and $M = S(X)$ and F as in Examples 10.10, 10.11, and 10.12, in the following table.

Table 10.1

	Example 10.10	Example 10.11	Example 10.12
$\sigma_p(F, B(X))$	$\mathbb{R} \setminus (-1, 1)$	$\{-1, 2\}$	$\mathbb{S}_{\sqrt{2}}$
$\sigma_p(F, S(X))$	$\{\pm 1\}$	$\{-1, 2\}$	$\mathbb{S}_{\sqrt{2}}$
$\sigma_q(F, B(X))$	$\mathbb{R} \setminus (-1, 1)$	$\{-1, 2\}$	$\mathbb{S}_{\sqrt{2}}$
$\sigma_q(F, S(X))$	$\{\pm 1\}$	$\{-1, 2\}$	$\mathbb{S}_{\sqrt{2}}$
$\sigma_q^0(F, B(X))$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{C}$
$\sigma_q^0(F, S(X))$	$\{\pm 1\}$	$\{-1, 2\}$	$\mathbb{S}_{\sqrt{2}}$

The first column in Table 10.2 shows that $\sigma_p(F, M)$, $\sigma_q(F, M)$ and $\sigma_q^0(F, M)$ are not necessarily bounded for bounded M, and that the set $\sigma_q^0(F, M)$ may be strictly larger than the sets $\sigma_p(F, M)$ and $\sigma_q(F, M)$ if M is compact but $\theta \in M$.

We close this section with another example which shows that the assumptions in Propositions 10.4 and 10.5 are necessary even for linear operators.

Example 10.17. In $X = l_2$ over $\mathbb{K} = \mathbb{R}$, consider the compact linear operator

$$L(x_1, x_2, x_3, \dots) := (x_1, \tfrac{1}{2}x_2, \tfrac{1}{3}x_3, \dots) \tag{10.69}$$

which we already considered in (1.41). An easy calculation shows then that

$$\sigma_p(F, B(X)) = \sigma_p(F, S(X)) = \{1, \tfrac{1}{2}, \tfrac{1}{3}, \dots\}, \quad \sigma_q^0(F, B(X)) = \mathbb{R}, \tag{10.70}$$

and

$$\sigma_q(F, B(X)) = \sigma_q(F, S(X)) = \sigma_q^0(F, S(X)) = \{0, 1, \tfrac{1}{2}, \tfrac{1}{3}, \dots\}. \tag{10.71}$$

The equalities (10.70) and (10.71) show that $\sigma_p(F, M)$ may be not closed even if M is both bounded and bounded away from zero, and so Proposition 10.5 (c) is not true for $\sigma_p(F, M)$. Moreover, Proposition 10.5 (b) fails if M is closed and bounded, but $\theta \in M$. ♡

10.5 Connected eigenvalues

Among all notions of eigenvalues discussed so far, the set of connected eigenvalues (10.51) seems to be the most natural. This may be seen, for instance, by Theorem 10.8 below which is completely analogous to the linear case. First we state a boundedness result for the point phantom. Recall that by $\mathfrak{OBC}(X)$ we denote the family of all open bounded connected subsets of X containing θ.

Theorem 10.7. *Suppose that*

$$d := \operatorname{dist}(\theta, J(\partial\Omega)) = \inf_{x\in\partial\Omega} \|J(x)\| > 0 \tag{10.72}$$

and

$$M := \sup_{x\in\partial\Omega} \|F(x)\| < \infty \tag{10.73}$$

for some $\Omega \in \mathfrak{OBC}(X)$. Then the point phantom $\phi_{\mathrm{p}}(J, F)$ is bounded by M/d.

Proof. Let $\lambda \in \phi_{\mathrm{p}}(J, F)$, and let $C \subseteq N(\lambda J - F)$ be an unbounded connected set with $\theta \in C$. Then $\emptyset \neq \Omega_C := \Omega \cap C \neq C$. Since Ω_C is open in C and C is connected, it cannot be closed in C. Consequently, the boundary $\partial_C(\Omega_C)$ of Ω_C with respect to C contains some point z which does not belong to Ω_C. Then $z \in \partial\Omega$ and $z \in C \subseteq N(\lambda J - F)$, and so

$$|\lambda| d = |\lambda| \inf_{x\in\partial\Omega} \|J(x)\| \leq \|\lambda J(z)\| = \|F(z)\| \leq \sup_{x\in\partial\Omega} \|F(x)\| = M,$$

which proves the assertion. □

Theorem 10.8. *Suppose that $F : X \to X$ is a compact operator. Then the set $\phi_{\mathrm{p}}(F) \cup \{0\}$ is compact. If X is finite dimensional then even the set $\phi_{\mathrm{p}}(F)$ is compact.*

Proof. By Theorem 10.7, the point phantom $\phi_{\mathrm{p}}(F)$ is bounded by

$$M := \sup\{\|F(x)\| : x \in B(X)\}.$$

Moreover, from Proposition 8.5 we know that the approximate point phantom $\phi_{\mathrm{q}}(F)$ is closed, while in Theorem 8.8 we have proved that $\phi_{\mathrm{q}}(F) \setminus \{0\} = \phi_{\mathrm{p}}(F) \setminus \{0\}$ for compact F. Combining these two results gives the compactness of the set $\phi_{\mathrm{p}}(F) \cup \{0\}$. If X is finite dimensional, Theorem 8.8 shows that even $\phi_{\mathrm{q}}(F) = \phi_{\mathrm{p}}(F)$, which implies the second statement. □

It is a striking fact that an analogue to Theorem 10.8 for the set (10.53) of pathwise connected eigenvalues is not true! This shows that the definition (10.51) of the point phantom is quite subtle. To see this, we consider a modification of Example 10.14.

Example 10.18. Let $\varphi\colon [0, 1) \times \mathbb{R} \to \mathbb{R}$ be defined by

$$\varphi(x_1, x_2) := \begin{cases} 0 & \text{if } x_1 = 0 \text{ and } |x_2| \le 1, \\ |\sin \frac{1}{x_1} - x_2|\, x_1 & \text{if } x_1 > 0 \text{ and } |x_2| \le 1, \\ |x_2| - 1 & \text{if } x_1 = 0 \text{ and } |x_2| > 1, \\ |x_2| - 1 + |\sin \frac{1}{x_1} - x_2|\, x_1 & \text{if } x_1 > 0 \text{ and } |x_2| > 1, \end{cases}$$

and extend φ periodically in the first argument to the whole real line. Define $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ as in (10.62).

For any $\lambda > 0$, we find two continuous unbounded paths starting from $(1, \pm(1+\lambda))$ on which $\varphi(x) = \lambda$. Indeed, for $x_1 > 0$ we have $\varphi(1 + x_1, x_2) = \lambda$ if and only if one of the following cases occurs:

(1) $|x_2| \le 1$ and either $x_2 = c_+(x_1)$ or $x_2 = c_-(x_1)$,

(2) $x_2 \ge 1$ and $x_2 = d_+(x_1)$, or

(3) $x_2 \le -1$ and $x_2 = d_-(x_1)$.

Here we have used the abbreviation

$$c_\pm(t) := \sin \frac{1}{t} \pm \frac{\lambda}{t}, \quad d_\pm(t) := \frac{t \sin \frac{1}{t} \pm (1 + \lambda)}{1 + t}.$$

Moreover, the equation $d_+(x_1) \ge 1$ is equivalent to $x_1 \le \lambda + x_1 \sin 1/x_1$ which in turn is equivalent to $c_+(x_1) \ge 1$. Similarly, $d_+(x_1) \le 1$ is equivalent to $c_+(x_1) \le 1$. Observe that $c_+(x_1) \ge -1$, and that the set $\{x_1 : d_+(x_1) = 1\} = \{x_1 : c_+(x_1) = 1\}$ has no accumulation points. Thus, putting

$$e_+(t) := \begin{cases} d_+(t) & \text{if } t > 0 \text{ and } d_+(t) \ge 1, \\ c_+(t) & \text{if } t > 0 \text{ and } c_+(t) \le 1, \\ 1 + \lambda & \text{if } t = 0, \end{cases}$$

we get a continuous function $e_+ : [0, \infty) \to \mathbb{R}$ which satisfies $\varphi(1+x_1, e_+(x_1)) = \lambda$. Similarly, the function $e_-\colon [0, \infty) \to \mathbb{R}$ given by

$$e_-(t) := \begin{cases} d_-(t) & \text{if } t > 0 \text{ and } d_-(t) \le 1, \\ c_-(t) & \text{if } t > 0 \text{ and } c_-(t) \ge 1, \\ -(1 + \lambda) & \text{if } t = 0, \end{cases}$$

is continuous with $\varphi(1 + x_1, e_-(x_1)) = \lambda$. The desired unbounded paths are thus given by $t \mapsto (1 + t, e_\pm(t))$ for $0 \le t < \infty$.

Now, let $G \subset \mathbb{R}^2$ be a Jordan domain whose border is the line from $(1, -2)$ to $(0, 0)$, the line from $(0, 0)$ to $(1, 2)$, and a curve joining $(1, 2)$ and $(1, -2)$ which lies to the left of the two lines. On ∂G define $\varphi(x) = 1$. By the Tietze–Uryson lemma, we may extend φ to a continuous function on $X = \mathbb{R}^2$. On the lines (t, ct) $(0 < t \leq 1)$ with $|c| < 2$ we redefine φ by putting $\varphi((t, ct)) := 0$ for $|c| \leq 1$, and $\varphi((t, ct)) := |c| - 1$ for $1 \leq |c| \leq 2$. Then $\varphi\colon X \to \mathbb{R}$ is continuous on $X \setminus \{\theta\}$. Moreover, φ is locally bounded at θ, and so the operator $F(x) = \varphi(x)x$ is continuous on X with $F(\theta) = \theta$. A straightforward calculation shows that

$$N(\lambda I - F) = \{x \in X : \varphi(x) = \lambda\} \cup \{\theta\}.$$

Thus, for $0 < \lambda \leq 1$, the set $N(\lambda I - F)$ contains, by construction, a continuous unbounded path starting from θ (even two paths: one passing through $(1, 1 + \lambda)$ and another one passing through $(1, -(1 + \lambda))$). Consequently, λ is a pathwise connected eigenvalue. On the other hand, 0 is not a pathwise connected eigenvalue. Indeed, the set

$$\tilde{N} := N(F) \cap \{(x_1, x_2) : x_1 \geq 1\} = (\{1\} \times [-1, 1]) \cup \{(1 + t, \sin \tfrac{1}{t}) : t > 0\}$$

does not contain an unbounded path which starts at the border $\{1\} \times [-1, 1]$ of $\tilde{N}$. But each unbounded path in $N(F)$ starting at θ would have to contain such a path, since $N(F) \cap \partial G = \{\theta\}$, by construction.

Altogether, we have shown that $(0, 1] \subseteq \phi_{\mathrm{pp}}(F)$, but $0 \notin \phi_{\mathrm{pp}}(F)$, and so Theorem 10.8 fails for $\phi_{\mathrm{p}}(F)$ replaced with $\phi_{\mathrm{pp}}(F)$. ♡

10.6 Notes, remarks and references

In contrast to nonlinear spectral theory, nonlinear eigenvalue has a long history. As a matter of fact, in the early work on this subject which started with the fundamental contributions of Nemytskij [202], [203], Krasnosel'skij [160], [161], and Vajnberg [256]–[258], the term "spectrum" always referred to the eigenvalue set (10.2). A completely new aspect came up in 1969 when Kachurovskij [156] and Neuberger [204] independently started studying spectral values which are not necessarily eigenvalues. This may be viewed as the beginning of nonlinear spectral theory as treated in this book.

Two basic techniques which are used throughout nonlinear analysis have turned out to be useful for studying nonlinear eigenvalue problems, viz., *topological methods* (fixed point theorems, degree theory) and *variational methods* (critical points, nonlinear functionals). The first technique is described in detail, together with many illuminating examples, in the (unpublished) manuscript [252], the second one in the survey [222]. Of course, one may also consult classical books on nonlinear analysis like [73], [163], [286].

The existence results for two eigenvalues of opposite sign contained in Theorems 10.1, 10.2 and 10.3 are now classical. Theorem 10.2 is usually referred to as the *Birkhoff–Kellogg Theorem* in the literature. We point out that one may prove such theorems also with the help of topological degree (see Section 3.5). A degree-theoretic proof of Theorem 10.2 for general $\Omega \in \mathfrak{OBC}(X)$ may be found, for example, in [72]. Theorem 10.2 and Theorem 10.3 both use the crucial assumption that $F(\partial\Omega)$ be bounded away from zero, see (10.7) and (10.25). One might wonder why Theorem 10.4 gives only one eigenvalue, while Theorem 10.2 guarantees the existence of two eigenvalues. The reason is, of course, that we have the additional information that $\lambda > 0$ and $x \in K$ in Theorem 10.4.

Interestingly, one may prove Theorem 2.5 as consequence of the generalized Birkhoff–Kellogg Theorem 10.3. In fact, (10.13) clearly holds for $F\colon S_r(X) \to S_r(X)$ satisfying $[F]_{\mathrm{A}} < 1$. If λ_+ is the positive eigenvalue whose existence is guaranteed by Theorem 10.3, then the equality

$$r = \|F(x_+)\| = |\lambda_+|\,\|x_+\| = r$$

implies that $\lambda_+ = 1$, and so x_+ is a fixed point of F. Similarly, if λ_- is the negative eigenvalue of F from Theorem 10.3, then the equality

$$r = \|F(x_-)\| = |\lambda_-|\,\|x_-\| = r$$

implies that $\lambda_- = -1$, and so F has not only a fixed point, but also an "antipodal point".

Actually, one can prove more. Fix a complex number $e^{i\alpha} \in \mathbb{S}$. Replacing the operator F in Theorem 10.3 by $e^{-i\alpha}F$ we see that

$$\inf_{\|x\|=r} \|e^{-i\alpha}F(x)\| > r[e^{-i\alpha}F|_{B_r(X)}]_{\mathrm{A}},$$

i.e., (10.13) holds for the operator $e^{-i\alpha}F$ as well. From Theorem 10.3 we deduce that there exist $\lambda_+ > 0$ and $x_+ \in S_r(X)$ such that $F(x_+) = \lambda_+ e^{i\alpha}x_+$. This means that F has "eigenvalues in each direction", i.e., for each complex number $\lambda \in \mathbb{S}$ one has $\rho\lambda \in \sigma_{\mathrm{p}}(F)$ for a suitable scalar $\rho > 0$.

The following result from [103] shows again that an operator F with $F(\theta) \neq \theta$ has "many" eigenvalues: *Let $F \in \mathfrak{A}(X)$ with $F(\theta) \neq \theta$, and suppose that $\lambda \in \mathbb{K}$ satisfies $|\lambda| > [F]_{\mathrm{A}}$ and $\lambda \notin \sigma_{\mathrm{p}}(F)$. Then for each $r > 0$ there exists some λ_r with $|\lambda_r| > |\lambda|$ and $\lambda_r \in \Lambda_r(F)$.* This shows, in particular, that every operator $F \in \mathfrak{A}(X)$ with $F(\theta) \neq \theta$ has an unbounded point spectrum. It should be noted, however, that the eigenvalues λ_r in the above result do not necessarily tend to infinity as $r \to \infty$. In fact, if F is in addition asymptotically linear with asymptotic derivative $F'(\infty)$, then $\lambda_r x_r = F'(\infty)x_r + R(x_r)$, where $[R]_{\mathrm{Q}} = [F - F'(\infty)]_{\mathrm{Q}} = 0$, and for each $\varepsilon > 0$ we can find $r(\varepsilon) >$ such that $|\lambda_r| \leq \|F'(\infty)\| + \varepsilon$ for $r \geq r(\varepsilon)$.

The definition of the s-quasinorm (10.18) and Proposition 10.1 are due to Pejsachowicz and Vignoli [213]. In [213, Cor. 2.1] it is shown that, if $F\colon H \to H$ satisfies $[F]_{\mathrm{A}} < 1$ and

$$\limsup_{\|x\|\to\infty} \frac{|\langle F(x), x\rangle|}{\|x\|^2} < 1, \tag{10.74}$$

then $I - F$ is surjective. It is not hard to see that (10.74) implies condition (10.19) in Example 10.5 (which is taken from [82]), and so Example 10.5 improves Corollary 2.1

in [213]. To see that these conditions are not equivalent, it suffices to take $F = -aI$ for $a \geq 1$ and to observe that

$$\lim_{\|x\|\to\infty} \frac{1}{\|x\|}\langle x - F(x), x\rangle = \lim_{\|x\|\to\infty} (1+a)\|x\| = \infty,$$

but

$$\limsup_{\|x\|\to\infty} \frac{|\langle F(x), x\rangle|}{\|x\|^2} = a \geq 1.$$

Much material from Sections 10.2 and 10.3 may be found in the paper [183] which, unfortunately, contains many errors. A more recent and correct version is the survey article [113] which also treats the case of multivalued operators. Example 10.9 is also taken from [113], [183]. The classical Krejn–Rutman theorem is contained in the survey article [165], a nice elementary proof may be found in [246]. A generalization to noncompact 1-homogeneous operators, together with applications to very general elliptic boundary value problems, is given in [185].

In the beginning of the theory of normed spaces with cones, a cone K satisfying condition (10.20) for some $x_0 \in K \cap S(X)$ and $\gamma > 0$ was called *quasinormal*. Later it turned out [169] that *every* cone in a Banach space is quasinormal, and so there is no need to keep a special name for this property. Some other constants which are related to the cone constant $\gamma(K)$ and describe some "degree of normality" of a cone K have been considered in [76]. Such constants arise from the study of nonlinear operator equations in ordered spaces and from fixed point theory.

The very natural definition of spectral radius (10.34) is due to Bonsall [43]. However, there are many other possibilities to define a spectral radius for nonlinear operators in cones, a comparative study of such spectral radii may be found in the recent survey [182].

Sometimes one is interested in eigenvectors which are not just positive (i.e., belong to some cone), but have additional properties; here the following special construction may be helpful [63]. Let X be a real Banach space with cone K and corresponding ordering $\preceq$, and $F\colon K \to K$ some (nonlinear) operator. Suppose that there exist some fixed element $e \in K \setminus \{\theta\}$ and functions $\alpha, \beta\colon K \to \mathbb{R}$ such that

$$\alpha(x)e \preceq F(x) \preceq \beta(x)e \quad (x \in K).$$

Then the authors of [63] introduce the special norm

$$\|x\|_e := \inf\{\gamma : \gamma > 0,\ -\gamma e \preceq x \preceq \gamma e\}$$

and compare the spectral properties of F with that of the operator F_e defined by

$$F_e(x) = \begin{cases} \|x\|_e^2 F\left(\frac{x}{\|x\|_e^2}\right) & \text{if } x \neq \theta, \\ \theta & \text{if } x = \theta. \end{cases}$$

The operator F_e coincides with F on the unit sphere $S_e(X)$ with respect to the norm $\|\cdot\|_e$ and is $(2-\tau)$-homogeneous if F is τ-homogeneous; compare this with the

1-homogeneous extension (3.10). We remark that the norm $\|\cdot\|_e$ is related to the so-called *part metric* (also called *Birkhoff metric* or *Thompson metric*) which is very useful for studying nonlinear problems in spaces with cones [247].

Sometimes one is interested in "localizing" positive eigenvectors more precisely. For example, in [290], [291] the author gives sufficient conditions on a positive operator F under which there exists a positive eigenvalue with corresponding eigenvector in the portion $K_{r,R} = \{x \in K : r \leq \|x\| \leq R\}$ of the cone. The concept of "cone compressing operators" is used in [95] to derive the existence of positive eigenvalues with eigenvectors of prescribed norm in a cone, i.e., in a set of the form $K \cap S_r(X)$. Finally, Filin [112] considers operators $F\colon K \to K$ with the property that $x \succeq y \succeq \theta$ implies $F(x) \preceq F(y)$, called "antitone operators". For such operators he proves the existence of a continuum of positive eigenvalues under suitable hypotheses.

A systematic investigation of the various concepts of eigenvalues discussed in Section 10.4 may be found in [232]. Lemma 10.5 seems to be new but is, of course, elementary. In [122] the authors remark that there are examples of bounded linear operators L which are injective but satisfy $[L]_{\mathrm{q}} = 0$. Examples of such operators are easily found; by Lemma 10.5, they cannot have a closed range.

The sets (10.64)–(10.66) of eigenvalues with "constraints" have been considered for the first time by Buryšek [50], [51], but they have not found yet applications to nonlinear problems. Our discussion in Section 10.5 shows that indeed the concept of connected eigenvalue, which leads to the point phantom and approximate point phantom, seems to be the most natural one in nonlinear spectral theory. We are convinced that this concept should give interesting new applications to specific nonlinear problems.

We point out that there is a vast literature on nonlinear operator equations of the type

$$F(x) = \lambda J(x), \tag{10.75}$$

and even on the more general equation

$$F(x) = J(\lambda, x). \tag{10.76}$$

Equation (10.76) gives rise to the so-called *bifurcation theory* which is one of the most important fields of nonlinear functional analysis and is intimately related to nonlinear eigenvalue problems. In [133] the authors develop a bifurcation theory for nonlinear eigenvalue problems connected to variational inequalities. We will consider a very special case of bifurcation problems in Section 12.4 below. Here we restrict ourselves to some biographical remarks for the interested reader.

A general discussion of equation (10.76) may be found in Berger's early work [37], [38], a detailed account of more recent results in the book [178] and in the survey [157]. The article [210] studies (10.75) in the spirit of the monograph [116], but proposes a different approach based on a new topological degree.

Even if the left-hand sides of the equations (10.75) and (10.76) are *linear*, i.e., have the form

$$Lx = \lambda J(x) \tag{10.77}$$

and

$$Lx = J(\lambda, x), \tag{10.78}$$

respectively, these equations have an enormous range of applications. There are essentially two kinds of problems where spectral methods apply successfully to these equations.

The first kind concerns *nonlinear elliptic differential equations*. Here in most cases the operator L is the *Laplace operator*, i.e., the problem (10.77) has the form

$$-\Delta u(x) = \lambda f(x, u(x)) \quad (x \in G), \tag{10.79}$$

where $G \subseteq \mathbb{R}^n$ is some sufficiently smooth domain, and the solutions u are supposed to satisfy some additional condition on ∂G (usually, the Dirichlet condition $u(x) \equiv 0$). (Here we write $u(x)$ instead of $x(t)$ for the unknown function, as is common use in partial differential equations.) For example, this problem has been studied with variational techniques in [292] and in spaces with cones in [96]. The particular cases $f(x, u) = g(x)h(u)$ and $f(x, u) = g(x)uh(u)$ are treated in [99], [100], where G is unbounded and some asymptotic condition on u is imposed.

A case which has received particular attention is that of *polynomial nonlinearities*. Thus, the author of [62] studies the eigenvalue problem

$$\begin{cases} -\Delta u(x) - \lambda u(x) = \alpha(x)|u(x)|^{p-2}u(x) & \text{in } G, \\ u(x) \equiv 0 & \text{on } \partial G. \end{cases} \tag{10.80}$$

The solutions of this eigenvalue problem may be equivalently obtained as critical points of the nonlinear functional Ψ_p defined by

$$\Psi_p(u) = \frac{2}{p} \int_G \alpha(x)|u(x)|^p \, dx.$$

We remark, however, that the problem (10.80) may appear somewhat artificial, inasmuch as the left-hand side of (10.80) is linear (in particular, 1-homogeneous), while the right-hand side is $(p-1)$-homogeneous. As one could expect, more satisfactory results may be obtained if both sides of the eigenvalue equation have the same degree of homogeneity. The by far most prominent example of such an equation is the *p-Laplace equation* which we will study in Section 12.5 below.

The second kind of applications concerns *Hammerstein integral equations*. A classical reference is here [140]–[142], where the following type of results are obtained. Assume that $F : X \to X$ is a compact operator which maps θ into itself and admits a Fréchet derivative $F'(\theta)$ at zero and in addition satisfies the coercivity condition $[F]_{\mathrm{Q}} = \infty$. Then the operator $\lambda I - F$ is surjective for all $\lambda \in \mathbb{K}$ which are not in the point spectrum $\sigma_{\mathrm{p}}(F'(\theta))$. In other words, for $\lambda \notin \sigma(F'(\theta))$ the equation

$$\lambda x - F(x) = y \tag{10.81}$$

has a solution for every $y \in X$. Since Fréchet derivatives of Hammerstein operators may be calculated rather easily (see Section 4.3), such existence results may be applied successfully to Hammerstein integral equations. Similarly, in [245] one may find conditions on the spectrum of the linear operator $F'(\theta)$ under which the equation (10.81) has at least *two solutions*, together with applications again to Hammerstein equations. Finally, the author of [289] studies the eigenvalue equation $\lambda x = F(x)$ for compact F under the "double coercivity condition"

$$\lim_{\|x\|\to\infty} \frac{\|F(x)\|}{\|x\|} = \lim_{\|x\|\to 0} \frac{\|F(x)\|}{\|x\|} = \infty$$

and applies this to Hammerstein integral equations. Applications of eigenvalue problems to nonlinear integral equations of Hammerstein–Volterra type may be found in [22].

Surprisingly, there exist also results in which eigenvalues of nonlinear operators may be obtained from *regular* values of linear operators. For instance [278], if an operator $L \in \mathfrak{L}(X)$ has a so-called "approximate inverse", then all $\lambda \in \rho(L)$ are eigenvalues of a *nonlinear* compact perturbation of L.

Equation (10.78) is of course much more complicated than equation (10.77), since the dependence of J on λ need not be linear. Here the existence of "connected branches" of solutions are of particular interest (see, e.g. [98]). This problem is related to the notion of connected eigenvalue which we studied in Section 10.5.

If one replaces the nonlinear operators F and J in the equation

$$F(x) + \lambda J(x) = y \quad (y \in Y) \tag{10.82}$$

by nonlinear *functionals* $\Phi, \Psi : X \to \mathbb{R}$, one ends up with the problem

$$\Phi(x) + \lambda \Psi(x) = r \quad (r \in \mathbb{R}). \tag{10.83}$$

In [227] it is shown that, under suitable semicontinuity conditions on Φ and Ψ, for each r in some real interval one can find $\lambda > 0$ and $x \in X$ which solve (10.83). The usefulness of such results lies in the usual "duality" of operators and (sufficiently regular) functionals which establishes a one-to-one correspondence between critical points of functionals and zeros of operators.

Still another type of equation, viz.,

$$\lambda I - Lx = F(x),$$

where L is a linear compact selfadjoint operator in a Hilbert space, and F is a nonlinear compact 1-homogeneous gradient operator, has been studied by Chiappinelli in [64]. This type of equation arises often in applications to problems like (10.79), and it is therefore interesting to compare the eigenvalues of operators L and $L + F$.

To conclude, we mention the special nonlinear eigenvalue problem

$$F(\lambda)z = \theta \quad (z = (z_1, \ldots, z_n) \in \mathbb{C}^n),$$

where $F\colon \Lambda \to \mathfrak{L}(\mathbb{C}^n)$ is a holomorphic matrix function defined on some domain $\Lambda \subseteq \mathbb{C}$. This problem is analyzed in [139], with a particular emphasis on the derivatives of the map $\lambda \mapsto F(\lambda)^{-1}(w)$ for fixed $w = (w_1, \ldots, w_n) \in \mathbb{C}^n$.

Chapter 11

Numerical Ranges of Nonlinear Operators

In this chapter we study some concepts of numerical range for both linear and nonlinear operators in Hilbert and Banach spaces. As in the linear case, numerical ranges provide a powerful tool for localizing the spectrum, but have also other interesting applications. In particular, it turns out that the Kachurovskij spectrum (5.9) of a Lipschitz continuous operator is contained in the convex closure of the numerical range. For special classes of spaces and operators, such as compact or differentiable operators, this may be slightly improved, as we shall show in the last section.

11.1 Linear operators in Hilbert spaces

Recall that the *numerical range* of a bounded linear operator L in a complex Hilbert space H with scalar product $\langle \cdot, \cdot \rangle$ is defined by

$$W(L) = \left\{ \frac{\langle Lx, x\rangle}{\|x\|^2} : x \in H,\ x \neq \theta \right\} = \{\langle Lx, x\rangle : x \in S(H)\}. \tag{11.1}$$

This definition goes essentially back to Toeplitz. We survey some fundamental properties of $W(L)$ in the following proposition. Since we are not primarily interested in linear operators, we drop the proof.

Proposition 11.1. *The numerical range* (11.1) *has the following properties* ($L, M \in \mathfrak{L}(H), \alpha, \beta, \lambda \in \mathbb{K}$):

(a) $W(\alpha I + \beta L) = \{\alpha + \beta\lambda : \lambda \in W(L)\}$.

(b) $W(L^*) = \{\overline{\lambda} : \lambda \in W(L)\}$.

(c) $W(U^*LU) = W(L)$ *for any unitary operator* U.

(d) $W(L + M) \subseteq W(L) + W(M)$.

(e) $W(L) = \{\lambda\}$ *if and only if* $L = \lambda I$.

(f) $W(L)$ *is convex.*

(g) *In case* $H = \mathbb{C}^n$, $W(L)$ *is compact.*

(h) *In case* $H = \mathbb{C}^2$, $W(L)$ *is the convex hull of an ellipse whose foci are the two complex eigenvalues of* L*; if there is only one eigenvalue* λ*, the ellipse is a circle with center* λ *and radius* $\|\lambda I - L\|/2$.

(i) *The inclusion*

$$\operatorname{co}\sigma(L) \subseteq \overline{W(L)} \tag{11.2}$$

is true.

(j) *If L is normal, then*

$$\operatorname{co}\sigma(L) = \overline{W(L)}. \tag{11.3}$$

(k) *If L is normal and $W(L)$ is closed, the extremal points of $W(L)$ are eigenvalues of L.*

(l) *L is selfadjoint if and only if $W(L) \subseteq \mathbb{R}$.*

(m) *If* $\operatorname{dist}(\lambda, W(L)) > 0$*, then the resolvent operator $R(\lambda; L) = (\lambda I - L)^{-1} : H \to H$ is bounded by*

$$\|R(\lambda; L)\| \le \frac{1}{\operatorname{dist}(\lambda, W(L))}. \tag{11.4}$$

We remark that Proposition 11.1 (m) implies that the spectrum $\sigma(L)$ is always contained in the closure of the numerical range $W(L)$, and so one may "localize" the spectrum in the complex plane by means of the numerical range. This is important because often one does not know the spectrum explicitly. Moreover, taking into account that the boundary $\partial\sigma(L)$ of the spectrum $\sigma(L)$ is contained in the approximate point spectrum $\sigma_{\mathrm{q}}(L)$ (see Proposition 1.4 (a)), from (11.2) one may deduce the inclusion

$$\sigma_{\mathrm{q}}(L) \subseteq \overline{W(L)}. \tag{11.5}$$

In fact, from $\partial\sigma(L) \subseteq \sigma_{\mathrm{q}}(L)$ and the convexity of $\overline{W(L)}$ we see that (11.2) and (11.5) are actually equivalent.

Simple examples show that the spectrum $\sigma(L+M)$ of the sum of two operators has in general nothing to do with the set $\sigma(L)+\sigma(M)$. In contrast, by Proposition 11.1 (d) we have

$$\sigma(L + M) \subseteq \overline{W(L + M)} \subseteq \overline{W(L)} + \overline{W(M)},$$

and so $W(L)$ and $W(M)$ may be very well used to localize $\sigma(L + M)$.

We point out that the numerical range $W(L)$ of a bounded linear operator may be essentially larger than the convex closure of the spectrum, as we show in the following Example 11.1. By Proposition 11.1 (j), such an operator cannot be normal. Afterwards we show in Example 11.2 that, in contrast to the spectrum, the numerical range need not be closed.

Example 11.1. Let $H = \mathbb{C}^2$ be the two-dimensional complex Hilbert space and $L \in \mathfrak{L}(H)$ be defined by

$$L(z, w) = (0, 2az),$$

where $a \in \mathbb{C}$ is fixed. A trivial calculation shows that $\sigma(L) = \{0\}$, and so from Proposition 11.1 (h) we know that

$$W(L) = \overline{\mathbb{D}}_{|a|} = \{\lambda \in \mathbb{C} : |\lambda| \le |a|\}.$$

Since we may choose $|a|$ arbitrarily large, the boundary of $W(L)$ can be as far as we want from the spectrum $\sigma(L)$. ♡

Example 11.2. Let $H = l_2$ be the space of all square-summable complex sequences, and let $L \in \mathfrak{L}(H)$ be the left shift operator (1.35). We already know that $\|L\| = 1$ and $\sigma(L) = \overline{\mathbb{D}}$ (see (1.36)). For $x = (x_1, x_2, x_3, \dots)$, where $x_1 \neq 0$, without loss of generality, we get

$$\begin{aligned} |\langle Lx, x\rangle| &\leq |x_1|\,|x_2| + |x_2|\,|x_3| + |x_3|\,|x_4| + \dots \\ &\leq \frac{1}{2}[|x_1|^2 + 2|x_2|^2 + 2|x_3|^2 + \dots] \\ &\leq \frac{1}{2}[2 - |x_1|^2] < 1. \end{aligned} \tag{11.6}$$

(In case $x_1 = x_2 = \dots = x_k = 0$ and $x_{k+1} \neq 0$ we replace $2 - |x_1|^2$ by $2 - |x_{k+1}|^2$ in (11.6).) This shows that $W(L)$ is contained in the *open* complex unit disc $\mathbb{D}$.

We claim that actually $W(L) = \mathbb{D}$, and so the closure in (11.3) cannot be removed. To see this, fix $z \in \mathbb{D}$, i.e., $z = re^{i\alpha}$ with $0 \leq r < 1$ and $0 \leq \alpha < 2\pi$. The sequence

$$x := (\sqrt{1-r^2}, r\sqrt{1-r^2}e^{-i\alpha}, r^2\sqrt{1-r^2}e^{-2i\alpha}, r^3\sqrt{1-r^2}e^{-3i\alpha}, \dots)$$

has then the norm

$$\|x\|^2 = 1 - r^2 + r^2(1-r^2) + r^4(1-r^2) + r^6(1-r^2) + \dots = (1-r^2)\sum_{k=0}^{\infty} r^{2k} = 1.$$

Furthermore,

$$\langle Lx, x\rangle = r(1-r^2)e^{i\alpha} + r^3(1-r^2)e^{i\alpha} + r^5(1-r^2)e^{i\alpha} + \dots = z,$$

and so $z \in W(L)$ as claimed. ♡

Given $L \in \mathfrak{L}(H)$, we call the nonnegative real number defined by

$$w(L) := \sup\{|\lambda| : \lambda \in W(L)\} \tag{11.7}$$

the *numerical radius* of L. This number plays a prominent role in the study of bounded linear operators. The most important property of (11.7) is that it defines an equivalent norm on $\mathfrak{L}(H)$, since

$$\frac{1}{2}\|L\| \leq w(L) \leq \|L\|. \tag{11.8}$$

We survey some other properties of the numerical radius in the following Proposition 11.2.

Proposition 11.2. *The numerical radius* (11.7) *has the following properties* ($L, M \in \mathfrak{L}(H)$, $\mu \in \mathbb{K}$).

(a) $w(L) = 0$ *if and only if* $L = \Theta$.

(b) $w(L + M) \leq w(L) + w(M)$.

(c) $w(\mu L) = |\mu| w(L)$.

(d) $r(L) \leq w(L)$, *where* $r(L)$ *denotes the spectral radius* (1.8).

(e) $r(L) = w(L) = \|L\|$ *if* L *is normal.*

(f) $w(L^n) \leq w(L)^n$ *for every* $n \in \mathbb{N}$.

Proof. The properties (a)–(c) are elementary. The bilateral estimate (11.8) implies, together with (f) and the Gel'fand formula (1.9), that

$$r(L) = \lim_{n\to\infty} \sqrt[n]{\|L^n\|} \leq \lim_{n\to\infty} \sqrt[n]{2w(L^n)} \leq \lim_{n\to\infty} \sqrt[n]{2w(L)^n} = w(L)$$

which proves (d). Alternatively, we could have used the inclusion (11.2) to deduce (d). Since $r(L) = \|L\|$ for normal operators (see Theorem 1.1 (e)), we get (e) from (d). Only the proof of (f) is more complicated.

First of all, by (c) it suffices to assume that $w(L) \leq 1$ and to prove that $w(L^n) \leq 1$. To this end, we recall the algebraic identity

$$1 - e^{in\tau}\langle L^n x, x\rangle = \frac{1}{n}\sum_{j=1}^{n} \|z_j\| \Big[1 - \frac{e^{2\pi ij/n} e^{i\tau}}{\|z_j\|}\langle Lz_j, z_j\rangle\Big] \quad (x \in S(H)),$$

for any $\tau \in \mathbb{R}$, where we have used the shortcut

$$z_j := \Big(\prod_{k=1,\, k\neq j}^{n} (1 - e^{2\pi ik/n} L)\Big)x \quad (j = 1, 2, \ldots, n).$$

Since $w(L) \leq 1$, the real part of each term on the right-hand side of this identity is nonnegative, so that $\mathrm{Re}(1 - e^{in\tau}\langle L^n x, x\rangle) \geq 0$. But this is true for all real τ, and thus $|\langle L^n x, x\rangle| \leq 1$. □

Observe that, by (11.8) and (1.9), the spectral radius of a bounded linear operator may be calculated through the numerical radius of its iterates, namely,

$$r(L) = \lim_{n\to\infty} \sqrt[n]{w(L^n)}.$$

We also point out that property (f) in Proposition 11.2 is relevant because the more general property $w(LM) \leq w(L)w(M)$ may be *not* true even if L and M commute (see Example 11.14 below for a counterexample).

We close this section with a general example which again illustrates the "geometric position" of the numerical range.

Example 11.3. Let $H = l_2$ over $\mathbb{C}$, $(a_n)_n$ a bounded sequence of complex numbers, and $A := \{a_n : n \in \mathbb{N}\}$. Consider in H the linear operator L defined by

$$L(x_1, x_2, x_3, \dots) = (a_1 x_1, a_2 x_2, a_3 x_3, \dots). \tag{11.9}$$

We know from Example 1.2 that

$$\sigma(L) = \overline{A} \tag{11.10}$$

in this case. We claim that

$$\operatorname{co} A \subseteq W(L) \subseteq \overline{W(L)} = \overline{\operatorname{co}}\, A \tag{11.11}$$

and

$$w(L) = r(L) = \|L\| = \sup\{|a_n| : n \in \mathbb{N}\}. \tag{11.12}$$

In fact, from the definition (11.1) it follows that $\lambda \in \mathbb{C}$ belongs to $W(L)$ if and only if

$$\lambda = \sum_{n=1}^{\infty} a_n |x_n|^2 \tag{11.13}$$

for some $x = (x_n)_n \in S(l_2)$. Fix $a_{n_1}, \dots, a_{n_m} \in A$ and $\mu_1, \dots, \mu_m \in [0, 1]$ with $\mu_1 + \cdots + \mu_m = 1$, and consider the element

$$x := \sum_{k=1}^{m} \sqrt{\mu_k} e_{n_k} = (0, \dots, 0, \sqrt{\mu_1}, 0, \dots, 0, \sqrt{\mu_k}, 0, \dots, 0, \sqrt{\mu_m}, 0, 0, 0, \dots),$$

with $\sqrt{\mu_k}$ at the n_k-th position ($k = 1, \dots, m$). We then have $\|x\| = 1$ and $\mu_1 a_{n_1} + \cdots + \mu_m a_{n_m} \in W(L)$, by (11.13). This proves the first inclusion in (11.11). The second inclusion is trivial, and the equality follows from (11.10) and Proposition 11.1 (j), since L is normal. Finally, the equality (11.12) is also a consequence of the normality of L and Proposition 11.2 (e). ♡

Choosing in Example 11.3, in particular, $a_n := 1/n$ we see again that $W(L)$ need not be closed. In fact, in this case we get $\sigma(L) = \{1/n : n \in \mathbb{N}\} \cup \{0\}$ and $W(L) = (0, 1]$.

11.2 Linear operators in Banach spaces

Defining the numerical range of a bounded linear operator in a Banach space requires some more definitions. Given a Banach space X with dual X^*, we use the notation $\langle x, \ell \rangle$ to denote the value of $\ell \in X^*$ at $x \in X$; this is compatible with the usual scalar product if X is Hilbert. Recall that the *duality map* $\mathcal{D} \colon X \to X^*$ is defined by

$$\mathcal{D}(x) := \{\ell_x \in X^* : \langle x, \ell_x \rangle = \|x\|^2,\ \|\ell_x\| = \|x\|\} \quad (x \in X). \tag{11.14}$$

Clearly, this is a 1-homogeneous multivalued map; if $\mathcal{D}$ is singlevalued, the space X is called *smooth*. For example, every Hilbert space is smooth with

$$\mathcal{D}(x) = \{\langle \cdot, x\rangle\}. \tag{11.15}$$

The following relation between the duality map (11.14) and special functionals from X^* is important in geometry of Banach spaces. If we denote by X^*_∞ the set of all functionals $\ell \in X^*$ which attain their norm in the unit sphere $S(X)$ of X, then

$$X^*_\infty = \bigcup_{x \in X} \mathcal{D}(x). \tag{11.16}$$

Given $L \in \mathfrak{L}(X)$, the *numerical range of L in the sense of Bauer* is defined by

$$W_{\mathrm{B}}(L) = \bigcup_{x \neq \theta} \left\{ \frac{\langle Lx, \ell_x\rangle}{\|x\|^2} : \ell_x \in \mathcal{D}(x) \right\}. \tag{11.17}$$

By (11.15), this gives precisely the definition (11.1) in the Hilbert space case. We remark that there are many important smooth Banach spaces which are not Hilbert. Two of them are discussed in the following example.

Example 11.4. First, the sequence space $X = l_p$ is smooth for $1 < p < \infty$ with $\mathcal{D}(x) = \{\ell_x\}$, where

$$\langle y, \ell_x\rangle := \frac{1}{\|x\|^{p-2}} \sum_{n=1}^{\infty} |x_n|^{p-2}\overline{x_n} y_n \quad (x \in l_p \setminus \{\theta\},\ y \in l_p).$$

Consequently, $\lambda \in \mathbb{C}$ belongs to $W_{\mathrm{B}}(L)$ if and only if

$$\sum_{n=1}^{\infty} |x_n|^{p-2}\overline{x_n}(Lx)_n = \lambda \|x\|^p.$$

Similarly, the corresponding function space $X = L_p[0,1]$ is also smooth for $1 < p < \infty$ with $\mathcal{D}(x) = \{\ell_x\}$, where

$$\langle y, \ell_x\rangle := \frac{1}{\|x\|^{p-2}} \int_0^1 |x(t)|^{p-2}\overline{x(t)} y(t)\, dt \quad (x \in L_p \setminus \{\theta\},\ y \in L_p).$$

Consequently, $\lambda \in \mathbb{C}$ belongs to $W_{\mathrm{B}}(L)$ if and only if

$$\int_0^1 |x(t)|^{p-2}\overline{x(t)}(Lx)(t)\, dt = \lambda \|x\|^p.$$

In case $p = 1$ or $p = \infty$, neither l_p nor $L_p[0,1]$ are smooth, and the formulas for the duality map become more complicated. ♡

We state now some elementary properties of the numerical range (11.17). In analogy to (11.7), we call the number

$$w_{\mathrm{B}}(L) := \sup\{|\lambda| : \lambda \in W_{\mathrm{B}}(L)\} \tag{11.18}$$

the *numerical radius of L in the sense of Bauer*.

Proposition 11.3. *The numerical range* (11.17) *has the following properties* ($K, L \in \mathfrak{L}(X)$, $\mu \in \mathbb{C}$).

(a) $W_{\mathrm{B}}(K + L) \subseteq W_{\mathrm{B}}(K) + W_{\mathrm{B}}(L)$.

(b) $W_{\mathrm{B}}(\mu L) = \mu W_{\mathrm{B}}(L)$.

(c) $W_{\mathrm{B}}(\mu I - L) = \{\mu\} - W_{\mathrm{B}}(L)$.

(d) $W_{\mathrm{B}}(L)$ *is bounded with* $w_{\mathrm{B}}(L) \leq \|L\|$.

We do not give the elementary proof of Proposition 11.3, because we will prove a more general result (see Proposition 11.4) in the next section. Instead, we discuss an alternative approach to numerical ranges in Banach spaces which is based on the concept of semi-inner products.

Given a linear space X over $\mathbb{K}$, a map $[\cdot, \cdot]\colon X \times X \to \mathbb{K}$ is called *semi-inner product* if it has the following properties ($x, y, z \in X, \alpha, \beta \in \mathbb{K}$):

(a) $[x, x] > 0$ if and only if $x \neq \theta$.

(b) $[x + y, z] = [x, z] + [y, z]$.

(c) $[\alpha x, y] = \alpha[x, y]$.

(d) $[x, \beta y] = \overline{\beta}[x, y]$.

(e) $|[x, y]|^2 \leq [x, x] \cdot [y, y]$.

Now, if $[\cdot, \cdot]$ is a semi-inner product on X, one may define a norm on X putting

$$\|x\| := \sqrt{[x, x]}. \tag{11.19}$$

Conversely, on every normed linear space one may define a semi-inner product by means of the duality map (11.14) in the following way. For any selection ℓ of the multivalued map $\mathcal{D}$ (i.e., $\ell_x \in \mathcal{D}(x)$ for all $x \in X$) we put

$$[x, y] := \langle y, \ell_x \rangle \quad (y \in X). \tag{11.20}$$

It is not hard to see that (11.20) has all properties of a semi-inner product, and that $[x, x] = \|x\|^2$, by (11.14).

Obviously, in general there are many ways to define a semi-inner product by means of a given norm on X, at least if X is non-smooth. In what follows, we write $\mathrm{sip}(X)$ for the set of all semi-inner products on X which generate the given norm by (11.19).

The definition (11.20) shows how to associate to each selection of the duality map a semi-inner product. Vice versa, one may recover the duality map of a normed space by means of the collection of all semi-inner products on this space. In fact, this follows from the important formula

$$\mathcal{D}(x) = \{[\,\cdot\,, x] : [\,\cdot\,,\,\cdot\,] \in \mathrm{sip}(X)\} \quad (x \in X) \tag{11.21}$$

which generalizes (11.15), inasmuch as in a Hilbert space H we simply have

$$\mathrm{sip}(H) = \{\langle\,\cdot\,,\,\cdot\,\rangle\}. \tag{11.22}$$

The calculations in Example 11.4 imply, together with (11.20), that

$$[y, x] = \frac{1}{\|x\|^{p-2}} \sum_{n=1}^{\infty} |x_n|^{p-2} \overline{x_n} y_n \quad (x \in l_p \setminus \{\theta\},\ y \in l_p)$$

is the unique element from $\mathrm{sip}(l_p)$, and

$$[y, x] = \frac{1}{\|x\|^{p-2}} \int_0^1 |x(t)|^{p-2} \overline{x(t)} y(t)\, dt \quad (x \in L_p \setminus \{\theta\},\ y \in L_p)$$

is the unique element in $\mathrm{sip}(L_p)$ for $1 < p < \infty$.

The preceding remarks show that there is an intimate relation between normed linear spaces and semi-inner product space. For example, a normed linear space X is smooth if and only if $\mathrm{sip}(X)$ is a singleton; in particular, the scalar product on a Hilbert space is the only semi-inner product on this space. In general, the duality map on X and the set $\mathrm{sip}(X)$ are connected by (11.21). So it is natural to define a numerical range on a Banach space alternatively by means of semi-inner products. Following Lumer, we define a numerical range for a fixed semi-inner product $[\,\cdot\,,\,\cdot\,] \in \mathrm{sip}(X)$ and $L \in \mathfrak{L}(X)$ by

$$W_{\mathrm{L}}(L; [\,\cdot\,,\,\cdot\,]) := \left\{ \frac{[Lx, x]}{\|x\|^2} : x \in X,\ x \neq \theta \right\} = \{[Lx, x] : \|x\| = 1\}. \tag{11.23}$$

Our preceding discussion shows then that the numerical range

$$W_{\mathrm{L}}(L) := \bigcup_{[\,\cdot\,,\,\cdot\,] \in \mathrm{sip}(X)} W_{\mathrm{L}}(L; [\,\cdot\,,\,\cdot\,]) \tag{11.24}$$

coincides with Bauer's numerical range (11.17), by (11.20) and (11.21). In particular, all properties of Proposition 11.3 hold true for $W_{\mathrm{B}}(L)$ replaced with $W_{\mathrm{L}}(L)$. In what follows, we will use the unifying notation $W_{\mathrm{BL}}(L)$ to denote either the numerical range (11.17) or (11.24).

We point out that some of the properties stated in Proposition 11.1 get lost when passing from Hilbert to Banach spaces. For instance, the numerical range $W_{\mathrm{BL}}(L)$ need not be convex, as the following example shows.

Example 11.5. For $1 < p < \infty$, we write X_p for the Banach space $\mathbb{C}^2$, equipped with the norm $\|(z, w)\|_p := (|z|^p + |w|^p)^{1/p}$. Let $L \in \mathfrak{L}(X_p)$ be defined by

$$L(z, w) := (iz + w, -z - iw).$$

Here the dual space is $X^* = X_q$, i.e., $\mathbb{C}^2$ with the conjugate norm $\|\cdot\|_q$ ($\frac{1}{p} + \frac{1}{q} = 1$). A trivial calculation shows that $\sigma(L) = \{\pm i\sqrt{2}\}$. Moreover, it is not hard to show that $W_{\mathrm{BL}}(L) = \{yi : -1 \le y \le 1\}$ in case $p = 2$. We claim that the numerical range $W_{\mathrm{BL}}(L)$ is not convex in case $p \ne 2$.

Given $(z, w) \in S(X_p)$, it is easy to see that $(\zeta, \omega) := (\overline{z}|z|^{p-2}, \overline{w}|w|^{p-2})$ is the unique element in $S(X_q)$ satisfying $z\zeta + w\omega = 1$. Therefore

$$W_{\mathrm{BL}}(L) = \{i|z|^p - i|w|^p + w\overline{z}|z|^{p-2} - \overline{w}z|w|^{p-2} : (z, w) \in S(X_p)\}.$$

Taking polar coordinates $z = re^{i\alpha}$ and $w = se^{i\beta}$ with $r^p + s^p = 1$, this may be rewritten in the form

$$\begin{aligned} W_{\mathrm{BL}}(L) = \big\{ & rs(r^{p-2} - s^{p-2})\cos(\beta - \alpha) \\ & + i[r^p - s^p + rs(r^{p-2} + s^{p-2})\sin(\beta - \alpha)] : \\ & r^p + s^p = 1,\ 0 \le \alpha, \beta < 2\pi \big\}. \end{aligned}$$

It follows that

$$\mu := \max\{\mathrm{Re}\,\lambda : \lambda \in W_{\mathrm{BL}}(L)\} = \max\{rs(r^{p-2} - s^{p-2}) : r^p + s^p = 1\}$$

and

$$\begin{aligned} \nu :&= \max\{\lambda : \lambda \in W_{\mathrm{BL}}(L) \cap \mathbb{R}\} \\ &= \max\big\{rs(r^{p-2} - s^{p-2})\cos(\beta - \alpha) : r^p + s^p = 1, \\ &\qquad 0 \le \alpha, \beta < 2\pi,\ r^p - s^p + rs(r^{p-2} + s^{p-2}) = 0\big\}. \end{aligned}$$

Choosing $\hat{r}, \hat{s}, \tilde{r}, \tilde{s}, \tilde{\alpha}$, and $\tilde{\beta}$ such that $\hat{r}^p + \hat{s}^p = \tilde{r}^p + \tilde{s}^p = 1$ and

$$\mu = \hat{r}\hat{s}(\hat{r}^{p-2} - \hat{s}^{p-2}), \quad \nu = \tilde{r}\tilde{s}(\tilde{r}^{p-2} - \tilde{s}^{p-2})\cos(\tilde{\beta} - \tilde{\alpha})$$

in case $p \ne 2$ we obtain $\mu > 0$ and $\nu < \mu$ unless $\cos(\tilde{\beta} - \tilde{\alpha}) = 1$, i.e., $\tilde{\alpha} = \tilde{\beta}$. But since $\tilde{r}\tilde{s}(\tilde{r}^{p-2} + \tilde{s}^{p-2})\sin(\tilde{\beta} - \tilde{\alpha}) = 0$, we have $\tilde{r} = \tilde{s}$ if $\cos(\tilde{\beta} - \tilde{\alpha}) = 0$, giving $\tilde{\beta} = 0$. Thus $\mu > \nu$ unless $p = 2$.

We conclude that the maximum μ is attained above and below the real axis, and so $W_{\mathrm{BL}}(L)$ cannot be convex in case $p \ne 2$. ♡

We remark that the set $W_{\mathrm{BL}}(L)$ is always connected in case of a complex Banach space, although it may be non-convex, by the preceding example. On the other hand, Lumer's numerical range (11.23) which depends on the particular choice of the semi-inner product may even be disconnected, as the following example shows.

Example 11.6. Slightly modifying the preceding Example 11.5, let X_∞ denote the Banach space $\mathbb{C}^2$ equipped with the norm $\|(z, w)\|_\infty = \max\{|z|, |w|\}$. This norm is generated, for example, by the semi-inner product

$$[(z, w), (\zeta, \omega)] := \begin{cases} z\overline{\zeta} & \text{if } |\zeta| \geq |\omega|, \\ w\overline{\omega} & \text{if } |\zeta| < |\omega|. \end{cases} \tag{11.25}$$

Indeed,

$$[(z, w), (z, w)] = \begin{cases} |z|^2 = \|(z, w)\|_\infty^2 & \text{if } |z| \geq |w|, \\ |w|^2 = \|(z, w)\|_\infty^2 & \text{if } |z| < |w|. \end{cases}$$

Define $L \in \mathfrak{L}(X_\infty)$ by $L(z, w) := (z, 0)$. Then for $(z, w) \in S(X_\infty)$ we have

$$[L(z, w), (z, w)] := \begin{cases} 1 & \text{if } |z| = 1, \\ 0 & \text{if } |z| < 1, \end{cases}$$

and so $w_{\mathrm{L}}(L; [\,\cdot\,,\,\cdot\,]) = \{0, 1\}$ for the semi-inner product (11.25). ♡

The following Theorem 11.1 gives a fundamental relation between the spectrum of a bounded linear operator in a Banach space and its numerical range in the sense of (11.17) or (11.24). In what follows, we will discuss several "nonlinear analogues" of this.

Theorem 11.1. *Let X be a Banach space and $L \in \mathfrak{L}(X)$. Then*

$$\sigma(L) \subseteq \overline{W_{\mathrm{BL}}(L)}. \tag{11.26}$$

Proof. Suppose that $\lambda \in \mathbb{K} \setminus \overline{W_{\mathrm{BL}}(L)}$. Then $0 \notin \overline{W_{\mathrm{BL}}(\lambda I - L)}$, by Proposition 11.3 (c), and so from Proposition 2.1 (a) it follows that $R(\lambda; L) = (\lambda I - L)^{-1}$ exists and is bounded on $R(\lambda I - L)$ with

$$\|R(\lambda; L)|_{R(\lambda I - L)}\| \leq \frac{1}{\operatorname{dist}(\lambda, \overline{W_{\mathrm{BL}}(L)})}.$$

It remains to show that $\lambda I - L$ is onto. As before, denote by X_∞^* the subset of all functionals $\ell \in X^*$ which obtain their norm in the unit sphere $S(X)$ of X. Since this set is dense in X^*, for fixed $\ell \in S(X^*)$ we may find a sequence $(\ell_n)_n$ in X_∞^* such that $\|\ell_n\| = 1$ for all $n \in \mathbb{N}$ and $\ell_n \to \ell$ as $n \to \infty$. By (11.16) and (11.21), there exist a sequence $(x_n)_n$ in $S(X)$ and a sequence of semi-inner products $[\,\cdot\,,\,\cdot\,]_n$ in $\operatorname{sip}(X)$ such that $\langle x, \ell_n \rangle = [x, x_n]_n$ for all $x \in X$. Consequently,

$$\begin{aligned}\|(\lambda I - L)^*\ell\| &= \lim_{n\to\infty} \|(\lambda I - L)^*\ell_n\| \\ &\geq \limsup_{n\to\infty} \|\langle x_n, (\lambda I - L)^*\ell_n\rangle\| \\ &= \limsup_{n\to\infty} \|\langle(\lambda I - L)(x_n), \ell_n\rangle\| \\ &= \limsup_{n\to\infty} \|[(\lambda I - L)(x_n), x_n]_n\| \\ &\geq \operatorname{dist}(\lambda, \overline{W_{\mathrm{BL}}(L)}) > 0.\end{aligned}$$

Thus $\|(\lambda I - L)^*\ell\| > 0$ for all $\ell \in S(X^*)$ and it follows that $(\lambda I - L)^*$ is injective. By well-known results on bounded linear operators and their adjoints, this implies that $\lambda I - L$ is onto, and so the proof is complete. □

11.3 Numerical ranges of nonlinear operators

The first definition of a numerical range for nonlinear operators seems to be due to Zarantonello and goes as follows. Let H be a Hilbert space and $F\colon H \to H$ be bounded and continuous. Put

$$W_{\mathrm{Z}}(F) = \left\{ \frac{\langle F(x) - F(y), x - y\rangle}{\|x - y\|^2} : x, y \in H,\ x \neq y \right\}. \tag{11.27}$$

Obviously, this definition coincides with (11.1) in the linear case. By the continuity of F, the set $W_{\mathrm{Z}}(F)$ is always connected. As in the linear case, $W_{\mathrm{Z}}(F)$ need not be closed.

Now let $F\colon H \to H$ be Lipschitz continuous and $\lambda \notin \overline{W_{\mathrm{Z}}(F)}$. Then the operator $\lambda I - F\colon H \to H$ is a *lipeomorphism*, i.e., $\lambda I - F \in \mathfrak{Lip}(H)$ and $(\lambda I - F)^{-1} \in \mathfrak{Lip}(H)$. In other words, for the Kachurovskij spectrum $\sigma_{\mathrm{K}}(F)$ defined in (5.9) we have the important inclusion

$$\sigma_{\mathrm{K}}(F) \subseteq \overline{W_{\mathrm{Z}}(F)} \tag{11.28}$$

which is of course analogous to (11.3). As a matter of fact, this can be made more precise. To see this, we first need a sufficient condition on a nonlinear operator to be a homeomorphism in a Hilbert space which is similar to Lemma 9.5.

Lemma 11.1. *Let H be a Hilbert space, and suppose that $F\colon H \to H$ satisfies $F(\theta) = \theta$ and*

$$|\langle Fx - Fy, x - y\rangle| \geq d\|x - y\|^2 \tag{11.29}$$

for some $d > 0$. Then F is a homeomorphism in H whose inverse is Lipschitz continuous with $[F^{-1}]_{\mathrm{Lip}} \leq 1/d$.

Proof. The hypothesis (11.29) implies that $[F]_{\mathrm{lip}} \geq d$, and so Proposition 2.1 (a) shows that F is closed and injective, and $F^{-1}\colon R(F) \to H$ is Lipschitz continuous with $[F^{-1}|_{R(F)}]_{\mathrm{Lip}} \leq 1/d$. It is the surjectivity of F which requires a careful analysis and is based on the fact that, if $F\colon B_r(\mathbb{R}^n) \to \mathbb{R}^n$ is continuous and satisfies $F(\theta) = \theta$ and (11.29), then F is one-to-one and $R(F) \supseteq B_{rd}(\mathbb{R}^n)$.

Let $y \in H$ be fixed with $\|y\| = R$, and put $r := R/d$. Denote by P_1 the projection of H onto the subspace generated by y, and let $x_1 \in B_r(H)$ be the solution of the equation $P_1 F(x) = P_1 y$ which exists by what we have just observed. Further, denote by P_2 the projection of H onto the subspace generated by y and $F(x_1) - y$, and let $x_2 \in B_r(H)$ be the solution of the equation $P_2 F(x) = P_2 y$. Continuing this way we may construct a sequence $(x_n)_n$ in $B_r(H)$ which converges, without loss of generality, weakly to some $x_* \in B_r(H)$. Now, for $n > m$ we have, by (11.29) and the fact that $F(x_m) - y$ is orthogonal to x_k for $k = 1, 2, \ldots, m$,

$$\begin{aligned} d\|x_n - x_m\|^2 &\leq |\langle F(x_n) - F(x_m), x_n - x_m\rangle| \\ &\leq |\langle F(x_n) - y, x_n - x_m\rangle| + |\langle F(x_m) - y, x_n - x_m\rangle| \\ &= |\langle F(x_m) - y, x_n\rangle|. \end{aligned}$$

Letting first $n \to \infty$ we obtain

$$d\|x_* - x_m\|^2 \leq d \limsup_{n\to\infty} \|x_n - x_m\|^2 \leq |\langle F(x_m) - y, x_*\rangle|,$$

and afterwards for $m \to \infty$ we arrive at the estimate

$$d \limsup_{m\to\infty} \|x_* - x_m\|^2 \leq \limsup_{m\to\infty} |\langle F(x_m) - y, x_*\rangle| = 0.$$

This shows that the sequence $(x_n)_n$ is actually strongly convergent to x_*. By continuity, this implies that $F(x_n) \to F(x_*)$ as $n \to \infty$. We claim that

$$F(x_n) \rightharpoonup y \quad (n \to \infty). \tag{11.30}$$

Denote by $H_0 = \mathrm{span}\{y, F(x_1)-y, F(x_2)-y, F(x_3)-y, \ldots\}$ the subspace generated by y and all elements $F(x_n) - y$. Then $\langle F(x_n) - y, z\rangle \to 0$ for every $z \in H_0$, and thus (11.30) follows. Since strong and weak convergence are compatible, we conclude that $F(x_*) = y$, and so F is onto as claimed. □

Lemma 11.1 immediately implies the following Theorem 11.2 which in turn yields (11.28).

Theorem 11.2. *Let H be a Hilbert space, $F\colon H \to H$ Lipschitz continuous with $F(\theta) = \theta$, and $\lambda \in \mathbb{K}$ with*

$$d_\lambda := \mathrm{dist}(\lambda, W_Z(F)) > 0.$$

Then $\lambda I - F\colon H \to H$ is a lipeomorphism with

$$[R(\lambda; F)]_{\mathrm{Lip}} = [(\lambda I - F)^{-1}]_{\mathrm{Lip}} \leq \frac{1}{d_\lambda}. \tag{11.31}$$

Now we pass to numerical ranges of nonlinear operators in Banach spaces. Passing from Hilbert to Banach spaces goes exactly as in the linear case, either by means of the duality map or using semi-inner products. So let X be a Banach space with dual X^*, and consider the duality map (11.14). Let $F\colon X \to X$ be bounded and continuous. Following Rhodius, we define a numerical range for F by

$$W_{\mathrm{R}}(F) = \bigcup_{x \neq y} \left\{ \frac{\langle F(x) - F(y), \ell_{x-y} \rangle}{\|x-y\|^2} : \ell_{x-y} \in \mathcal{D}(x-y) \right\}. \tag{11.32}$$

A comparison of (11.15), (11.27) and (11.32) shows that the numerical ranges in Zarantonello's and Rhodius' sense coincide in the Hilbert case, and so the latter is an extension of the former. As before, we call the number

$$w_{\mathrm{R}}(F) := \sup\{|\lambda| : \lambda \in W_{\mathrm{R}}(F)\} \tag{11.33}$$

the *numerical radius of F in the sense of Rhodius*. The following proposition contains Proposition 11.3 as a special case.

Proposition 11.4. *The numerical range* (11.32) *has the following properties* ($F, G\colon X \to X$ *continuous and bounded,* $\mu \in \mathbb{K}$):

(a) $W_{\mathrm{R}}(F+G) \subseteq W_{\mathrm{R}}(F) + W_{\mathrm{R}}(G)$.

(b) $W_{\mathrm{R}}(\mu F) = \mu W_{\mathrm{R}}(F)$.

(c) $W_{\mathrm{R}}(F_z) = W_{\mathrm{R}}(F)$, *where* $F_z(x) = F(x) + z$.

(d) $W_{\mathrm{R}}(\mu I - F) = \{\mu\} - W_{\mathrm{R}}(F)$.

(e) $W_{\mathrm{R}}(F)$ *is bounded for* $F \in \mathfrak{Lip}(X)$ *with* $w_{\mathrm{R}}(F) \leq [F]_{\mathrm{Lip}}$.

Proof. The assertions (a) and (b) follow from the linearity of each functional $\ell_{x-y} \in \mathcal{D}(x-y)$, while (c) follows from the fact that we consider differences in (11.32).

To prove (d), fix $\lambda \in W_{\mathrm{R}}(\mu I - F)$. Then for some $\ell_{x-y} \in \mathcal{D}(x-y)$, with $x \neq y$, we get

$$\begin{aligned}\lambda &= \frac{\langle \mu(x-y) - F(x) + F(y), \ell_{x-y} \rangle}{\|x-y\|^2} \\ &= \mu \frac{\langle x-y, \ell_{x-y} \rangle}{\|x-y\|^2} - \frac{\langle F(x) - F(y), \ell_{x-y} \rangle}{\|x-y\|^2} \\ &= \mu - \frac{\langle F(x) - F(y), \ell_{x-y} \rangle}{\|x-y\|^2},\end{aligned}$$

by definition of $\mathcal{D}(x-y)$, and so $\mu - \lambda \in W_{\mathrm{R}}(F)$.

Finally, for $F \in \mathfrak{Lip}(X)$, $x \neq y$, and $\ell_{x-y} \in \mathcal{D}(x-y)$ we obtain

$$\frac{|\langle F(x) - F(y), \ell_{x-y} \rangle|}{\|x-y\|^2} = \frac{\|F(x) - F(y)\|}{\|x-y\|} \leq [F]_{\mathrm{Lip}}, \tag{11.34}$$

which proves (e). ◻

A certain simplification of the numerical range (11.32) for continuous operators F in a Banach space X was defined by Feng, namely

$$W_{\mathrm{F}}(F) = \bigcup_{x \neq \theta} \left\{ \frac{\langle F(x), \ell_x \rangle}{\|x\|^2} : \ell_x \in \mathfrak{D}(x) \right\} \cup \{0\}, \tag{11.35}$$

together with the corresponding numerical radius

$$w_{\mathrm{F}}(F) := \sup\{|\lambda| : \lambda \in W_{\mathrm{F}}(F)\}. \tag{11.36}$$

Of course, the inclusion

$$W_{\mathrm{F}}(F) \subseteq W_{\mathrm{R}}(F) \tag{11.37}$$

holds in case $F(\theta) = \theta$. The following theorem relates the numerical range (11.35) to the Feng spectrum which we studied in detail in Chapter 7.

Theorem 11.3. *Let X be a Banach space and $F \in \mathfrak{A}(X)$. Fix $\lambda \in \sigma_{\mathrm{F}}(F)$ with $|\lambda| > [F]_{\mathrm{A}}$, where $\sigma_{\mathrm{F}}(F)$ denotes the Feng spectrum* (7.19). *Then $\lambda \in \overline{\mathrm{co}}\, W_{\mathrm{F}}(F)$.*

Proof. Suppose that $|\lambda| > [F]_{\mathrm{A}}$ and $\mathrm{dist}(\lambda, \overline{\mathrm{co}}\, W_{\mathrm{F}}(F)) > 0$; we show that then $\lambda \in \rho_{\mathrm{F}}(F)$ (see (7.18)).

First of all, from $|\lambda| > [F]_{\mathrm{A}}$ it follows that $\lambda \notin \sigma_{\mathrm{a}}(F)$, by Proposition 2.5. Next, for $x \in X \setminus \{\theta\}$ and $\ell_x \in \mathfrak{D}(x)$ we get

$$\begin{aligned} 0 &< \mathrm{dist}(\lambda, \overline{\mathrm{co}}\, W_{\mathrm{F}}(F)) \\ &\leq \left| \lambda - \frac{\langle F(x), \ell_x \rangle}{\|x\|^2} \right| \\ &= \frac{|\langle \lambda x, \ell_x \rangle - \langle F(x), \ell_x \rangle|}{\|x\|^2} \\ &= \frac{|\langle \lambda x - F(x), \ell_x \rangle|}{\|x\|^2} \\ &\leq \frac{\|\lambda x - F(x)\|}{\|x\|}. \end{aligned}$$

which implies that $\lambda \notin \sigma_{\mathrm{b}}(F)$.

It remains to show that $\lambda \notin \sigma_\nu(F)$, i.e., $\nu(\lambda I - F) > 0$ (see (7.3) and (7.21)). Consider the set

$$S = \{x \in X : \lambda x - tF(x) = \theta \text{ for some } t \in [0, 1]\}.$$

Clearly, $\theta \in S$. Given $x \in S \setminus \{\theta\}$ we have $\lambda x = tF(x)$ for some $t \in (0, 1]$, hence

$$\frac{\langle F(x), \ell_x \rangle}{\|x\|^2} = \frac{\lambda}{t} \frac{\|x\|^2}{\|x\|^2} = \frac{\lambda}{t} \quad (\ell_x \in \mathfrak{D}(x)).$$

But this implies that

$$\lambda = t\frac{\langle F(x), \ell_x\rangle}{\|x\|^2} \in \operatorname{co} W_{\mathrm{F}}(F),$$

a contradiction. So we have proved that $S = \{\theta\}$. By Property 7.4 of k-epi operators, applied to $F_0 = \lambda I$ and $H(\cdot, t) = -tF$, we conclude that $\lambda I - F$ is k-epi on $\overline{\Omega}$ for any $\Omega \in \mathfrak{O}\mathfrak{B}\mathfrak{C}(X)$ and $0 \leq k \leq |\lambda| - [F]_{\mathrm{A}}$. This means that $\nu(\lambda I - F) > 0$, and we are done. □

We remark that the corresponding definition of a numerical range for F building on semi-inner products, namely

$$W_{\mathrm{M}}(F) = \bigcup_{[\cdot,\cdot]\in \mathrm{sip}(X)} W_{\mathrm{M}}(F; [\cdot, \cdot]), \tag{11.38}$$

where

$$W_{\mathrm{M}}(F; [\cdot, \cdot]) = \left\{ \frac{[F(x) - F(y), x - y]}{\|x - y\|^2} : x, y \in X,\ x \neq y \right\}, \tag{11.39}$$

is essentially due to Martin. Of course, the numerical range (11.38) is the same as (11.32). In what follows, we will use the unifying notation $W_{\mathrm{MR}}(F)$ to denote either the numerical range (11.32) or (11.38).

We establish now a fundamental link between the numerical range $W_{\mathrm{MR}}(F)$, on the one hand, and the Kachurovskij spectrum for Lipschitz continuous operators, on the other. To this end, we first need two lemmas.

Lemma 11.2. *Let X be a Banach space and $F \in \mathfrak{Lip}(X)$. Suppose that*

$$d_\lambda := \operatorname{dist}(\lambda, W_{\mathrm{MR}}(F)) > 0 \tag{11.40}$$

for some $\lambda \in \mathbb{K}$. Then $\lambda I - F\colon X \to X$ is injective, $R(\lambda I - F)$ is closed, and $R(\lambda; F) = (\lambda I - F)^{-1}\colon R(\lambda I - F) \to X$ is Lipschitz continuous with (11.31).

Proof. The assertion is an immediate consequence of Proposition 2.1 (a) and is proved as Theorem 11.2 by means of Lemma 11.1. □

Lemma 11.3. *Let X be a Banach space, $[\cdot, \cdot] \in \mathrm{sip}(X)$, and $F \in \mathfrak{Lip}(X)$. Suppose that there is a number $\beta > 0$ such that*

$$\frac{\mathrm{Re}[F(x) - F(y), x - y]}{\|x - y\|^2} \leq -\beta \quad (x \neq y). \tag{11.41}$$

Then F is a lipeomorphism, i.e., F is a bijection with $F^{-1} \in \mathfrak{Lip}(X)$.

Proof. Since (11.41) implies (11.40) for $\lambda = 0$, we know from Lemma 11.2 that F is injective, $R(F)$ is closed, and $F^{-1}: R(F) \to X$ is Lipschitz continuous. So it remains to show that $R(F) = X$. Since $W_{\mathrm{MR}}(F_z) = W_{\mathrm{MR}}(F)$ for every $z \in X$, it obviously suffices to prove that $\theta \in R(F)$.

Fix $h > 0$ with $h < 1/[F]_{\mathrm{Lip}}$; then $(I - hF)^{-1} \in \mathfrak{Lip}(X)$, by Proposition 5.1. The algebraic identity

$$\frac{1}{h}[(I - hF)^{-1} - I] = \frac{1}{h}[I - (I - hF)](I - hF)^{-1} = F(I - hF)^{-1}$$

shows that we only have to prove that there exists some $z \in X$ such that $(I - hF)^{-1}(z) - z = \theta$, i.e., $z = (I - hF)(z)$.

From Proposition 11.4 (b), (d) we know that $W_{\mathrm{MR}}(I - hF) = \{1\} - hW_{\mathrm{MR}}(F)$, and so (11.41) implies that

$$\inf\{\mathrm{Re}\,\lambda : \lambda \in W_{\mathrm{MR}}(I - hF)\} \geq 1 + h\beta.$$

Consequently, for $x \neq y$ and $[\,\cdot\,,\,\cdot\,] \in \mathrm{sip}(X)$ we have

$$\begin{aligned}\|(I - hF)(x) - (I - hF)(y)\| &\geq \frac{\mathrm{Re}[(I - hF)(x) - (I - hF)(y), x - y]}{\|x - y\|} \\ &\geq \sup\{\|x - y\|\ \mathrm{Re}\,\lambda : \lambda \in W_{\mathrm{MR}}(I - hF)\} \\ &\geq (1 + h\beta)\|x - y\|.\end{aligned}$$

This is easily seen to imply that

$$[(I - hF)^{-1}]_{\mathrm{Lip}} \leq \frac{1}{1 + h\beta} < 1,$$

and so from Proposition 5.1 we conclude that $\theta \in R(I - (I - hF)^{-1}) = X$. □

We are now in a position to prove the announced relation between the Kachurovskij spectrum of a Lipschitz continuous operator and its numerical range in the sense of Rhodius or Martin. In the linear case, this gives precisely (11.26).

Theorem 11.4. *Let X be a Banach space and $F \in \mathfrak{Lip}(X)$. Then*

$$\sigma_{\mathrm{K}}(F) \subseteq \overline{\mathrm{co}}\, W_{\mathrm{MR}}(F). \tag{11.42}$$

Proof. Suppose that $\lambda \notin \overline{\mathrm{co}}\, W_{\mathrm{MR}}(F)$. Clearly, it suffices to show that there is an $\alpha \in \mathbb{K}$ with $|\alpha| = 1$ such that $\alpha(\lambda I - F): X \to X$ is a lipeomorphism. To this end we use Lemma 11.3 with F replaced by $\alpha(\lambda I - F)$. Now, for fixed $[\,\cdot\,,\,\cdot\,] \in \mathrm{sip}(X)$

we have

$$\begin{aligned}
&\overline{\mathrm{co}}\left\{\frac{[\alpha(\lambda I - F)(x) - \alpha(\lambda I - F)(y), x - y]}{\|x - y\|^2} : x \neq y\right\} \\
&\quad = \overline{\mathrm{co}}\left\{\alpha\frac{[(\lambda I - F)(x) - (\lambda I - F)(y), x - y]}{\|x - y\|^2} : x \neq y\right\} \\
&\quad = \alpha\ \overline{\mathrm{co}}\left(\{\lambda\} - \left\{\frac{[F(x) - F(y), x - y]}{\|x - y\|^2} : x \neq y\right\}\right) \\
&\quad = \alpha\left(\{\lambda\} - \overline{\mathrm{co}}\left\{\frac{[F(x) - F(y), x - y]}{\|x - y\|^2} : x \neq y\right\}\right).
\end{aligned}$$

Since

$$0 \notin \{\lambda\} - \overline{\mathrm{co}}\left\{\frac{[F(x) - F(y), x - y]}{\|x - y\|^2} : x \neq y\right\},$$

it suffices to show that, if $\Lambda \subset \mathbb{K}$ is compact and convex with $0 \notin \Lambda$, we find an $\alpha \in \mathbb{K}$ with $|\alpha| = 1$ such that $\mathrm{Re}(\alpha\lambda) < 0$ for all $\lambda \in \Lambda$. This is trivial if $\mathbb{K} = \mathbb{R}$. Identifying $\mathbb{C}$ with $\mathbb{R}^2$, a geometric reasoning shows that there is a line L in $\mathbb{C}$ containing 0 which does not meet Λ. By choosing $\alpha \in \mathbb{S}$ in such a way that αL coincides with the imaginary axis, it is clear that either $\alpha\Lambda$ or $-\alpha\Lambda$ lies entirely in the open left half-plane. This completes the proof. □

In view of (11.28) one could expect that (11.42) may be replaced by the stronger inclusion

$$\sigma_{\mathrm{K}}(F) \subseteq \overline{W_{\mathrm{MR}}(F)}. \tag{11.43}$$

The following theorem gives several additional conditions on X or F under which this is in fact true. Let us first make a general observation. Recall that we defined the subspectrum $\sigma_{\mathrm{lip}}(F)$ in (2.28) by

$$\sigma_{\mathrm{lip}}(F) = \{\lambda \in \mathbb{K} : [\lambda I - F]_{\mathrm{lip}} = 0\},$$

with $[F]_{\mathrm{lip}}$ being the lower Lipschitz constant (2.2). Now, if X is a Banach space and $F \in \mathfrak{Lip}(X)$, then it is not hard to see that

$$\sigma_{\mathrm{lip}}(F) \subseteq \overline{W_{\mathrm{MR}}(F)}. \tag{11.44}$$

In fact, suppose that $\lambda \in \mathbb{K}$ satisfies (11.40). By Lemma 11.1 we know then that

$$|\langle \lambda x - F(x) - \lambda y + F(y), \ell_{x-y}\rangle| \geq d_\lambda \|x - y\|^2$$

for $x \neq y$ and $\ell_{x-y} \in \mathfrak{D}(x - y)$. Since $\|\ell_{x-y}\| = 1$, this implies that $[\lambda I - F]_{\mathrm{lip}} \geq d_\lambda > 0$, i.e., $\lambda \notin \sigma_{\mathrm{lip}}(F)$.

Theorem 11.5. *Let X be a Banach space and $F \in \mathfrak{Lip}(X)$. Then* (11.43) *holds true provided that one of the following conditions is satisfied*:

(a) *X is a real Banach space.*

(b) *X is finite dimensional.*

(c) *X is a Hilbert space.*

(d) *F is compact.*

(e) *F belongs to $\mathfrak{C}^1(X)$.*

Proof. The assertion (a) is almost trivial, since the numerical range $W_{\mathrm{MR}}(F)$ is always an interval in the real case, while the statements (b) and (c) are consequences of Theorem 11.2.

To prove (d), suppose that F is compact. Then from Proposition 5.3 we know that $\sigma_{\mathrm{K}}(F) = \sigma_{\mathrm{lip}}(F)$ which together with (11.44) immediately implies (11.43).

The only nontrivial condition which has to be checked is (e). So suppose that F is continuously Fréchet differentiable on X, and let $\lambda \in \sigma_{\mathrm{K}}(F)$. Theorem 4.2 implies that then either $\lambda \in \sigma(F'(x_0))$ for some $x_0 \in X$, or $\lambda I - F$ is not proper.

Suppose that $\lambda \in \sigma(F'(x_0))$; we show that

$$W_{\mathrm{MR}}(F'(x_0)) \subseteq \overline{W_{\mathrm{MR}}(F)}. \tag{11.45}$$

Fix $[\cdot, \cdot] \in \mathrm{sip}(X)$ and $\mu \in W_{\mathrm{MR}}(F'(x_0); [\cdot, \cdot]) = w_{\mathrm{L}}(F'(x_0); [\cdot, \cdot])$ (see (11.39) and (11.23)), i.e., $\mu = [F'(x_0)h, h]$ for some $h \in S(X)$. Choose a positive sequence $(\tau_n)_n$ such that

$$\frac{1}{\tau_n}\|F(h + \tau_n h) - F(h) - F'(x_0)(\tau_n h)\| \leq \frac{1}{n},$$

and define

$$\mu_n := \frac{1}{\tau_n}[F(h + \tau_n h) - F(h), h] \quad (n \in \mathbb{N}).$$

Since $\|(h + \tau_n h) - h\| = \|\tau_n h\| = \tau_n$, it follows that $\mu_n \in W_{\mathrm{MR}}(F; [\cdot, \cdot])$ for all $n \in \mathbb{N}$. Moreover,

$$\begin{aligned} |\mu_n - \mu| &= \left| \frac{[F(h + \tau_n h) - F(h), h]}{\tau_n} - \frac{[F'(x_0)(\tau_n h), h]}{\tau_n} \right| \\ &= \frac{|[F(h + \tau_n h) - F(h) - F'(x_0)(\tau_n h), h]|}{\tau_n} \\ &\leq \frac{\|F(h + \tau_n h) - F(h) - F'(x_0)(\tau_n h)\|}{\tau_n} \leq \frac{1}{n}. \end{aligned}$$

Since the semi-inner product $[\cdot, \cdot]$ was arbitrary, we conclude that $\mu \in \overline{W_{\mathrm{MR}}(F)}$, and so (11.45) holds true.

Suppose now that $\lambda I - F$ is not proper. Then $\lambda \in \overline{W_{\mathrm{MR}}(F)}$, since otherwise $\lambda I - F$ would be proper, by Lemma 11.2. □

We illustrate the inclusions (11.42)–(11.44) by means of an operator which we already considered several times before.

Example 11.7. For $1 < p < \infty$, let X be the complex sequence space l_p, and let $F \in \mathfrak{Lip}(X)$ be defined by

$$F(x_1, x_2, x_3, \dots) := (\|x\|, x_1, x_2, \dots).$$

We already know from Example 5.3 that

$$\sigma_{\mathrm{K}}(F) = \overline{\mathbb{D}}_{2^{1/p}} = \{\lambda \in \mathbb{C} : |\lambda| \leq 2^{1/p}\}$$

and

$$\sigma_{\mathrm{lip}}(F) = \overline{\mathbb{D}}_{2^{1/p}} \setminus \mathbb{D} = \{\lambda \in \mathbb{C} : 1 \leq |\lambda| \leq 2^{1/p}\}.$$

For $x \neq \theta$, denote by ℓ_x the functional from Example 11.4 which is the unique element in $\mathcal{D}(x)$. For $a \in \overline{\mathbb{D}}$ and $x := (\overline{a}, 0, (1 - |a|^p)^{1/p}, 0, 0, \dots) \in X$ we have $\|x\| = 1$ and $F(x) = (1, \overline{a}, 0, (1 - |a|^p)^{1/p}, 0, \dots)$. Calculating ℓ_x at this element we obtain

$$\langle F(x), \ell_x \rangle = \sum_{n=1}^{\infty} |x_n|^{p-2} \overline{x_n} F(x)_n = |a|^{p-2} a,$$

and so we see that

$$W_{\mathrm{MR}}(F) \supseteq \{|a|^{p-2} a : |a| \leq 1\} = \overline{\mathbb{D}}. \tag{11.46}$$

But from (11.44) and (11.46) we may conclude that even

$$\overline{\mathbb{D}} \cup (\overline{\mathbb{D}}_{2^{1/p}} \setminus \mathbb{D}) \subseteq \overline{W_{\mathrm{MR}}(F)}.$$

So the inclusion (11.43) holds true, although none of the hypotheses (a)–(e) from Theorem 11.5 is satisfied for F in this example. ♡

Now we discuss yet another numerical range which was introduced by Furi, Martelli and Vignoli in connection with the FMV-spectrum (see Chapter 6) in Hilbert spaces.

Let H be a real or complex Hilbert space and $F\colon H \to H$ continuous and bounded with $F(\theta) = \theta$. We associate with F another operator $F^{\perp}\colon H \to H$ defined by

$$F^{\perp}(x) := \varphi_F(x) x, \tag{11.47}$$

where

$$\varphi_F(x) := \begin{cases} \frac{\langle F(x), x \rangle}{\|x\|^2} & \text{if } x \neq \theta, \\ 0 & \text{if } x = \theta. \end{cases} \tag{11.48}$$

Since $F(\theta) = \theta$, by assumption, the operator $F^{\perp}$ is also continuous and bounded. Moreover, if F is τ-homogeneous (see (7.29)), then $F^{\perp}$ is also τ-homogeneous. However, neither the linearity nor the compactness of F carry over to $F^{\perp}$, as the following simple example shows.

Example 11.8. In $H = l_2$, let $L \in \mathfrak{L}(H)$ be the projection on the first coordinate, i.e.,

$$L(x_1, x_2, x_3, \dots) := (x_1, 0, 0, \dots).$$

Then L is certainly compact, but

$$L^{\perp}(x_1, x_2, x_3, \dots) := \frac{|x_1|^2}{\|x\|^2}(x_1, x_2, x_3, \dots).$$

is not. In fact, $L^{\perp}$ maps the element $e_1 + e_k$, with $e_k = (\delta_{k,n})_n$ denoting the k-th basis sequence, into the element $(e_1 + e_k)/2$. ♡

Of course, the reason for the noncompactness of $L^{\perp}$ in Example 11.8 is that the identity is never compact in an infinite dimensional space. Consequently, the operator $F^{\perp}$ can be compact only if $\langle F(x), x\rangle \neq 0$ at most on a finite dimensional subspace of H.

For further reference we collect now some properties of the operator (11.47) in the following

Lemma 11.4. *For $F, G \in \mathfrak{C}(H)$ and $\lambda \in \mathbb{K}$, the following is true.*

(a) $F^{\perp\perp} = F^{\perp}$.

(b) $(F + G)^{\perp} = F^{\perp} + G^{\perp}$.

(c) $(\lambda F)^{\perp} = \lambda F^{\perp}$.

(d) $F^{\perp} = F$ *if and only if* $F(x) = \varphi(x)x$ *for some continuous function* $\varphi\colon H \to \mathbb{K}$ *with* $\varphi(\theta) = 0$.

(e) $[F^{\perp}]_{\mathrm{Q}} \leq [F]_{\mathrm{Q}}$.

(f) $[F^{\perp}]_{\mathrm{q}} \leq [F]_{\mathrm{q}}$.

Proof. The properties (a)–(d) are easy to verify, while the estimate

$$\frac{\|F^{\perp}(x)\|}{\|x\|} = |\varphi_F(x)| \leq \frac{\|F(x)\|}{\|x\|}$$

implies both (e) and (f). □

Following Furi, Martelli and Vignoli, we define a numerical range for $F \in \mathfrak{C}(H)$ by

$$W_{\mathrm{FMV}}(F) := \sigma_{\mathrm{q}}(F^{\perp}), \tag{11.49}$$

where σ_{q} denotes the spectral set (2.29). In other words, we have

$$W_{\mathrm{FMV}}(F) = \{\lambda \in \mathbb{K} : \varphi_F(x_n) \to \lambda \text{ for some unbounded sequence } (x_n)_n\}. \tag{11.50}$$

To illustrate this definition, let us check what (11.49) means for $L \in \mathfrak{L}(H)$. In this case $L^\perp$ is 1-homogeneous and we get

$$\begin{aligned} W_{\mathrm{FMV}}(L) &= \sigma_{\mathrm{q}}(L^\perp) \\ &= \left\{\lambda \in \mathbb{K} : \liminf_{\|x\|\to\infty} \frac{\|\lambda x - L^\perp(x)\|}{\|x\|} = 0\right\} \\ &= \left\{\lambda \in \mathbb{K} : \inf_{\|x\|=1} \|\lambda x - L^\perp(x)\| = 0\right\} \\ &= \left\{\lambda \in \mathbb{K} : \inf_{\|x\|=1} |\lambda - \langle Lx, x\rangle| = 0\right\} \\ &= \overline{W(L)}, \end{aligned}$$

with $W(L)$ given by (11.1). So, for linear operators we get, up to closure, the familiar definition of numerical range. Putting as before

$$w_{\mathrm{FMV}}(F) := \sup\{|\lambda| : \lambda \in W_{\mathrm{FMV}}(F)\}, \tag{11.51}$$

we have the following properties which are parallel to Proposition 11.4.

Proposition 11.5. *The numerical range* (11.49) *has the following properties* ($F, G\colon X \to X$ *continuous and bounded,* $\mu \in \mathbb{K}$):

(a) $W_{\mathrm{FMV}}(F + G) \subseteq W_{\mathrm{FMV}}(F) + W_{\mathrm{FMV}}(G)$ if $[F^\perp]_{\mathrm{Q}} < \infty$.

(b) $W_{\mathrm{FMV}}(\mu F) = \mu W_{\mathrm{FMV}}(F)$.

(c) $W_{\mathrm{FMV}}(F_z) = W_{\mathrm{FMV}}(F)$, *where* $F_z(x) = F(x) + z$.

(d) $W_{\mathrm{FMV}}(\mu I - F) = \{\mu\} - W_{\mathrm{FMV}}(F)$.

(e) $W_{\mathrm{FMV}}(F)$ *is bounded for* $F^\perp \in \mathfrak{Q}(X)$, *with* $w_{\mathrm{FMV}}(F) \le [F^\perp]_{\mathrm{Q}}$.

Proof. (a) Given $\lambda \in W_{\mathrm{FMV}}(F+G) = \sigma_{\mathrm{q}}(F^\perp + G^\perp)$, choose an unbounded sequence $(x_n)_n$ in H such that $\varphi_F(x_n) + \varphi_G(x_n) \to \lambda$ as $n \to \infty$. Since the sequence $(\varphi_F(x_n))_n$ is bounded, by our hypothesis $[F^\perp]_{\mathrm{Q}} < \infty$, we may assume that $\varphi_F(x_n) \to \mu \in \mathbb{K}$, passing to a subsequence if necessary. Consequently, $\varphi_G(x_n) \to \lambda - \mu$. By (11.50), this means precisely that $\mu \in W_{\mathrm{FMV}}(F)$ and $\lambda - \mu \in W_{\mathrm{FMV}}(G)$.

The assertion (b) follows from the equality $\varphi_{\mu F} = \mu\varphi_F$, while (c) follows from the equality

$$\varphi_{F_z}(x) = \frac{\langle F(x) + z, x\rangle}{\|x\|^2} = \varphi_F(x) + \frac{\langle z, x\rangle}{\|x\|^2}$$

and the fact that the last term tends to zero as $\|x\| \to \infty$. Finally, the equality $\varphi_{\mu I - F}(x) \equiv \mu - \varphi_F(x)$ implies (d), and (e) is a direct consequence of the definition of $W_{\mathrm{FMV}}(F)$ and (2.33). □

Observe that from Lemma 11.4 (b), (c) and (f) we may deduce that

$$[\lambda I - F^\perp]_{\mathrm{q}} = [(\lambda I - F)^\perp]_{\mathrm{q}} \le [\lambda I - F]_{\mathrm{q}},$$

and so

$$\sigma_q(F) \subseteq W_{FMV}(F)$$

which is analogous to (11.5) and (11.44). The following Theorem 11.6 gives some information on the topological structure of the numerical range (11.49).

Theorem 11.6. *Let H be a complex Hilbert space, and suppose that $F \in \mathfrak{C}(H)$ satisfies $[F^{\perp}]_Q < \infty$. Then the set $W_{FMV}(F)$ is nonempty, connected, and compact.*

Proof. We claim that

$$W_{FMV}(F) = \bigcap_{n \in \mathbb{N}} \overline{\varphi_F(H \setminus B_n(H))}. \tag{11.52}$$

Indeed, for $\lambda \in W_{FMV}(F)$ we find a sequence $(x_n)_n$ in H with $\|x_n\| > n$ and $\varphi_F(x_n) \to \lambda$ as $n \to \infty$. Now, if $\lambda \notin \overline{\varphi_F(H \setminus B_m(H))}$ for some $m \in \mathbb{N}$, we may find a neighbourhood U_λ of λ such that $U_\lambda \cap \varphi_F(H \setminus B_n(H)) = \emptyset$ for $n \geq m$. On the other hand, $\varphi_F(x_n) \in \varphi_F(H \setminus B_n(H))$, because $\|x_n\| > n$, and so $\varphi_F(x_n) \notin U_\lambda$, contradicting the fact that $\varphi_F(x_n) \to \lambda$.

Conversely, if λ belongs to the intersection in (11.52), we may find a sequence $(x_n)_n$ with $\|x_n\| > n$ and $\varphi_F(x_n) \to \lambda$. This implies that $\lambda \in \sigma_q(F^{\perp})$, and so (11.52) is proved.

Now, the assumption $[F^{\perp}]_Q < \infty$ implies that the set $\varphi_F(H \setminus B_n(H))$ is bounded for sufficiently large $n \in \mathbb{N}$. Moreover, $H \setminus B_n(H)$ is nonempty and connected, and so is $\varphi_F(H \setminus B_n(H))$. Thus, the assertion follows from the representation (11.52) and the fact that a decreasing sequence of nonempty, connected, and compact sets is also nonempty, connected, and compact. □

Let us compare now the numerical ranges (11.27) and (11.49). Since $W_Z(L) = W(L)$ for $L \in \mathfrak{L}(H)$, but $W_{FMV}(L) = \overline{W(L)}$, we have the strict inclusion $W_Z(L) \subset W_{FMV}(L)$ for the operator L from Example 11.2, and so $W_{FMV}(F)$ need not be contained in $W_Z(F)$. The following example shows that $W_Z(F)$ need not be contained in $W_{FMV}(F)$ either.

Example 11.9. Let $F : \mathbb{R} \to \mathbb{R}$ be defined by

$$F(x) := \begin{cases} |x| & \text{if } |x| \leq 1, \\ \sqrt{|x|} & \text{if } |x| > 1. \end{cases}$$

Since F is Lipschitz continuous with $F(0) = 0$, both $W_Z(F)$ and $W_{FMV}(F)$ are defined. Clearly,

$$\varphi_F(x) = \begin{cases} \operatorname{sign} x & \text{if } |x| \leq 1, \\ \frac{\operatorname{sign} x}{\sqrt{|x|}} & \text{if } |x| > 1 \end{cases}$$

in this case, and so

$$W_{\text{FMV}}(F) = \left\{ \lim_{|x|\to\infty} \frac{\operatorname{sign} x}{\sqrt{|x|}} \right\} = \{0\}.$$

On the other hand, the set

$$W_{\text{Z}}(F) = \left\{ \frac{(F(x) - F(y))(x - y)}{|x - y|^2} : x \neq y \right\} = [-1, 1]$$

is strictly larger than $W_{\text{FMV}}(F)$. ♡

Let us briefly illustrate an application of the numerical range (11.49) to complex systems of differential equations. In the following proposition we denote by $\mathbb{C}_-$ the left half-plane of all $z \in \mathbb{C}$ with $\operatorname{Re} z < 0$.

Proposition 11.6. *Let $F\colon \mathbb{C}^n \to \mathbb{C}^n$ be continuous with $[F^\perp]_{\text{Q}} < \infty$. Assume that there exists a linear isomorphism $L\colon \mathbb{C}^n \to \mathbb{C}^n$ such that $W_{\text{FMV}}(LFL^{-1}) \subseteq \mathbb{C}_-$. Then all solutions $z = z(t)$ of the autonomous system of differential equations*

$$\dot{z} = F(z) \tag{11.53}$$

are bounded as $t \to \infty$.

Proof. Putting $w := Lz$, the system (11.53) is transformed into the system

$$\dot{w} = LFL^{-1}(w) =: G(w). \tag{11.54}$$

Therefore it suffices to show that all solutions $w = w(t)$ of (11.54) remain bounded as $t \to \infty$. Since the numerical range $W_{\text{FMV}}(G)$ is compact, by Theorem 11.6, and contained in $\mathbb{C}_-$, by assumption, we find a $\beta > 0$ such that $\operatorname{Re}\lambda < -\beta$ for all $\lambda \in W_{\text{FMV}}(G)$. On the other hand, from the definition of the numerical range $W_{\text{FMV}}(G) = \sigma_{\text{q}}(G^\perp)$ it follows that there exists $R > 0$ such that

$$\operatorname{Re}\langle G(w), w\rangle < -\beta\|w\|^2 \quad (\|w\| \geq R).$$

Now, for any solution $w = w(t)$ of (11.54) we have

$$\frac{d}{dt}\|w(t)\|^2 = \langle \dot{w}(t), w(t)\rangle + \langle w(t), \dot{w}(t)\rangle = 2\operatorname{Re}\langle G(w)(t), w(t)\rangle.$$

The last term in this equality is negative if $\|w(t)\| \geq R$, and this certainly implies that $w = w(t)$ is bounded. □

For the remaining part of this section, we discuss a certain analogue to the numerical range (11.49) for Banach spaces. Let X be a Banach space with duality map (11.14), and consider the multivalued map $\varphi_F\colon X \setminus \{\theta\} \to 2^{\mathbb{K}}$ defined by

$$\varphi_F(x) := \left\{ \frac{\langle F(x), \ell_x\rangle}{\|x\|^2} : \ell_x \in \mathcal{D}(x) \right\}. \tag{11.55}$$

The *numerical range in the sense of Conti and De Pascale* is then defined by

$$W_{\mathrm{CD}}(F) := \bigcap_{r>0} \overline{\varphi_F(B_r(X) \setminus \{\theta\})} \tag{11.56}$$

In other words, we have $\lambda \in W_{\mathrm{CD}}(F)$ if and only if there exist sequences $(x_n)_n$ in $X \setminus \{\theta\}$ and $(\ell_n)_n$ in X^* such that $x_n \to \theta$, $\ell_n \in \mathcal{D}(x_n)$, and

$$\frac{\langle F(x_n), \ell_n \rangle}{\|x_n\|^2} \to \lambda \quad (n \to \infty).$$

If $X = H$ is a Hilbert space, we have of course $\langle F(x), \ell_x \rangle = \langle F(x), x \rangle$, and so the Conti–De Pascale numerical range (11.56) may be considered as a "local analogue" to the "asymptotic" Furi–Martelli–Vignoli numerical range (11.50).

Theorem 11.7. *Let X be a real Banach space with* $\dim X \geq 2$ *and $F: X \to X$ be continuous and bounded. Then the set $W_{\mathrm{CD}}(F)$ is nonempty, connected and compact.*

Proof. By construction, the closure of the set $\varphi_F(B_r(X) \setminus \{\theta\})$ is a closed interval for all $r > 0$. On the other hand, the boundedness of F implies that the map (11.55) is bounded on $X \setminus \{\theta\}$. So it remains to observe that the numerical range (11.56) is representable as a decreasing family of nonempty compact intervals. □

It seems somewhat strange that Theorem 11.6 is formulated in complex Hilbert spaces, while Theorem 11.7 is formulated in real Banach spaces. The following example shows that, surprisingly, Theorem 11.7 is in fact false even for $X = \mathbb{C}$.

Example 11.10. Let $F: \mathbb{C} \to \mathbb{C}$ be defined by

$$F(z) = F(x + iy) := \begin{cases} 0 & \text{if } z = 0, \\ \frac{z^2}{|z|^{3/2}} & \text{if } |y| \leq |z|^{3/2}, \\ \frac{x^2}{|z|^{3/2}} - |y| + ix\left(\operatorname{sign} y + \frac{y}{|z|^{3/2}}\right) & \text{if } |y| > |z|^{3/2}. \end{cases}$$

Then F is continuous on the whole complex plane, and

$$\varphi_F(z) = \varphi_F(x + iy) = \begin{cases} 0 & \text{if } z = 0, \\ \frac{z}{|z|^{1/2}} & \text{if } |y| \leq |z|^{3/2}, \\ \frac{x}{|z|^{3/2}} + i \ \operatorname{sign} y & \text{if } |y| > |z|^{3/2}. \end{cases}$$

By considering sequences $z_n := iy_n$ one sees that $\pm i \in W_{\mathrm{CD}}(F)$. More precisely, one can show that $W_{\mathrm{CD}}(F)$ consists precisely of the two horizontal lines $\operatorname{Im} z = \pm 1$, and so it is neither compact nor connected. ♡

11.4 Numerical ranges and Jordan domains

The question arises whether or not (11.43) may be sharpened for other operators than those stated in Theorem 11.5. To study this problem, we need a further definition.

Recall that a closed set $J \subset \mathbb{C}$ is called *Jordan domain* if $\mathbb{C} \setminus J$ is connected. Observe that an analogous definition in $\mathbb{R}$ is not interesting, since a bounded nonempty subset of $\mathbb{R}$ has never a connected complement.

Given a bounded set $M \subset \mathbb{C}$, we call the set

$$\operatorname{jo} M := \bigcap \{J : M \subseteq J \subset \mathbb{C},\ J \text{ closed Jordan domain}\} \tag{11.57}$$

the *Jordan hull* of M.

Lemma 11.5. *The Jordan hull* (11.57) *has the following properties*:

(a) $M = \operatorname{jo} M$ *implies that* M *is a Jordan domain.*

(b) $M \subseteq N$ *implies that* $\operatorname{jo} M \subseteq \operatorname{jo} N$.

(c) $\overline{M} \subseteq \operatorname{jo} M \subseteq \overline{\operatorname{co}} M$.

Proof. The assertions (a) and (b) are immediate. The first inclusion in (c) follows from the fact that we only consider closed Jordan domains in (11.57), while the second inclusion is a consequence of the fact that every compact convex subset of the complex plane is a Jordan domain. □

It is easy to find examples for strict inclusion in (c). Thus, the following result is, at least theoretically, an improvement of Theorem 11.4.

Theorem 11.8. *Let X be a Banach space and $F \in \mathfrak{Lip}(X)$. Then*

$$\sigma_{\mathrm{K}}(F) \subseteq \operatorname{jo} W_{\mathrm{MR}}(F). \tag{11.58}$$

Proof. Suppose that there exists a $\lambda \in \sigma_{\mathrm{K}}(F) \setminus \operatorname{jo} W_{\mathrm{MR}}(F)$. Fix $\mu \in \mathbb{C} \setminus [\sigma_{\mathrm{K}}(F) \cup \operatorname{jo} W_{\mathrm{MR}}(F)]$. Being connected and open, the set $\mathbb{C} \setminus \operatorname{jo} W_{\mathrm{MR}}(F)$ is pathwise connected, and hence we may find a continuous map $\gamma : [0, 1] \to \mathbb{C} \setminus \operatorname{jo} W_{\mathrm{MR}}(F)$ such that $\gamma(0) = \lambda$ and $\gamma(1) = \mu$. But since $\lambda \in \sigma_{\mathrm{K}}(F)$ and $\mu \notin \sigma_{\mathrm{K}}(F)$, there is some $\tau \in (0, 1)$ such that $\nu := \gamma(\tau) \in \partial\sigma_{\mathrm{K}}(F)$. From (5.13) and (11.44) it follows that $\nu \in \overline{W_{\mathrm{MR}}(F)} \subseteq \operatorname{jo} W_{\mathrm{MR}}(F)$, a contradiction. □

We conclude this section with a series of further examples which we already considered before.

Example 11.11. Let $X = \mathbb{C}^2$ and $F(z, w) = (\overline{w}, -\overline{z})$. Then $\sigma_{\mathrm{K}}(F) = \emptyset$, since the operator

$$(\lambda I - F)^{-1}(z, w) = \left(\frac{\overline{\lambda} z + \overline{w}}{|\lambda|^2 + 1}, \frac{\overline{\lambda} w - \overline{z}}{|\lambda|^2 + 1} \right)$$

is a lipeomorphism for each $\lambda \in \mathbb{C}$. On the other hand,

$$W_Z(F) = W_{MR}(F) = W_{FMV}(F) = W_{CD}(F) = \{0\},$$

as one sees by a direct calculation. Note that $\langle F(z, w), (z, w)\rangle \equiv 0$ in this example. ♡

Example 11.12. Let $X = \mathbb{C}^2$ and $F(z, w) = (\overline{w}, i\overline{z})$. Then $\sigma_K(F) = \emptyset$, as we have seen in Example 3.18. On the other hand,

$$\begin{aligned} W_Z(F) &= W_{MR}(F) = W_{FMV}(F) = W_{CD}(F) \\ &= \{(1 + i)\overline{zw} : |z|^2 + |w|^2 = 1\} \\ &= \{\lambda \in \mathbb{C} : |\lambda| \leq 1/\sqrt{2}\} \end{aligned}$$

again by a direct calculation. ♡

Example 11.13. Let H be an infinite dimensional Hilbert space, $e \in S(H)$, and $F(x) := \|x\|e$. Then

$$\sigma_K(F) = \overline{\mathbb{D}}. \tag{11.59}$$

To see this, observe that the operator $\lambda I - F$ is not onto for $|\lambda| < 1$. In fact, fix $z \perp e$, and suppose that the equation $\lambda x - F(x) = z$ has a solution $\hat{x}$. Scalar multiplication by $\langle \cdot, e\rangle$ yields then $\lambda\langle \hat{x}, e\rangle - \|\hat{x}\| = 0$, and so

$$\|\hat{x}\| = |\lambda|\, |\langle \hat{x}, e\rangle| \leq |\lambda|\, \|\hat{x}\|,$$

a contradiction. So $\mathbb{D} \subseteq \sigma_K(F)$; since $[F]_{Lip} \leq 1$ and $\sigma_K(F)$ is closed, we have proved equality (11.59).

Moreover, from Theorem 11.5 (c) or (d) we know that $\overline{W_Z(F)} = \overline{W_{MR}(F)} \supseteq \overline{\mathbb{D}}$. But one may show easily that

$$W_Z(F) = W_{MR}(F) = \left\{ \frac{\|x\| - \|y\|}{\|x - y\|^2}\langle e, x - y\rangle : x \neq y \right\} = \overline{\mathbb{D}},$$

$$W_{FMV}(F) = \left\{ \lambda \in \mathbb{C} : \frac{\langle e, x_n\rangle}{\|x_n\|} \to \lambda \text{ for some } \|x_n\| \to \infty \right\} = \overline{\mathbb{D}},$$

and

$$W_{CD}(F) = \left\{ \lambda \in \mathbb{C} : \frac{\langle e, x_n\rangle}{\|x_n\|} \to \lambda \text{ for some } \|x_n\| \to 0 \right\} = \overline{\mathbb{D}},$$

and so all numerical ranges coincide. ♡

11.5 Notes, remarks and references

Numerical ranges and radii for linear operators have applications in matrix theory, numerical analysis, operator theory, system theory and even quantum mechanics. There are not only useful for "localizing" the spectrum, as mentioned above, but also for studying the convergence of iterative processes (see, e.g., [104] for a recent example of such an application). Now there exist many definitions of numerical ranges for linear and nonlinear operators; we try to describe some of them in what follows.

As remarked in Section 11.1, the first definition (11.1) of a numerical range for bounded linear operators in Hilbert spaces is due to Toeplitz [251]. We intentionally have confined ourselves to *complex* Hilbert spaces in Section 11.1, because in the real case many of the elementary properties of the numerical range are not true. Moreover, the structure of the numerical range $W(L)$ is rather poor in real Hilbert spaces: it is always an interval whose endpoints are $\inf\{\langle Lx, x\rangle : \|x\| = 1\}$ and $\sup\{\langle Lx, x\rangle : \|x\| = 1\}$.

The properties of $W(L)$ stated in Proposition 11.1 (f), (h), (j) and (k) have been proved in [251], [80], [243], and [190], respectively, the other properties are rather elementary. The properties of the numerical radius $w(L)$ given in Proposition 11.2 may be found in [212], together with an elementary proof of the algebraic identity which we used for the assertion (f). The paper [212] also contains the following interesting counterexample to the natural conjecture $w(LM) \le w(L)w(M)$.

Example 11.14. Let $L\colon \mathbb{R}^4 \to \mathbb{R}^4$ be defined by

$$L(x, y, u, v) := (y + u + v, u + v, v, 0).$$

Then L, L^2, and L^3 correspond to the (nilpotent) matrices

$$L = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad L^2 = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad L^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

A straightforward calculation shows that $w(L) < 1$ and $w(L^2) = w(L^3) = 1/2$. Consequently, putting $M = L^2$ we have $w(LM) > w(L)w(M)$. ♡

From Proposition 11.2 it follows that $w(L^n) = w(L)^n$ if and only if $w(L) = r(L)$. Clearly, the operator L in Example 11.14 has spectral radius zero.

We remark that Németh [201] defines some kind of Dini derivative for nonlinear operators in the Euclidean space $\mathbb{R}^n$ which seems to be related to Toeplitz' numerical range. Given a continuous operator $F\colon \mathbb{R}^n \to \mathbb{R}^n$, the limits

$$F^{\wedge}(x) := \limsup_{y \to x} \frac{\langle F(x) - F(y), x - y\rangle}{\|x - y\|^2}$$

and

$$F^{\vee}(x) := \liminf_{y \to x} \frac{\langle F(x) - F(y), x - y \rangle}{\|x - y\|^2}$$

are called the *upper and lower scalar derivative*, respectively, of F at x. This derivative allows one to get information on the “monotonicity behaviour” of F on convex subsets of $\mathbb{R}^n$. Moreover, if F is Fréchet differentiable at x with derivative $L := F'(x)$, then

$$F^{\wedge}(x) = \sup W(L) = \max\{|\lambda| : \lambda \in \sigma(\tfrac{1}{2}(L + L^*))\}$$

and

$$F^{\vee}(x) = \inf W(L) = \min\{|\lambda| : \lambda \in \sigma(\tfrac{1}{2}(L + L^*))\}.$$

There are several books and monographs on numerical ranges, mainly for bounded linear operators in Hilbert or Banach spaces. Among them we mention the classical books [45], [46], [146] and the more recent monograph [143].

Toeplitz’ definition was extended to the Banach space case by Bauer [34], again for bounded linear operators, by means of the duality map (11.14). We remark that in the thesis [82], from which we have taken some results and examples in this chapter, the duality map of a Banach space X is defined by

$$\hat{\mathfrak{D}}(x) := \{\ell_x \in S(X^*) : \langle x, \ell_x \rangle = \|x\|\} \quad (x \neq \theta). \tag{11.60}$$

By the classical Hahn–Banach theorem, the set $\hat{\mathfrak{D}}(x)$ is nonempty for every $x \in X$. The usual duality map (11.14) is of course related to the map (11.60) by

$$\hat{\mathfrak{D}}(x) = \frac{\mathfrak{D}(x)}{\|x\|} \quad (x \neq \theta).$$

A Banach space X is smooth if and only if its norm is Gâteaux differentiable on $X \setminus \{\theta\}$; this is sometimes taken as definition of smoothness. For example, the spaces l_p and L_p are smooth for $1 < p < \infty$, but not smooth for $p = 1$ or $p = \infty$. Similarly, the spaces c of convergent sequences and C of continuous functions are also non-smooth. A useful sufficient condition states that a space X is smooth if its dual X^* is strictly convex. This and much more information on continuity properties of duality maps may be found in the monograph [77].

Semi-inner products on linear spaces have been introduced and studied by Lumer [179]; in our presentation we followed Martin’s book [184]. Lumer’s original definition (11.23) of the numerical range $W_{\mathrm{L}}(L; [\cdot\,,\,\cdot])$, which was announced in [180], depends on $[\cdot\,,\,\cdot] \in \mathrm{sip}(X)$. However, in [179] Lumer proved the remarkable formula

$$\sup\{\mathrm{Re}\,\lambda : \lambda \in W_{\mathrm{L}}(L; [\cdot\,,\,\cdot])\} = M[L],$$

where $M[L]$ denotes the logarithmic norm (5.45) of L in the sense of Dahlquist [69]. From this formula it follows in particular that the set $\overline{\mathrm{co}}\, W_{\mathrm{L}}(L; [\cdot\,,\,\cdot])$ is actually independent of the choice of a semi-inner product $[\cdot\,,\,\cdot]$ on X. The numerical range in the sense of our definition (11.24), i.e.,

$$W_{\rm L}(L) = \bigcup_{[\cdot\,,\,\cdot]\in {\rm sip}(X)} W_{\rm L}(L;[\cdot\,,\,\cdot])$$

was apparently considered first by Williams [276]; he also proved that

$$\sigma(L) \subseteq \overline{W_{\rm L}(L;[\cdot\,,\,\cdot])} \tag{11.61}$$

for all $[\cdot\,,\,\cdot] \in {\rm sip}(X)$, which of course implies our Theorem 11.1. Zenger [288] proved independently the inclusion

$$\operatorname{co}\sigma(L) \subseteq \overline{W_{\rm B}(L)} \tag{11.62}$$

for Bauer's numerical range (11.17); this is analogous to the inclusion (11.2) for Toeplitz' numerical range. However, in contrast to the numerical range $W(L)$ in Hilbert spaces, Bauer's and Lumer's numerical ranges need not be convex, as we have seen in Example 11.5 which is Example 1 in Section 11 of the book [45]. An analogous example in the space X_∞ (see Example 11.6) was given earlier by Nirschl and Schneider [207]. On the other hand, it was shown in [44] that the numerical range $W_{\rm BL}(L)$ is always connected. Example 11.6 which shows that this is not true for the numerical range (11.23) is also taken from [44]. The fundamental Theorem 11.1 is Theorem III.7.2 in the book [184].

The first one who defined a numerical range for nonlinear operators in Hilbert spaces seems to be Zarantonello [282], this explains the subscript in our definition (11.27). He also announced and proved Theorem 11.1 in [282], [283]. In these papers one may also find a certain "local" version of (11.27), namely

$$W_{\rm Z}^{\rm loc}(F) := \bigcap_{r>0} \overline{W_{\rm Z}(F;r)}, \tag{11.63}$$

where

$$W_{\rm Z}(F;r) := \left\{ \frac{\langle F(x)-F(y), x-y\rangle}{\|x-y\|^2} : 0 < \|x-y\| \leq r \right\}.$$

This local numerical range seems to be of interest for some nonlinear problems and was considered first, apparently, by Minty. As definition (11.56) shows, the numerical range $W_{\rm CD}(F)$ introduced by Conti and De Pascale is also of local nature. Lemma 11.1 and Theorem 11.2 are taken from [282].

It is difficult to say who was the first to define a numerical range for nonlinear operators in Banach spaces through the duality map (11.14); our definition (11.32) is close to that given by Rhodius [223] for locally convex spaces. The definition (11.38) which builds on semi-inner products is taken from the book [184]. We remark that Verma [264]–[267] replaces (11.38) by

$$W_{\rm V}(F;[\cdot\,,\,\cdot]) := \left\{ \frac{[Fx,x]+[F(y)-F(z),y-z]}{\|x\|^2+\|y-z\|^2} : x \neq \theta,\ y \neq z \right\} \tag{11.64}$$

for a *fixed* $[\cdot,\cdot] \in \mathrm{sip}(X)$ and repeats many results from the book [184] for (11.64). For instance, he proves in [266] that

$$\sigma_{\mathrm{K}}(F) \subseteq \overline{\mathrm{co}}\, W_{\mathrm{V}}(F; [\cdot,\cdot]) \tag{11.65}$$

for every $[\cdot,\cdot] \in \mathrm{sip}(X)$. Our Lemmas 11.2 and 11.3, as well as Theorem 11.4, may be found in [184]. Theorem 11.5 gives various sufficient conditions for the inclusion (11.43) which is apparently stronger than (11.42). In particular, condition (e) was considered by Edmunds in [101] in case of reflexive complex spaces. The general case is Proposition IV.1.4 in the book [184]; our proof is based on Theorem 4.2. Theorem 11.5 (d) and Example 11.7 are taken from Dörfner's thesis [82].

One of the earliest papers on numerical ranges for nonlinear operators on spheres in Banach spaces is [44]. The authors of [44] first introduce another duality map on $S(X)$ by

$$\mathcal{B}(e) := \{\ell \in S(X^*) : \langle e, \ell\rangle = 1\} \quad (e \in S(X)). \tag{11.66}$$

Given a continuous operator $F\colon S(X) \to X$, they define then a numerical range by

$$W_{\mathrm{BCS}}(F) := \bigcup_{e \in S(X)} \{\langle F(e), \ell\rangle : \ell \in \mathcal{B}(X)\}. \tag{11.67}$$

This numerical range is related to Feng's numerical range (11.35) in the following way. If $F_1\colon X \to X$ denotes the 1-homogeneous extension (6.7) of $F\colon S(X) \to X$, i.e.,

$$F_1(x) = \begin{cases} \|x\| F\big(\frac{x}{\|x\|}\big) & \text{if } x \neq \theta, \\ \theta & \text{if } x = \theta, \end{cases}$$

then $\langle F_1(x), \ell\rangle = \|x\| \langle F(x/\|x\|), \ell\rangle$, hence

$$\mathcal{B}\big(\tfrac{x}{\|x\|}\big) = \{\ell \in S(X^*) : \langle x, \ell\rangle = \|x\|\} = \mathcal{D}(x)\|x\|.$$

Consequently,

$$\begin{aligned} W_{\mathrm{F}}(F_1) &= \bigcup_{x \neq \theta} \left\{ \frac{\langle F_1(x), \ell\rangle}{\|x\|} : \ell \in \mathcal{D}(x)\|x\| \right\} \\ &= \bigcup_{x \neq \theta} \left\{ \big\langle F\big(\tfrac{x}{\|x\|}\big), \ell \big\rangle : \ell \in \mathcal{B}\big(\tfrac{x}{\|x\|}\big) \right\} \\ &= \bigcup_{e \in S(X)} \{\langle F(e), \ell\rangle : \ell \in \mathcal{B}(e)\} = W_{\mathrm{BCS}}(F). \end{aligned} \tag{11.68}$$

The numerical range (11.50) was introduced in the Hilbert space case by Furi, Martelli and Vignoli in [122], where one may also find Theorem 11.6 and the application to differential equations which we stated as Proposition 11.6. It is well known that all solutions $z = z(t)$ of the linear differential equation

$$\dot{z} = Az, \quad (A \in \mathfrak{L}(\mathbb{C}^n))$$

are bounded as $t \to \infty$ if $\sigma(A) \subseteq \mathbb{C}_-$. However, this condition does not imply that $W(A) \subseteq \mathbb{C}_-$. On the other hand, the following useful result is proved in [122]: for any open set $U \supseteq \overline{\text{co}}\,\sigma(A)$ there exists an isomorphism $L \in \mathfrak{L}(\mathbb{C}^n)$ such that $W(LAL^{-1}) \subseteq U$.

The extension (11.56) of the numerical range of Furi, Martelli and Vignoli was proposed by Conti and De Pascale [66], even for multivalued operators F. The relation

$$W_{\text{CD}}(F_1) = W_{\text{BCS}}(F) \tag{11.69}$$

for an operator $F\colon S(X) \to X$ and its 1-homogeneous extension F_1 is analogous to (11.68).

We may summarize our discussion on numerical ranges in the hierarchy contained in the following table.

Table 11.1

	F linear	F nonlinear
X Hilbert space	Toeplitz (1918)	Zarantonello (1964)
X Hilbert space (asymptotic range)		Furi–Martelli–Vignoli (1978)
X Banach space (via duality map)	Bauer (1962)	Rhodius (1976) Dörfner (1996) Feng (1997)
X Banach space (via semi-inner products)	Lumer (1961)	Martin (1976) Verma (1991)
X Banach space (local range)		Conti–De Pascale (1978)

Moreover, the following tables (for linear operators) and (for nonlinear operators) contain the most important topological properties of the numerical ranges discussed so far.

Table 11.2

	closed	convex	connected
$W(L)$	no (a)	yes	yes
$W_{\text{L}}(L; [\,\cdot\,,\,\cdot\,])$	no (a)	no (b)	no (c)
$W_{\text{BL}}(L)$	no (a)	no (b)	yes (d)

Table 11.3

	closed	connected
$W_Z(F)$	no (a)	yes
$W_{MR}(F)$	no (a)	yes (d)
$W_F(F)$	no (a)	no
$W_V(F; [\cdot, \cdot])$	no (a)	
$W_{BCS}(F)$	no (a)	yes (d)
$W_{FMV}(F)$	yes (f)	yes (f)
$W_{CD}(F)$	yes (d)/(e)	yes (d)/(e)

(a) see Example 11.2; (b) see Example 11.5; (c) see Example 11.6;
(d) unless $X = \mathbb{R}$; (e) if $\mathbb{K} = \mathbb{R}$; (f) if $\mathbb{K} = \mathbb{C}$.

All definitions and results in Section 11.4 are taken from Dörfner's paper [81], see also [10]. To see that Theorem 11.8 is a proper extension of Theorem 11.4, one should find a Banach space X and an operator $F: X \to X$ such that

$$\overline{W_{MR}(F)} \subset \sigma_K(F) = \text{jo}\, W_{MR}(F) \subset \overline{\text{co}}\, W_{MR}(F). \tag{11.70}$$

By Theorem 11.5, in such an example X cannot be Hilbert, real, or finite dimensional, and F cannot be differentiable or compact. Moreover, (11.44) implies that the strict inclusion $\sigma_{\text{lip}}(F) \subset \sigma_K(F)$ must hold for such an operator. We do not know of any such example, and probably this is a hard problem.

On the other hand, we do not know either if (11.43) is true for every $F \in \mathfrak{Lip}(X)$ in a smooth Banach space X. Dörfner has shown in [82] that the set of all operators $F \in \mathfrak{Lip}_0(X)$ satisfying (11.43) is a *closed subset* of $\mathfrak{Lip}_0(X)$. It is not clear, however, whether or not this set is also open in $\mathfrak{Lip}_0(X)$, and hence coincides with $\mathfrak{Lip}_0(X)$.

Numerical ranges have been generalized in various directions. For example, Williams [276] has proved the following theorem for "pairs" of operators.

Theorem 11.9. *Let H be a complex Hilbert space, $F: H \to H$ bounded and continuous, and $G \in \mathfrak{Lip}(H)$ with $0 \notin \overline{W_Z(G)}$. Suppose that $\lambda \in \mathbb{C}$ satisfies*

$$\lambda \notin \left\{ \frac{\mu}{\nu} : \mu \in \overline{W_Z(F)}, \nu \in \overline{W_Z(G)} \right\}.$$

Then the operator $\lambda I - FG^{-1}: H \to H$ is a lipeomorphism.

Theorem 11.9 implies that

$$\sigma_K(FG^{-1}) \subseteq \left\{ \frac{\mu}{\nu} : \mu \in \overline{W_Z(F)}, \nu \in \overline{W_Z(G)} \right\}.$$

Of course, putting $G = I$ in Theorem 11.9 we get for $F \in \mathfrak{Lip}(H)$ again the inclusion (11.28) which states that $\lambda I - F$ is a lipeomorphism for $\lambda \notin \overline{W_Z(F)}$.

As soon as one defines numerical ranges in a Banach space X through duality maps, one has to take into account the dual space X^*. This was a motivation for Canavati [59] to introduce a very general numerical range for operators $F\colon X \times X^* \to X$ in the spirit of [122] and [66]. To this end, Canavati replaces the auxiliary maps (11.48) and (11.55) by the map $\varphi_F\colon X \times X^* \to \mathbb{K}$ defined by

$$\varphi_F(x, \ell) := \frac{\ell(F(x), \ell)}{\|x\| \, \|\ell\|} \quad (x \in X,\ \ell \in X^*).$$

Finally, we mention that Pietschmann and Rhodius [220], [223]–[226] have defined and studied numerical ranges also in locally convex spaces; we briefly describe their construction. Let E be a locally convex space with a generating family $\mathcal{P}$ of seminorms $p\colon E \to \mathbb{K}$, and let $F\colon E \to E$ be continuous and bounded. For each $p \in \mathcal{P}$, denote by $N(p)$ the set of all $x \in E$ such that $p(x) = 0$, and consider the duality map

$$\mathcal{D}_p(x) := \{\ell_x \in E^* : \langle x, \ell_x\rangle = 1,\ |\langle y, \ell_x\rangle| \leq p(y)/p(x)\ (y \in E)\} \qquad (11.71)$$

for $x \in E \setminus N(p)$. Using this duality map, Rhodius defines a p-dependent numerical range by

$$W_{R,p}(F) := \bigcup_{p(x-y)\neq 0} \{\langle F(x) - F(y), \ell_{x-y}\rangle : \ell_{x-y} \in \mathcal{D}_p(x-y)\}. \qquad (11.72)$$

Of course, in a Banach space X with $p(x) = \|x\|$ the map (11.71) simply becomes

$$\mathcal{D}_{\|\cdot\|}(e) := \{\ell \in S(X^*) : \langle e, \ell\rangle = 1\} = \mathcal{B}(e) \quad (e \in S(X)),$$

which connects (11.72) to the numerical range (11.67) of Bonsall–Cain–Schneider. Interestingly, the numerical range (11.72) is related to the Rhodius spectrum $\sigma_{\mathrm{R}}(F)$ for continuous operators F which we discussed in detail in Chapter 4. In fact, Rhodius [225] proves that

$$\sigma_{\mathrm{R}}(F) \setminus \sigma_{\mathrm{R}}(F|_{N(p)}) \subseteq \overline{W_{R,p}(F)} \quad (p \in \mathcal{P}),$$

where $N(p) = \{x \in E : p(x) = 0\}$ as above. Finally, we remark that applications of numerical ranges to the geometry of Banach and locally convex spaces may be found in [220].

Chapter 12
Some Applications

In this chapter we discuss some selected applications of various nonlinear spectra, with a particular emphasis on the Furi–Martelli–Vignoli spectrum and the Feng spectrum. In the first two sections we consider general solvability results for equations involving a single nonlinear operator, or a "semilinear pair" of operators. Afterwards we illustrate the applicability of such results to boundary value problems. Bifurcation and asymptotic bifurcation points of nonlinear operators are treated in the following section. Finally, we show how the spectral theory for homogeneous nonlinear operators from Section 9.6 can be applied to derive a nonlinear Fredholm type alternative which in turn gives an existence and perturbation theorem for the eigenvalue problem of a nonlinear elliptic equation involving the p-Laplace operator.

12.1 Solvability of nonlinear equations

As we have seen in Chapter 2, the spectral sets (2.28)–(2.31) have a natural meaning in the linear case. Thus, the three sets $\sigma_{\mathrm{lip}}(L)$, $\sigma_{\mathrm{b}}(L)$ and $\sigma_{\mathrm{q}}(L)$ all coincide with the approximate point spectrum (1.50), while the set $\sigma_{\mathrm{a}}(L)$ coincides with the "left Fredholm spectrum" from (1.65). For $L \in \mathfrak{L}(X)$, these subspectra give precise information on the solvability of the linear equation

$$\lambda x - Lx = y \quad (y \in X). \tag{12.1}$$

So one could ask to what extent the spectral sets (2.28)–(2.31) also provide information on the solvability of the nonlinear equation

$$\lambda x - F(x) = y \quad (y \in X). \tag{12.2}$$

We already know that $\lambda \in \sigma_{\mathrm{lip}}(F)$ implies that the operator $\lambda I - F$ is injective (Proposition 2.1 (a)), and hence that the equation (12.2) has at most one solution for fixed y. On the other hand, none of the relations $\lambda \in \sigma_{\mathrm{lip}}(F)$, $\lambda \in \sigma_{\mathrm{b}}(F)$, or $\lambda \in \sigma_{\mathrm{q}}(F)$ implies the surjectivity of $\lambda I - F$ even in the linear case, as is shown by the right-shift operator (1.38). Nevertheless, we may give a positive result for all scalars which are "far enough" from each of these spectral sets. As before, for a nonempty compact subset $\Sigma \subseteq \mathbb{C}$ we denote by $c_\infty[\Sigma]$ the unbounded connected component of $\mathbb{C} \setminus \Sigma$.

Theorem 12.1. *Suppose that $F \in \mathfrak{A}(X)$ and put $[F]_{\mathrm{A}} =: \rho$. Then equation* (12.2) *has a solution for all $y \in X$ provided that one of the following* 6 *conditions on F and λ is satisfied*:

(a) $F \in \mathfrak{Q}(X)$ *and* $\lambda \in c_\infty[\sigma_q(F) \cup \mathbb{D}_\rho]$.

(b) $F \in \mathfrak{K}(X) \cap \mathfrak{Q}(X)$ *and* $\lambda \in c_\infty[\sigma_q(F) \cup \{0\}]$.

(c) $F \in \mathfrak{B}(X)$ *and* $\lambda \in c_\infty[\sigma_b(F) \cup \mathbb{D}_\rho]$.

(d) $F \in \mathfrak{K}(X) \cap \mathfrak{B}(X)$ *and* $\lambda \in c_\infty[\sigma_b(F) \cup \{0\}]$.

(e) $F \in \mathfrak{Lip}(X)$ *and* $\lambda \in c_\infty[\sigma_{lip}(F) \cup \mathbb{D}_\rho]$.

(f) $F \in \mathfrak{K}(X) \cap \mathfrak{Lip}(X)$ *and* $\lambda \in c_\infty[\sigma_{lip}(F) \cup \{0\}]$.

Proof. The statement (a) has already been proved by degree-theoretical methods in Theorem 3.7, and (b) is of course a special case of (a). The statements (c)–(f) are proved similarly, taking into account that $\sigma_q(F)$ is the smallest of the three subspectra (2.28)–(2.30). □

As we have seen in Chapter 9, to enlarge the applicability of spectral theory one has to replace the identity in (12.2) by some "well-behaved" nonlinear operator J. Consequently, we consider now the more general problem

$$\lambda J(x) - F(x) = y \quad (y \in Y), \tag{12.3}$$

where F and J are continuous nonlinear operators between two Banach spaces X and Y satisfying some additional conditions to be specified below.

Theorem 12.2. *Suppose that* $F \in \mathfrak{A}(X, Y)$, *and* $J : X \to Y$ *satisfies* $\nu(J) > 0$. *Fix* $\lambda \in \mathbb{K}$ *with* $|\lambda|\nu(J) > [F]_A$, *and let*

$$S = \{x \in X : \lambda J(x) = tF(x) \text{ for some } t \in (0, 1]\}. \tag{12.4}$$

Then either S is unbounded, or the operator $\lambda J - F$ is k-epi on $\overline{\Omega}$ for some $\Omega \in \mathfrak{OBC}(X)$ *and every* $k \leq |\lambda|\nu(J) - [F]_A$.

Proof. We apply Property 7.4 of k-epi operators to the operator $F_0 := \lambda J$ and the homotopy $H(x, t) := -tF(x)$. For $M \subseteq \Omega$ we get the estimates

$$\alpha(H(M \times [0, 1])) \leq \alpha(\mathrm{co}(F(M) \cup \{\theta\})) = \alpha(F(M)) \leq [F]_A \alpha(M).$$

So, if the set (12.4) is bounded, we may find $\Omega \in \mathfrak{OBC}(X)$ such that $S \cap \partial\Omega = \emptyset$. From Property 7.4 we conclude that the operator $F_0 + H(\cdot, 1) = \lambda J - F$ is k-epi on $\overline{\Omega}$ for $0 \leq k \leq |\lambda|\nu(J) - [F]_A$ as claimed. □

Observe that we could have also used the Rouché type estimate (7.17) for the proof of Theorem 12.2. This would require to show that

$$|\lambda| \sup_{x \in \partial\Omega} \|J(x)\| < \inf_{x \in \partial\Omega} \|F(x)\|$$

for some $\Omega \in \mathfrak{OBC}(X)$, but without using the set S in (12.4). The following gives a parallel result for stably solvable operators and may be proved as a direct consequence of the Rouché type estimate (6.6).

Theorem 12.3. *Suppose that* $F \in \mathfrak{A}(X,Y) \cap \mathfrak{Q}(X,Y)$*, and* $J: X \to Y$ *satisfies* $\mu(J) > 0$*. Fix* $\lambda \in \mathbb{K}$ *with* $|\lambda|\mu(J) > \max\{[F]_{\mathrm{A}}, [F]_{\mathrm{Q}}\}$*. Then the operator* $\lambda J - F$ *is* k*-stably solvable for every* $k \leq |\lambda|\mu(J) - \max\{[F]_{\mathrm{A}}, [F]_{\mathrm{Q}}\}$*. In particular, equation* (12.3) *has a solution* $x \in X$ *for every* $y \in Y$*.*

We illustrate Theorems 12.2 and 12.3 by means of an application to nonlinear integral equations. We start with a Hammerstein integral equation of the form

$$\lambda x(s) - \int_0^1 k(s,t) f(t, x(t))\, dt = y(s) \quad (0 \leq s \leq 1). \tag{12.5}$$

The nonlinear Hammerstein operator

$$H(x)(s) = \int_0^1 k(s,t) f(t, x(t))\, dt \tag{12.6}$$

defined by the right-hand side of (12.5) may be viewed as composition $H = KF$ of the nonlinear Nemytskij operator

$$F(x)(t) = f(t, x(t)) \tag{12.7}$$

generated by the nonlinearity f and the linear integral operator

$$Ky(s) = \int_0^1 k(s,t) y(t)\, dt \tag{12.8}$$

generated by the kernel function k. We suppose that $k: [0,1] \times [0,1] \to \mathbb{R}$ is continuous and $f: [0,1] \times \mathbb{R} \to \mathbb{R}$ satisfies a Carathéodory condition. Moreover, we assume that f satisfies a growth condition of the form

$$|f(t,u)| \leq a(t) + b(t)|u| \quad (0 \leq t \leq 1,\ u \in \mathbb{R}), \tag{12.9}$$

with two functions $a, b \in L_1[0,1]$. In what follows, we write $\|x\|_1$ for the L_1-norm and $\|x\|_\infty$ for the C-norm of a function x. Moreover, we define a scalar function κ by

$$\kappa(t) := \max_{0 \leq s \leq 1} |k(s,t)| \quad (0 \leq t \leq 1). \tag{12.10}$$

Observe that, by well-known formulas for the norm of a linear integral operator, the norm $\|\kappa\|_1$ is then precisely the operator norm of the linear operator (12.8) in the space $C[0,1]$.

Proposition 12.1. *Suppose that* $|\lambda| > \|\kappa b\|_1$*, where* $\kappa(t)$ *is given by* (12.10) *and* b *is from* (12.9)*. Then the equation* (12.5) *has a solution* $x \in C[0,1]$ *for* $y(s) \equiv 0$*. Moreover, if* $a(t) \equiv 0$ *in* (12.9)*, then equation* (12.5) *has a solution* $x \in C[0,1]$ *for every* $y \in C[0,1]$*.*

Proof. We apply Theorem 12.2 in $X = Y = C[0, 1]$ with $J = I$. Since the operator (12.6) is compact in X, we see that $[\lambda I - H]_{\mathrm{a}} = |\lambda| > 0$. Now we distinguish two cases for λ.

Suppose first that $\lambda \notin \sigma_{\mathrm{b}}(H)$, i.e., $[\lambda I - H]_{\mathrm{b}} > 0$. Consider the set

$$S = \{x \in X : \lambda x = tH(x) \text{ for some } t \in (0, 1]\}. \tag{12.11}$$

By assumption, for $x \in S$ we have

$$|\lambda|\, \|x\|_\infty \le \|H(x)\|_\infty \le \|\kappa a\|_1 + \|\kappa b\|_1 \|x\|_\infty,$$

hence

$$\|x\|_\infty \le \frac{\|\kappa a\|_1}{|\lambda| - \|\kappa b\|_1},$$

which shows that the set (12.11) is bounded. From Theorem 12.2 we conclude that the operator $\lambda I - H$ is k-epi on X for $k < |\lambda|$, i.e., $\nu(\lambda I - H) > 0$. Together with our assumption $[\lambda I - H]_{\mathrm{b}} > 0$ this implies that $\lambda \in \rho_{\mathrm{F}}(H)$, and so the equation $H(x) = \lambda x$ has a solution.

Suppose now that $\lambda \in \sigma_{\mathrm{b}}(H)$, i.e., $[\lambda I - H]_{\mathrm{b}} = 0$. Then we may find a sequence $(x_n)_n$ in X such that

$$\|\lambda x_n - H(x_n)\|_\infty \le \frac{1}{n}\|x_n\|_\infty$$

and thus

$$|\lambda|\, \|x_n\|_\infty - \|\kappa a\|_1 - \|\kappa b\|_1 \|x_n\|_\infty \le \frac{1}{n}\|x_n\|_\infty,$$

hence

$$\left(|\lambda| - \|\kappa b\|_1 - \frac{1}{n}\right)\|x_n\|_\infty \le \|\kappa a\|_1.$$

This shows that the sequence $(x_n)_n$ is bounded, because $|\lambda| > \|\kappa b\|_1$. Consequently, $\|\lambda x_n - H(x_n)\|_\infty \to 0$ as $n \to \infty$. Denoting $M := \{x_1, x_2, x_3, \ldots\}$, from

$$[\lambda I - H]_{\mathrm{a}} \alpha(M) \le \alpha((\lambda I - H)(M)) = 0$$

we see that $(x_n)_n$ has a convergent subsequence, and the limit of this subsequence is certainly a solution of the equation $H(x) = \lambda x$.

To prove the last assertion, assume that $a(t) \equiv 0$. Then the Feng spectral radius (7.22) satisfies the estimate

$$r_{\mathrm{F}}(H) \le \max\{[H]_{\mathrm{A}}, [H]_{\mathrm{B}}\} = \sup_{x \neq \theta} \frac{\|H(x)\|_\infty}{\|x\|_\infty} \le \|\kappa b\|_1,$$

and so $\lambda \in \rho_{\mathrm{F}}(H)$ for $|\lambda| > \|\kappa b\|_1$. □

As second example let us consider the Uryson integral equation of second kind

$$\lambda x(s) - \int_0^1 k(s,t,x(t))\,dt = 0 \quad (0 \le s \le 1). \tag{12.12}$$

We are going to study the nonlinear Uryson operator

$$U(x)(s) = \int_0^1 k(s,t,x(t))\,dt \tag{12.13}$$

generated by (12.12) in the space $L_2[0,1]$. To this end, we make the following assumptions on the (continuous) nonlinear kernel function $k\colon [0,1]\times[0,1]\times\mathbb{R} \to \mathbb{R}$:

$$\sup_{|u|\le r} |k(s,t,u)| \le \beta_r(s,t), \quad M_r := \sup_{0\le s\le 1} \int_0^1 \beta_r(s,t)\,dt < \infty, \tag{12.14}$$

$$\sup_{|u|\le r} |k(s,t,u) - k(\sigma,t,u)| \le \gamma_r(s,\sigma,t), \quad \lim_{s\to\sigma} \int_0^1 \gamma_r(s,\sigma,t)\,dt = 0, \tag{12.15}$$

and

$$|k(s,t,u)| \le \psi(s,t)(1+|u|), \quad M := \iint_0^1 \psi(s,t)^2\,dt\,ds < \infty. \tag{12.16}$$

Proposition 12.2. *Suppose that* $|\lambda|^2 > 4M$. *Then the equation* (12.12) *has a solution* $x \in L_2[0,1]$.

Proof. We apply Theorem 12.2 in $X = Y = L_2[0,1]$ with $J = I$. It is well known that, under our assumptions (12.14)–(12.16), the operator (12.13) is compact in X. Moreover, for any $x \in X$ one has the estimates

$$\begin{aligned}
|U(x)(s)|^2 &\le \left(\int_0^1 k(s,t,x(t))\,dt\right)^2 \\
&\le \left(\int_0^1 \psi(s,t)(1+|x(t)|)\,dt\right)^2 \\
&\le \left(\int_0^1 \psi(s,t)^2\,dt\right)\left(\int_0^1 (1+|x(t)|)^2\,dt\right) \\
&\le 4\left(\int_0^1 \psi(s,t)^2\,dt\right)(1+\|x\|^2),
\end{aligned}$$

where in the last step we used the fact that $(a+b)^p \le 2^p(a^p+b^p)$ for $a, b \ge 0$ and $p \ge 1$. Consequently,

$$\|U(x)\|^2 \le 4\left(\iint_0^1 \psi(s,t)^2\,dt\,ds\right)(1+\|x\|^2) \le 4M(1+\|x\|^2).$$

We now distinguish again the two cases $[\lambda I - U]_{\mathrm{b}} > 0$ and $[\lambda I - U]_{\mathrm{b}} = 0$. In the first case the set

$$S = \{x \in X : \lambda x = tU(x) \text{ for some } t \in (0, 1]\}.$$

is again bounded, because every $x \in S$ satisfies

$$|\lambda|^2 \, \|x\|^2 \le \|U(x)\|^2 \le 4M(1 + \|x\|^2),$$

hence

$$\|x\|^2 \le \frac{4M}{|\lambda|^2 - 4M}.$$

By Theorem 12.2, the operator $\lambda I - U$ is k-epi for $k < |\lambda|$, and so $\lambda \in \rho_{\mathrm{F}}(U)$ as before.

Assume now that $[\lambda I - U]_{\mathrm{b}} = 0$. Then we may find a sequence $(x_n)_n$ in X such that

$$\|\lambda x_n - U(x_n)\| \le \frac{1}{n} \|x_n\|.$$

We claim that the sequence $(x_n)_n$ is bounded. In fact, the estimate

$$\frac{1}{n}\|x_n\| \ge |\lambda| \, \|x_n\| - \|U(x_n)\| \ge |\lambda| \, \|x_n\| - 2\sqrt{M}\sqrt{1 + \|x_n\|^2}$$

implies that

$$|\lambda| - 2\sqrt{M}\left(\frac{1}{\|x_n\|^2} + 1\right)^{1/2} \le \frac{1}{n}.$$

Letting $n \to \infty$, the unboundedness of $(x_n)_n$ would give $|\lambda| \le 2\sqrt{M}$, contradicting our choice of λ.

So we have proved that the sequence $(x_n)_n$ is bounded. The remaining part of the proof goes as in Proposition 12.1. □

12.2 Solvability of semilinear equations

Now we are going to discuss some applications of the spectral theory for semilinear pairs (L, F) introduced in Sections 9.1 and 9.2. So let X and Y be two Banach spaces, $L\colon X \to Y$ a linear Fredholm operator of index 0, and $F\colon X \to Y$ nonlinear. We will be interested in solvability results for the equation

$$\lambda Lx - F(x) = y \quad (y \in Y). \tag{12.17}$$

We use the same terminology as in Chapter 9. So we consider the decompositions $X = N(L) \oplus X_0$ and $Y = Y_0 \oplus R(L)$, and denote by $P\colon X \to N(L)$ and $Q\colon Y \to Y_0$ the corresponding projections. Moreover, we write $L_P := L|_{X_0}\colon X_0 \to R(L)$ and $K_{PQ} := L_P^{-1}(I - Q)\colon Y \to X_0$, and denote by $\Pi\colon Y \to Y/R(L)$ the natural quotient map and by $\Lambda\colon Y/R(L) \to N(L)$ the natural isomorphism induced by L. Of course, if L is a bijection between X and Y, we simply have $X_0 = X$, $Y_0 = \{\theta\}$, $Px = Qy \equiv \theta$, and $K_{PQ} = L^{-1}$.

Recall that we have called $F\colon X \to Y$ an (L, α)-Lipschitz operator (and used the notation $F \in \mathfrak{A}_L(X, Y)$) if $[K_{PQ}F]_{\mathrm{A}} < \infty$. In particular, F is called (L, α)-contractive if $[K_{PQ}F]_{\mathrm{A}} < 1$, and L-compact if $[K_{PQ}F]_{\mathrm{A}} = 0$. As in Section 9.1 we associate with L and F the map $\Phi_\lambda(L, F)\colon X \to X$ defined by

$$\Phi_\lambda(L, F)(x) = \lambda(I - P)x - (\Lambda\Pi + K_{PQ})F(x) \quad (\lambda \in \mathbb{K}). \tag{12.18}$$

In case of a bijection L this operator simply reduces to $\Phi_\lambda(L, F) = \lambda I - L^{-1}F$. Before stating the first theorem, let us recall that the semilinear Feng spectrum is defined as union

$$\sigma_{\mathrm{F}}(L, F) = \sigma_\nu(L, F) \cup \sigma_{\mathrm{a}}(L, F) \cup \sigma_{\mathrm{b}}(L, F),$$

where $\lambda \in \sigma_\nu(L, F)$ if $\Phi_\lambda(L, F)$ is strictly epi, $\lambda \in \sigma_{\mathrm{a}}(L, F)$ if $[\Phi_\lambda(L, F)]_{\mathrm{a}} = 0$, and $\lambda \in \sigma_{\mathrm{b}}(L, F)$ if $[\Phi_\lambda(L, F)]_{\mathrm{b}} = 0$. Moreover, λ belongs to the point spectrum (9.14) of the pair (L, F) if and only if the equation (12.17) has a nontrivial solution $x \in X$ for $y = \theta$.

Similarly, the semilinear FMV-spectrum is defined as union

$$\sigma_{\mathrm{FMV}}(L, F) = \sigma_\delta(L, F) \cup \sigma_{\mathrm{a}}(L, F) \cup \sigma_{\mathrm{q}}(L, F),$$

where $\lambda \in \sigma_\delta(L, F)$ if $\Phi_\lambda(L, F)$ is not stably solvable, and $\lambda \in \sigma_{\mathrm{q}}(L, F)$ if $[\Phi_\lambda(L, F)]_{\mathrm{q}} = 0$. Every element in $\sigma_{\mathrm{q}}(L, F)$ is called an asymptotic eigenvalue of the pair (L, F).

The following two theorems show that "in the direction of every sufficiently large spectral value" one may find an eigenvalue of the pair (L, F) (in a sense to be made precise).

Theorem 12.4. *Suppose that $F \in \mathfrak{A}_L(X, Y)$ is 1-homogeneous. Let $\lambda \in \sigma_{\mathrm{F}}(L, F)$ with $|\lambda| > [K_{PQ}F]_{\mathrm{A}}$. Then there exists some $\tau \in (0, 1]$ such that λ/τ is an eigenvalue of the pair $(L, (I - Q)F)$.*

Proof. Fix $\lambda \in \mathbb{K}$ with $|\lambda| > [K_{PQ}F]_{\mathrm{A}}$, and suppose first that $\lambda \in \sigma_{\mathrm{b}}(\Phi_\lambda(L, F))$. Then without loss of generality we may choose a sequence $(e_n)_n$ in $S(X)$ such that

$$\|\lambda e_n - \lambda P e_n - (\Lambda\Pi + K_{PQ})F(e_n)\| \leq \frac{1}{n} \quad (n = 1, 2, 3, \ldots).$$

From $|\lambda| > [K_{PQ}F]_{\mathrm{A}}$ it follows that $[\Phi_\lambda(L, F)]_{\mathrm{a}} > 0$. Consequently, the sequence $(e_n)_n$ has a convergent subsequence, say $e_{n_k} \to e \in S(X)$ as $k \to \infty$. By continuity, we have $\Phi_\lambda(L, F)(e) = \theta$, and so from the equivalence of the equations (9.5) and (9.6) we conclude that

$$\lambda L e = F(e) = QF(e) + (I - Q)F(e).$$

From $F(e) \in R(L)$, hence $QF(e) = \theta$, we see that $\lambda \in \sigma_p(L, (I-Q)F)$, and so we may choose $\tau = 1$ in this case.

Now suppose that $\lambda \notin \sigma_b(\Phi_\lambda(L, F))$, i.e., $[\Phi_\lambda(L, F)]_b > 0$. Consider the set

$$S := \{x \in X : \lambda x - t\lambda Px = t(\Lambda\Pi + K_{PQ})F(x) \text{ for some } t \in [0, 1]\}. \quad (12.19)$$

We distinguish two cases. First, assume that the set (12.19) is unbounded, and choose a sequence $(x_n)_n$ in S with $\|x_n\| \to \infty$ as $n \to \infty$. Putting $e_n := x_n/\|x_n\|$ and using the 1-homogeneity of F we obtain

$$\lambda e_n - t_n\lambda Pe_n - t_n(\Lambda\Pi + K_{PQ})F(e_n) = \theta \quad (12.20)$$

for some sequence $(t_n)_n$ in $[0, 1]$. Without loss of generality we may suppose that $t_n \to \tau \in [0, 1]$ as $n \to \infty$. From (12.20) it follows that

$$\sup_n \frac{|\lambda|}{t_n} \leq |\lambda| \, \|P\| + \|\Lambda\Pi + K_{PQ}\| \sup_{\|x\|=1} \|F(x)\| < \infty,$$

which means that the sequence $(t_n)_n$ is bounded away from zero, and so $\tau > 0$. From (12.20) we also deduce that

$$\lambda e_n - \tau\lambda Pe_n - \tau(\Lambda\Pi + K_{PQ})F(e_n) = (t_n - \tau)\lambda Pe_n + (t_n - \tau)(\Lambda\Pi + K_{PQ})F(e_n) \to \theta$$

as $n \to \infty$. So the set $M := \{e_1, e_2, e_3, \ldots\} \subset S(X)$ satisfies

$$[\lambda I - \tau\lambda P - \tau(\Lambda\Pi + K_{PQ})F]_a \alpha(M) \leq \alpha((\lambda I - \tau\lambda P - \tau(\Lambda\Pi + K_{PQ})F)(M)) = 0.$$

From $|\lambda| > [K_{PQ}F]_A$ and the compactness of P and $\Lambda\Pi$ we conclude that

$$[\lambda I - \tau\lambda P - \tau(\Lambda\Pi + K_{PQ})F]_a > 0,$$

and so $\alpha(M) = 0$. This means that $e_{n_k} \to e$, as $k \to \infty$, for some subsequence $(e_{n_k})_k$ and $e \in S(X)$, and so

$$\lambda e - \tau\lambda Pe - \tau(\Lambda\Pi + K_{PQ})F(e) = \theta.$$

Since $\|e\| = 1$ and $\lambda \neq 0$ we get $\tau \neq 0$; moreover,

$$\lambda(I - P)e - \tau K_{PQ}F(e) = \tau\lambda Pe - \lambda Pe + \tau\Lambda\Pi F(e) \in N(L).$$

This means nothing else but

$$\theta = \lambda Le - \lambda LPe - \tau LK_{PQ}F(e) = \lambda Le - \tau(I - Q)F(e),$$

i.e., $\lambda/\tau \in \sigma_p(L, (I-Q)F)$, and so the assertion is proved in case the set (12.19) is unbounded.

Finally, suppose that the set (12.19) is bounded, and choose $R > 0$ sufficiently large such that S is contained in the open ball $B_R^o(X)$. Then for every $x \in S_R(X)$ and $0 \le t \le 1$ we have

$$\lambda x - t\lambda Px - t(\Lambda\Pi + K_{PQ})F(x) \neq \theta.$$

From Property 7.4 of epi operators we conclude that the operator (12.18) is strictly epi on $B_R(X)$. Moreover, our assumption $[\Phi_\lambda(L, F)]_{\mathrm{b}} > 0$ and the 1-homogeneity of F imply that $\Phi_\lambda(L, F)$ is strictly epi on *every* ball $B_r(X)$. So we have shown that $\lambda \in \rho_{\mathrm{F}}(L, F)$, contradicting our assumption $\lambda \in \sigma_{\mathrm{F}}(L, F)$. The proof is complete. □

Theorem 12.5. *Suppose that $F \in \mathfrak{A}_L(X, Y)$ satisfies $[K_{PQ}F]_{\mathrm{Q}} < \infty$. Let $\lambda \in \sigma_{\mathrm{FMV}}(L, F)$ with $|\lambda| > [K_{PQ}F]_{\mathrm{A}}$. Then there exists some $\tau \in (0, 1]$ such that λ/τ is an asymptotic eigenvalue of the pair $(L, (I - Q)F)$.*

Proof. Fix $\lambda \in \mathbb{K}$ with $|\lambda| > [K_{PQ}F]_{\mathrm{A}}$, and suppose that $\lambda \notin \sigma_{\mathrm{q}}(L, (I - Q)F)$, i.e.,

$$\liminf_{\|x\|\to\infty} \frac{\|\lambda Lx - (I - Q)F(x)\|}{\|x\|} > 0. \tag{12.21}$$

We claim that this implies that

$$\liminf_{\|x\|\to\infty} \frac{\|\Phi_\lambda(L, F)(x)\|}{\|x\|} > 0 \tag{12.22}$$

as well, where $\Phi_\lambda(L, F)$ is the operator (12.18). In fact, if (12.22) is false, we may find an unbounded sequence $(x_n)_n$ such that

$$\lim_{n\to\infty} \frac{\|\Phi_\lambda(L, F)(x_n)\|}{\|x_n\|} = \lim_{n\to\infty} \frac{\|\lambda(I - P)x_n - (\Lambda\Pi + K_{PQ})F(x_n)\|}{\|x_n\|} = 0,$$

and so also

$$\lim_{n\to\infty} \frac{\|\lambda L(I - P)x_n - L(\Lambda\Pi + K_{PQ})F(x_n)\|}{\|x_n\|} = 0.$$

But

$$\lambda L(I - P) - L(\Lambda\Pi + K_{PQ})F = \lambda L - (I - Q)F,$$

which yields a contradiction to (12.21). So we see that (12.21) implies (12.22), i.e., $\lambda \in \sigma_{\mathrm{q}}(L, F)$.

Given any operator $G\colon X \to Y$ with $[G]_{\mathrm{A}} = [G]_{\mathrm{Q}} = 0$, we define a homotopy $H_G\colon X \times [0, 1] \to Y$ by

$$H_G(x, t) := t[\lambda Px + (\Lambda\Pi + K_{PQ})F(x) + G(x)].$$

Obviously, $H_G(x, 0) = \theta$ and $H_G(x, 1) = \lambda x + G(x) - \Phi_\lambda(L, F)(x)$. We define a set S_G by

$$S_G := \{x \in X : \lambda x = H_G(x, t) \text{ for some } t \in [0, 1]\}$$

and distinguish two cases. Assume first that S_G is bounded for all G with $[G]_{\mathrm{A}} = [G]_{\mathrm{Q}} = 0$. Then $(\lambda I)(S_G)$ is also bounded for all such G, and hence the equation $\lambda x = H_G(x, 1)$ has a solution $\tilde{x} \in X$, by Proposition 6.2. But this means that

$$\Phi_\lambda(L, F)(\tilde{x}) = \lambda\tilde{x} - H_G(\tilde{x}, 1) + G(\tilde{x}) = \acute{}G(\tilde{x}).$$

It follows that $\Phi_\lambda(L, F)$ is stably solvable, and so we have $\lambda \in \rho_{\mathrm{FMV}}(L, F)$, a contradiction.

Suppose now that the set S_G is unbounded for some G with $[G]_{\mathrm{A}} = [G]_{\mathrm{Q}} = 0$. Then we may find sequences $(x_n)_n$ in X and $(t_n)_n$ in $[0, 1]$ such that $\|x_n\| \to \infty$ and $\lambda x_n = H_G(x_n, t_n)$. As in the proof of Theorem 12.4 we may suppose that $t_n \to \tau$ for some $\tau \in (0, 1]$. The equality

$$\lambda L - (I - Q)F = L\Phi_\lambda(L, F) = \lambda L - LH_G(\,\cdot\,, 1) + LG$$

and the fact that $tH_G(x, 1) = H_G(x, t)$ imply that

$$t_n(I - Q)F(x_n) = LH_G(x_n, t_n) - t_nLG(x_n) = \lambda Lx_n - t_nLG(x_n).$$

So we have

$$\begin{aligned}
&\lambda Lx_n - \tau(I - Q)F(x_n)\\
&\quad = \lambda Lx_n - t_n(I - Q)F(x_n) - t_nG(x_n) + (t_n - \tau)(I - Q)F(x_n) + t_nG(x_n)\\
&\quad = (t_n - \tau)(I - Q)F(x_n) + t_nG(x_n).
\end{aligned}$$

Taking norms and dividing by $\|x_n\|$ yields

$$\frac{\|\lambda Lx_n - \tau(I - Q)F(x_n)\|}{\|x_n\|} \le |t_n - \tau|\,\|I - Q\|\,\frac{\|F(x_n)\|}{\|x_n\|} + t_n\frac{\|G(x_n)\|}{\|x_n\|} \to 0$$

as $n \to \infty$, because $t_n \to \tau$, $[F]_{\mathrm{Q}} < \infty$, and $[G]_{\mathrm{Q}} = 0$. This shows that $[\lambda L - \tau(I - Q)F]_{\mathrm{q}} = 0$, and so $\lambda/\tau \in \sigma_{\mathrm{q}}(L, (I - Q)F)$ as claimed. □

In the following example we consider again the boundary value problem (10.44) from the viewpoint of Theorem 12.4.

Example 12.1. In $X = C[0, 1]$, consider the 1-homogeneous compact operator F defined by

$$F(x)(s) = \int_0^s \sqrt{x(t)^2 + x(1 - t)^2}\,dt. \tag{12.23}$$

It is evident that $\|F(x)\| \le \sqrt{2}\|x\|$ and F preserves the order $\preceq$ induced by the natural cone of nonnegative functions in X.

We put $L = I$ and claim that

$$\sigma_{\mathrm{F}}(F) \supseteq \sigma_{\mathrm{FMV}}(F) \supseteq [-1/\sqrt{2}, 1/\sqrt{2}]. \tag{12.24}$$

To see this, we first show that $\lambda I - F$ is not onto for $0 < \lambda < 1/\sqrt{2}$. In fact, otherwise the equation $\lambda x - F(x) = \lambda e$, with $e(t) \equiv 1$, would have a solution $\hat{x} \in X$. From $\lambda(\hat{x}(t) - 1) = F(\hat{x})(t) \geq 0$ it follows that $\hat{x}(t) \geq 1$, i.e., $e \preceq \hat{x}$, and so also $F^n(e) \preceq F^n(\hat{x})$ for all $n \in \mathbb{N}$.

Estimating the integrand in (12.23) one may easily show that

$$F^n(e)(t) \geq \frac{1}{(\sqrt{2})^n} t \quad (n = 1, 2, 3, \dots). \tag{12.25}$$

From $F(\hat{x}) \preceq \lambda \hat{x}$ it follows that $F^n(\hat{x}) \preceq \lambda^n \hat{x}$ for each $n \in \mathbb{N}$. Combining these estimates we obtain

$$F^n(e) \preceq F^n(\hat{x}) \preceq \lambda^n \hat{x},$$

and thus

$$\theta \preceq (\sqrt{2})^n F^n(e) \preceq (\sqrt{2})^n \lambda^n \hat{x} \to \theta \quad (n \to \infty),$$

contradicting (12.25). We conclude that there exists no such $\hat{x}$, and so $\lambda I - F$ is not surjective for $0 < \lambda < 1/\sqrt{2}$.

Similarly, $\lambda I - F$ is not surjective for $-1/\sqrt{2} < \lambda < 0$. In fact, this is an easy consequence of what we have just proved and the fact that F is even, i.e., $F(-x) = F(x)$. Since both the FMV-spectrum and the Feng spectrum are closed, (12.24) follows.

By Theorem 12.4, the operator (12.23) has an eigenvalue $\lambda_+ \geq 1/\sqrt{2}$ and another eigenvalue $\lambda_- \leq -1/\sqrt{2}$. Moreover, Theorem 7.7 implies that

$$\liminf_{n\to\infty} [F^n]_{\mathrm{B}}^{1/n} \geq \frac{1}{\sqrt{2}},$$

compare this with (10.46). ♡

We point out again that Theorem 10.2 does not apply to the operator (12.23) since

$$\inf\{\|F(x)\| : x \in S(X)\} = \inf\left\{\int_0^1 \sqrt{x(t)^2 + x(1-t)^2}\, dt : \max|x(t)| = 1\right\} = 0.$$

As already observed at the end of Section 10.3, the only positive eigenvalue of (12.23) is $\lambda_+ = 1/\sqrt{2}\log(1+\sqrt{2})$, and $\lambda_+ \geq 1/\sqrt{2}$, since $1 + \sqrt{2} \leq e$. Since the operator (12.23) is even, $\lambda_- = -\lambda_+ \leq -1/\sqrt{2}$ is the other eigenvalue whose existence is guaranteed by Theorem 10.2. Observe that the inclusion (12.24) shows that Theorem 7.8 is *not* true if we drop the oddness assumption on F.

Now we apply the spectral theory discussed in Section 9.1 more systematically to prove three existence results for semilinear operator equations. In particular, we will be interested in the semilinear equation

$$\lambda L x = F(x), \tag{12.26}$$

which is of course a special case of (12.17). Recall that $\mathfrak{OBC}(X)$ denotes the family of all bounded open connected subsets of X containing θ. Given a linear Fredholm operator $L\colon X \to Y$ of index zero, we write as before $\mathfrak{A}_L(X, Y)$ for the set of all operators $F\colon X \to Y$ such that $[K_{PQ}F]_{\mathrm{A}} < \infty$, where $K_{PQ} = L|_{X_0}^{-1}(I - Q)$.

Theorem 12.6. *Let $F, G \in \mathfrak{A}_L(X, Y)$, where G is odd and 1-homogeneous with $N(\lambda L - G) = \{\theta\}$ for some $\lambda \in \mathbb{K}$ satisfying $|\lambda| > [K_{PQ}G]_{\mathrm{A}}$. Suppose that*

$$\lambda Lx \neq (1-t)G(x) + tF(x) \quad (x \in \partial\Omega,\ 0 \leq t \leq 1), \tag{12.27}$$

where $\Omega \in \mathfrak{OBC}(X)$ is fixed. Then the equation (12.26) *has at least one solution in Ω, provided that $2[K_{PQ}G]_{\mathrm{A}} + [K_{PQ}F]_{\mathrm{A}} < |\lambda|$.*

Proof. Since $N(\lambda L - G) = \{\theta\}$, we see that λ is not an eigenvalue of the pair (L, G), i.e., $\lambda \notin \sigma_{\mathrm{p}}(L, G)$. By Theorem 9.2, this implies that $\lambda \notin \sigma_{\mathrm{F}}(L, G)$. In particular, the operator $\Phi_\lambda(L, G) = \lambda(I - P) - (\Lambda\Pi + K_{PQ})G$ is k_0-epi on $\overline{\Omega}$ for $0 \leq k_0 < |\lambda| - [K_{PQ}G]_{\mathrm{A}}$. By Lemma 9.2 (e) we know that then $\lambda L - G$ is (L, k_0)-epi on $\overline{\Omega}$.

Define a homotopy $H\colon \overline{\Omega} \times [0, 1] \to Y$ by

$$H(x, t) := t(G(x) - F(x)).$$

Obviously, $H(x, 0) \equiv \theta$ and

$$\alpha(H(M \times [0, 1])) \leq ([K_{PQ}F]_{\mathrm{A}} + [K_{PQ}G]_{\mathrm{A}})\alpha(M) \quad (M \subseteq \overline{\Omega}).$$

Moreover, $\lambda Lx - G(x) + H(x, t) \neq \theta$ for all $x \in \partial\Omega$ and $t \in [0, 1]$, by (12.27). By Property 9.4 of (L, k)-epi operators, $\lambda L - G + H(\cdot, 1) = \lambda L - F$ is k_1-epi on $\overline{\Omega}$ for $0 < k_1 < |\lambda| - 2[K_{PQ}G]_{\mathrm{A}} - [K_{PQ}F]_{\mathrm{A}}$. Thus there exists $\hat{x} \in \Omega$ such that $\lambda L\hat{x} = F(\hat{x})$ as claimed. □

Theorem 12.7. *Suppose that $F \in \mathfrak{A}_L(X, Y)$ is asymptotically linear with asymptotic derivative $F'(\infty)$. Let $\lambda \in \mathbb{K}$ with $|\lambda| > 3[F]_{\mathrm{A}}/[L]_{\mathrm{a}}$. Then equation* (12.26) *has a solution provided $\lambda \notin \sigma_{\mathrm{p}}(L, F'(\infty))$.*

Proof. Suppose that λ is not an eigenvalue of the pair $(L, F'(\infty))$. By Lemma 4.2 and our assumption on λ we have

$$[F'(\infty)]_{\mathrm{A}} \leq [F]_{\mathrm{A}} < \frac{1}{3}|\lambda|\,[L]_{\mathrm{a}} < |\lambda|\,[L]_{\mathrm{a}}, \tag{12.28}$$

So $L_\infty := \lambda L - F'(\infty)$ is a Fredholm operator of index zero, and $[L_\infty]_{\mathrm{a}} > 0$, by Proposition 2.4 (d). Let $P\colon X \to N(L_\infty)$ be the projection of X onto the nullspace of L_∞, and $\Lambda\colon Y/R(L_\infty) \to N(L_\infty)$ and $h\colon Y/R(L_\infty) \to Y_0$ canonical linear isomorphisms as in Section 9.1, where Y_0 is now of course a complementary subspace of Y to the range $R(L_\infty)$. Denote by R the remainder term of the derivative $F'(\infty)$, i.e.,

$$R(x) := F(x) - F'(\infty)x, \tag{12.29}$$

and consider the set

$$S := \{x \in X : L_\infty x - h\Lambda^{-1}Px + t[h\Lambda^{-1}Px - R(x)] = \theta \text{ for some } t \in [0,1]\}.$$

We claim that the set S is bounded. In fact, assume that there exist $x_n \in S$ with $\|x_n\| \to \infty$ as $n \to \infty$. Put $e_n = x_n/\|x_n\|$ and choose $t_n \in [0,1]$ such that

$$L_\infty e_n - h\Lambda^{-1}Pe_n + t_n h\Lambda^{-1}Pe_n = t_n R(e_n) \to 0, \quad (n \to \infty).$$

As $h\Lambda^{-1}P$ is compact, we can assume that $h\Lambda^{-1}Pe_n - t_n h\Lambda^{-1}Pe_n \to y_0$ for some $y_0 \in Y_0$. Then $L_\infty e_n \to y_0$. Since $|\lambda| > [F'(\infty)]_{\mathrm{A}}/[L]_{\mathrm{a}}$, by (12.28), we obtain $[L_\infty]_{\mathrm{a}} > 0$ and so $\alpha(M) = 0$, where $M := \{e_1, e_2, e_3, \ldots\}$. This shows that $e_{n_k} \to e$, as $k \to \infty$, for some subsequence $(e_{n_k})_k$ and $e \in S(X)$. By continuity, $\lambda L e_{n_k} \to y_0 + F'(\infty)e$ and $L_\infty e = y_0 \in R(L_\infty)$. Since $R(L_\infty) \cap Y_0 = \{\theta\}$, this ensures that $\lambda L e - F'(\infty)e = L_\infty e = \theta$, and so λ is an eigenvalue of the pair $(L, F'(\infty))$, contradicting our assumption.

So we have shown that the set S is indeed bounded. Fix $R > 0$ such that

$$L_\infty x - h\Lambda^{-1}Px + t[h\Lambda^{-1}Px - Rx] \neq \theta \quad (x \in S_R(X),\ t \in [0,1]).$$

Since $L_\infty - h\Lambda^{-1}P$ is one-to-one and $h\Lambda^{-1}P$ is a compact linear operator, by Theorem 9.2 we know that $1 \in \rho_{\mathrm{F}}(L_\infty, h\Lambda^{-1}P)$. In particular, the operator

$$\Phi_1(L_\infty, h\Lambda^{-1}P) = I - P - (\Lambda\Pi + K_{PQ})h\Lambda^{-1}P$$

is strictly epi. So, by Lemma 9.2 (e), the operator $L_\infty - h\Lambda^{-1}P$ is (L_∞, k)-epi for sufficiently small $k \geq 0$.

Define a homotopy $H\colon X \times [0,1] \to Y$ by

$$H(x,t) = t[h\Lambda^{-1}Px - R(x)],$$

with R as in (12.29), and observe that $H(x,0) \equiv \theta$ and $\alpha(H(M \times [0,1])) \leq \alpha(R(M))$ for any bounded set $M \subset X$. Since $|\lambda| > 3[F]_{\mathrm{A}}/[L]_{\mathrm{a}}$, by assumption, we further get from (12.28)

$$[R]_{\mathrm{A}} = [F - F'(\infty)]_{\mathrm{A}} \leq 2[F]_{\mathrm{A}} < |\lambda|\,[L]_{\mathrm{a}} - [F]_{\mathrm{A}} \leq |\lambda|\,[L]_{\mathrm{a}} - [F'(\infty)]_{\mathrm{a}} \leq [L_\infty]_{\mathrm{a}}.$$

Consequently, we have

$$\alpha(H(M \times [0,1])) \leq \alpha(R(M)) \leq [R]_{\mathrm{A}}\alpha(M) < [L_\infty]_{\mathrm{a}}\alpha(M)$$

for any bounded set $M \subset X$. Now fix $k_0 < 1$ with $[L_\infty]_{\mathrm{a}}k_0 > [R]_{\mathrm{A}}$. Then $L_\infty - h\Lambda^{-1}P$ is (L_∞, k_0)-epi on $B_R(X)$. Applying Property 9.4 of (L,k)-epi operators to $F_0 := L_\infty - h\Lambda^{-1}P$ and H as in (12.28) we obtain that the operator

$$F_0 + H(\cdot, 1) = L_\infty - h\Lambda^{-1}P + h\Lambda^{-1}P - R = L_\infty - R = \lambda L - F$$

is (L_∞, k_1)-epi for some $k_1 > 0$ on $B_R(X)$. So there exists $\hat{x} \in B_R(X)$ such that $\lambda L\hat{x} - F(\hat{x}) = \theta$, and we are done. □

Theorem 12.8. *Let $F \in \mathfrak{A}_L(X,Y)$ and $T \in \mathfrak{L}(X,Y)$. Moreover, given a convex domain $\Omega \in \mathfrak{OBC}(X)$, suppose that $G\colon \overline{\Omega} \to Y$ is L-compact with $G(\partial\Omega) \subseteq (\lambda L - T)(\Omega)$ and $N(\lambda L - T) = \{\theta\}$ for some $\lambda \in \mathbb{K}$. Finally assume that*

$$-\lambda Lx + (1-t)(Tx - G(x)) + tF(x) \neq \theta \quad (x \in \partial\Omega,\ 0 < t < 1). \tag{12.30}$$

Then equation (12.26) *has a solution in* Ω *provided that* $[K_{PQ}F]_{\mathrm{A}} + 2[K_{PQ}T]_{\mathrm{A}} < |\lambda|$.

Proof. The condition $N(\lambda L - T) = \{\theta\}$ implies that $\lambda \notin \sigma_{\mathrm{F}}(L,T)$, since $[K_{PQ}T]_{\mathrm{A}} < 1$. So the linear operator $\lambda L - T$ is (L,k)-epi for $0 \leq k < [\Phi_\lambda(L,T)]_{\mathrm{a}}$, where $\Phi_\lambda(L,T) = \lambda(I-P) - (\Lambda\Pi + K_{PQ})T$ and $[\Phi_\lambda(L,T)]_{\mathrm{a}} \leq |\lambda| - [K_{PQ}T]_{\mathrm{A}} > 0$. Consider the homotopy $H(x,t) := -tG(x)$. Clearly, $H\colon \overline{\Omega} \times [0,1] \to Y$ is an L-compact operator. Assume that there exists $\tilde{x} \in \partial\Omega$ such that

$$\lambda L\tilde{x} - T\tilde{x} - \tilde{t}G(\tilde{x}) = \theta$$

for some $\tilde{t} \in [0,1]$. Then

$$\lambda L\tilde{x} - T\tilde{x} = \tilde{t}G(\tilde{x}) = \tilde{t}(\lambda L - T)\hat{x},$$

where $\hat{x} \in \Omega$, by our assumption $G(\partial\Omega) \subseteq (\lambda L - T)(\Omega)$. Thus $\tilde{x} = \tilde{t}\hat{x}$, contradicting the convexity of Ω. So for any $x \in \partial\Omega$ and $t \in [0,1]$ we have $Lx + Tx \neq tG(x)$. By Property 9.4, we conclude that

$$\lambda L - T + H(\,\cdot\,,1) = \lambda L - T - G$$

is (L,k)-epi on $\overline{\Omega}$ for every $k < [\Phi_\lambda(L,T)]_{\mathrm{a}}$.

Now we introduce still another homotopy $\hat{H}\colon \overline{\Omega} \times [0,1] \to Y$ defined by

$$\hat{H}(x,t) := t(Tx + G(x) - F(x)).$$

Then $\hat{H}(x,0) \equiv \theta$ and

$$\alpha((K_{PQ}H)(M \times [0,1])) \leq ([K_{PQ}F]_{\mathrm{A}} + [K_{PQ}T]_{\mathrm{A}})\alpha(M)$$

for every $M \subseteq \overline{\Omega}$ (recall that $[K_{PQ}G]_{\mathrm{A}} = 0$). Moreover, $[K_{PQ}F]_{\mathrm{A}} + [K_{PQ}T]_{\mathrm{A}} < |\lambda| - [K_{PQ}T]_{\mathrm{A}} \leq [\Phi_\lambda(L,T)]_{\mathrm{a}}$. Again Property 9.4 and (12.30) imply that the operator

$$\lambda L - T - G + \hat{H}(\,\cdot\,,1) = \lambda L - T - G + T + G - F = \lambda L - F$$

is (L,ε)-epi on $\overline{\Omega}$ for some $\varepsilon > 0$. Hence there exists $\hat{x} \in \Omega$ which is a solution of equation (12.26), and we are done. □

12.3 Applications to boundary value problems

In this section we start our application of spectral methods to boundary value problems for ordinary differential equations. We start with the problem

$$\left.\begin{array}{l} \dot{x}(t) - A(t)x(t) = \varepsilon g(t, x(t)) \quad (0 \le t \le T), \\ Lx = \theta. \end{array}\right\} \tag{12.31}$$

Here $A\colon [0, T] \to \mathbb{R}^{n\times n}$ is continuous matrix valued function, $g\colon [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory function, $L\colon C([0, T], \mathbb{R}^n) \to \mathbb{R}^n$ is a bounded linear operator which associates to each continuous function $x\colon [0, T] \to \mathbb{R}^n$ a vector $Lx \in \mathbb{R}^n$, and $\varepsilon \neq 0$ is a scalar parameter.

Putting

$$Dx(t) = \frac{dx}{dt} - A(t)x \tag{12.32}$$

and

$$G(x)(t) = g(t, x(t)), \tag{12.33}$$

we may write (12.31) concisely as operator equation $Dx = \varepsilon G(x)$ in the Banach space

$$X = \{x \in C([0, 1], \mathbb{R}^n) : Lx = \theta\}.$$

In what follows, we write $\|x\|_1$ for the norm of a function x in $L_1([0, 1], \mathbb{R}^n)$, and $\|x\|_\infty$ for its norm in $C([0, 1], \mathbb{R}^n)$. By $U(t, s)$ we denote the *Cauchy function* of the operator family $A(t)$, i.e., the unique solution of the linear Volterra integral equation

$$U(t, s) = I + \int_s^t A(\tau)U(\tau, s)\, d\tau \quad (0 \le t, s \le T), \tag{12.34}$$

and by

$$Ez(t) = \int_0^t U(t, s)z(s)\, ds \quad (0 \le t \le T) \tag{12.35}$$

the associated *evolution operator*. A well-known fact from the theory of linear differential equations asserts that $DE = I$, i.e., the operator (12.35) is the right inverse to the differential operator (12.32).

Assume now that the composition $L_U := LU_0$ of the boundary operator L in (12.31) and the operator $U_0\colon \mathbb{R}^n \to C([0, T], \mathbb{R}^n)$ defined by

$$(U_0x)(t) := U(t, 0)x \quad (x \in \mathbb{R}^n)$$

is an *isomorphism* in $\mathbb{R}^n$. This assumption seems to be quite restrictive, but is fulfilled in many applications.

We claim that the nonlinear operator F defined by

$$F(x) := (I - U_0L_U^{-1}L)EG(x) \tag{12.36}$$

maps the Banach space X into itself. In fact, for any $x \in X$ we have

$$LF(x) = LEG(x) - LU_0L_U^{-1}LEG(x) = LEG(x) - LEG(x) = \theta,$$

and so $F(x) \in X$. Finally, we put

$$M = \sup_{0\le t,s\le T} \|U(t,s)\| \tag{12.37}$$

and we denote by

$$\mu_G(r) = \sup_{\|x\|\le r} \|G(x)\| \tag{12.38}$$

the so-called *growth function* of the Nemytskij operator (12.33).

Proposition 12.3. *Suppose that the nonlinearity $g\colon [0,T]\times\mathbb{R}^n \to \mathbb{R}^n$ satisfies a growth condition*

$$|g(t,u)| \le a(t) + b(t)|u| \quad (0\le t\le T,\ u\in\mathbb{R}^n) \tag{12.39}$$

for some $a, b \in L_1([0,T])$. Define a scalar function $\varphi\colon (0,\infty)\to(0,\infty)$ by

$$\varphi(r) = M^2\|L_U^{-1}\|\,\|L\|\mu_G(r) \quad (r>0),$$

with M given by (12.37) *and $\mu_G(r)$ given by* (12.38). *Then the problem* (12.1) *admits a solution $x\in X$ if and only if $1/\varepsilon$ belongs to the point spectrum of the operator* (12.36). *Moreover, the asymptotic point spectrum of this operator satisfies the inclusion*

$$\sigma_{\mathrm{q}}(F) \subseteq \left\{\lambda\in\mathbb{R} : \lambda\exp(-M\|b\|_1/\lambda) \le \lim_{r\to\infty}\frac{\varphi(r)}{r}\right\}. \tag{12.40}$$

Proof. We put $\lambda := 1/\varepsilon$. The fact that every solution of the boundary value problem (12.31) solves the eigenvalue equation $F(x) = \lambda x$, and vice versa, is well known. So we only have to prove the inclusion (12.40).

Let $x\in X$ be a solution of the equation (12.2) for some $\lambda>0$ and $y\in X$. Then

$$\begin{aligned}|\lambda|\,|x(t)| &\le |y(t)| + |EG(x)(t)| + |U_0L_U^{-1}LEG(x)(t)|\\ &\le |y(t)| + M\|a\|_1 + M\int_0^t b(s)|x(s)|\,ds + M^2\|L_U^{-1}\|\,\|L\|\,|g(t,x(t))|,\end{aligned}$$

and so

$$|x(t)| \le c_r + \frac{M}{\lambda}\int_0^t b(s)|x(s)|\,ds \quad (\|x\|_\infty \le r), \tag{12.41}$$

where we have put

$$c_r := \frac{1}{\lambda}\big[\|y\|_\infty + M\|a\|_1 + M^2\|L_U^{-1}\|\,\|L\|\mu_G(r)\big].$$

Applying Gronwall's lemma to (12.41) yields

$$|x(t)| \leq c_r \exp(M\|b\|_1/\lambda) \quad (\|x\|_\infty \leq r),$$

hence

$$\lambda \exp(-M\|b\|_1/\lambda) \leq \frac{c_r\lambda}{\|x\|_\infty} + \frac{\|\lambda x - F(x)\|}{\|x\|_\infty} + \frac{M\|a\|_1}{\|x\|_\infty} + \frac{\varphi(r)}{\|x\|_\infty}$$

for $0 < \|x\|_\infty \leq r$. Passing to the limit $r \to \infty$ we conclude that $[\lambda I - F]_{\mathrm{q}} > 0$ for any λ such that

$$\lim_{r\to\infty} \frac{\varphi(r)}{r} < \lambda \exp(-M\|b\|_1/\lambda), \tag{12.42}$$

and this is precisely what we have claimed. □

The hypothesis (12.42) seems to be rather technical and hard to verify. In some cases, however, it is easily checked.

Example 12.2. Suppose that $g(t, u) = a(t)+b(t)u$ with $a, b \in L_1[0, T]$. Obviously, the growth function in this case satisfies the trivial estimate

$$\mu_G(r) \leq \|a\|_1 + \|b\|_1 r,$$

and so condition (12.42) becomes

$$M^2\|L_U^{-1}\|\,\|L\|\,\|b\|_1 < \frac{1}{\varepsilon}\exp(-M\varepsilon\|b\|_1). \tag{12.43}$$

Putting $M\|b\|_1\varepsilon =: \eta$ and $\omega(\eta) := 1/\eta e^\eta$, one may rewrite (12.43) as $\omega(\eta) > M\|L_U^{-1}\|\,\|L\|$. But the function $\omega : (0, \infty) \to (0, \infty)$ is strictly decreasing, hence invertible, with

$$\lim_{\eta\to 0+} \omega(\eta) = \infty, \qquad \lim_{\eta\to\infty} \omega(\eta) = 0.$$

Consequently, condition (12.43) is valid precisely for

$$0 < \varepsilon < \frac{\omega^{-1}(M\|L_U^{-1}\|\,\|L\|)}{M\|b\|_1}.$$

Passing to $\lambda := 1/\varepsilon$, this gives an explicit bound of type (12.40) for the asymptotic point spectrum $\sigma_{\mathrm{q}}(F)$. ♡

Example 12.3. Suppose we are interested in finding solutions of the periodic problem

$$\left.\begin{array}{l} \ddot{u}(t) + \alpha^2 u(t) = \varepsilon h(t, u(t), \dot{u}(t)) \quad (0 \leq t \leq \pi), \\ u(0) = u(\pi), \\ \dot{u}(0) = \dot{u}(\pi), \end{array}\right\} \tag{12.44}$$

where α and ε are positive parameters. We may write this problem in the form (12.31). Indeed, putting $n = 2$, $T = \pi$, and

$$x = \begin{pmatrix} u \\ \dot{u} \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ -\alpha^2 & 0 \end{pmatrix},$$
$$g(t,x) = \begin{pmatrix} 0 \\ h(t,x) \end{pmatrix}, \quad L\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \end{pmatrix},$$

with $\Delta z = z(\pi) - z(0)$, the problem (12.44) has exactly the form (12.31). Now, the operator $U(t,0)$ is here given by

$$U(t,0) = \begin{pmatrix} \cos\alpha t & \frac{1}{\alpha}\sin\alpha t \\ -\alpha\sin\alpha t & \cos\alpha t \end{pmatrix}$$

and the operator L_U by

$$L_U\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} (c-1)x_1 + \frac{s}{\alpha}x_2 \\ -\alpha s x_1 + (c-1)x_2 \end{pmatrix},$$

respectively, where we have used the shortcut $s = \sin\alpha\pi$ and $c = \cos\alpha\pi$. So in this case $L_U : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear isomorphism if and only if $\alpha \notin 2\mathbb{Z}$.

A simple calculation shows that $M = \max\{\alpha, 1/\alpha\}$ and $\|L\| = 2$. The norm $\|L_U^{-1}\|$ may be calculated as usual from the square root of the largest eigenvalue Λ of the selfadjoint matrix $L_U L_U^*$. As a result, we obtain

$$\Lambda = (c-1)^2 + \frac{s^2}{2}\left(\alpha^2 + \frac{1}{\alpha^2}\right) + \frac{s}{2}\sqrt{4(c-1)^2\left(\alpha^2 + \frac{1}{\alpha^2} - 2\right) + s^2\left(\alpha^4 - \frac{1}{\alpha^4} - 2\right)},$$

with $s = \sin\alpha\pi$ and $c = \cos\alpha\pi$ as above. The growth condition (12.39) is valid for our choice of g if

$$|h(t,u,v)| \le a(t) + b(t)\sqrt{u^2 + v^2} \quad (a, b \in L_1[0,\pi]).$$

Of course, for verifying the inequality (12.42) one has to know the nonlinearity h in (12.44). For example, if

$$h(t,u,v) = tu + 2\sqrt{|v|}, \tag{12.45}$$

then $\mu_G(r) = h(1, u_r, v_r)$, where v_r is the unique positive solution of the cubic equation $v^3 + v^2 = r^2$, and $u_r = \sqrt{r^2 - v_r^2}$. On the other hand, if

$$h(t,u,v) = tu + \log(1 + |v|), \tag{12.46}$$

then $\mu_G(r) = h(1, u_r, v_r)$, where v_r is the smallest positive solution of the fourth order equation $v^4 + 2v^3 + 2v^2 = r^2$, and $u_r = \sqrt{r^2 - v_r^2}$. ♡

Now we consider another type of boundary value problems to show how the results of Section 12.1 apply. Consider the second order differential equation

$$\ddot{x}(t) + g(t)f(x(t)) = 0, \tag{12.47}$$

subject to either the condition

$$x(0) = 0, \quad x(1) = \alpha x(\eta), \tag{12.48}$$

or the condition

$$\dot{x}(0) = 0, \quad x(1) = \alpha x(\eta), \tag{12.49}$$

where $\eta \in (0, 1)$ is fixed. The problems (12.47)/(12.48) and (12.47)/(12.49) are usually referred to as *three point boundary value problems*, inasmuch as they involve conditions on x (or $\dot{x}$) at the three arguments 0, η, and 1. Many existence results have been obtained for such problems; in particular, it is known that, when $\alpha\eta \neq 1$ in (12.48) or $\alpha \neq 1$ in (12.49), these boundary value problems may be written equivalently as Hammerstein integral equation

$$x(s) = \int_0^1 k(s, t)g(t)f(x(t))\, dt \tag{12.50}$$

where the kernel function k depends on the boundary condition (12.48) or (12.49). More precisely, the kernel (Green's function) k of (12.50) in case of the boundary condition (12.48) is given by

$$k(s, t) = \frac{s(1-t)}{1-\alpha\eta} - l(s, t; \alpha, \eta), \tag{12.51}$$

where

$$l(s, t; \alpha, \eta) := \begin{cases} \frac{\alpha s(\eta - t)}{1-\alpha\eta} + s - t & \text{if } t \le \min\{\eta, s\}, \\ \frac{\alpha s(\eta - t)}{1-\alpha\eta} & \text{if } s < t \le \eta, \\ s - t & \text{if } \eta < t \le s, \\ 0 & \text{if } t > \max\{\eta, s\}. \end{cases}$$

Similarly, the kernel (Green's function) k of (12.50) in case of the boundary condition (12.49) is given by

$$k(s, t) = \frac{1-t}{1-\alpha} - m(s, t; \alpha, \eta), \tag{12.52}$$

where

$$m(s, t; \alpha, \eta) := \begin{cases} \frac{\alpha(\eta - t)}{1-\alpha} + s - t & \text{if } t \le \min\{\eta, s\}, \\ \frac{\alpha(\eta - t)}{1-\alpha} & \text{if } s < t \le \eta, \\ s - t & \text{if } \eta < t \le s, \\ 0 & \text{if } t > \max\{\eta, s\}. \end{cases}$$

Let us show how to apply Proposition 12.1 to the three point boundary value problems (12.47)/(12.48) and (12.47)/(12.49) in the spaces $L_1[0, 1]$ and $L_2[0, 1]$. Again, we write $\|x\|_1$ for the norm of x in the space $L_1[0, 1]$, $\|x\|_2$ for its norm in $L_2[0, 1]$, and $\|x\|_\infty$ for its norm in $L_\infty[0, 1]$. We define the function κ as in (12.10), and suppose as before that $\|\kappa g\|_1 < \infty$, where g is the function in (12.47). The function f in (12.47) is assumed to be continuous and positive and to satisfy the growth condition (12.9), where we may suppose without loss of generality that the functions a and b are constant, i.e.,

$$|f(u)| \leq a + b|u|. \tag{12.53}$$

By what we have observed before, solving the three point boundary value problems (12.47)/(12.48) and (12.47)/(12.49) may be reduced to solving equation (12.5) with $\lambda = 1$ and $y(s) \equiv 0$.

Proposition 12.4. *Suppose that $\alpha\eta \neq 1$, and $f : \mathbb{R} \to \mathbb{R}$ is a Carathéodory function which satisfies the growth condition* (12.53)*. Then the boundary value problem* (12.47)/(12.48) *has at least one solution provided that*

$$\|bg\|_1 < \begin{cases} \frac{4(1-\alpha\eta)}{1-\alpha} & \text{if } \alpha\eta \leq 0, \\ \frac{4(1-\alpha\eta)}{\max\{\alpha,1\}} & \text{if } 0 < \alpha\eta < 1, \\ \frac{4(\alpha\eta-1)}{\alpha} & \text{if } \alpha\eta > 1, \end{cases} \tag{12.54}$$

or

$$\|bg\|_2 < \begin{cases} \frac{\sqrt{30}(1-\alpha\eta)}{1-\alpha} & \text{if } \alpha\eta \leq 0, \\ \frac{\sqrt{30}(1-\alpha\eta)}{\max\{\alpha,1\}} & \text{if } 0 < \alpha\eta < 1, \\ \frac{\sqrt{30}(\alpha\eta-1)}{\alpha} & \text{if } \alpha\eta > 1, \end{cases} \tag{12.55}$$

Proof. Calculating the kernel function (12.51) and the corresponding majorant (12.10) in explicit form we distinguish three cases. First, suppose that $\alpha\eta \leq 0$. Then

$$\kappa(t) = \frac{1-\alpha}{1-\alpha\eta} t(1-t),$$

hence

$$\|\kappa\|_2 = \frac{1}{\sqrt{30}} \frac{1-\alpha}{1-\alpha\eta}, \quad \|\kappa\|_\infty = \frac{1}{4} \frac{1-\alpha}{1-\alpha\eta}.$$

By Hölder's inequality, we have $\|\kappa bg\|_1 < 1$ if either $\|bg\|_1 < 1/\|\kappa\|_\infty$ or $\|bg\|_2 < 1/\|\kappa\|_2$. This leads to the estimates

$$\|bg\|_1 < \frac{4(1-\alpha\eta)}{1-\alpha}, \quad \|bg\|_2 < \frac{\sqrt{30}(1-\alpha\eta)}{1-\alpha}$$

which are the first conditions in (12.54) and (12.55).

Second, suppose that $0 < \alpha\eta < 1$. Then

$$\kappa(t) = \frac{\max\{\alpha, 1\}}{1-\alpha\eta} t(1-t),$$

hence

$$\|\kappa\|_2 = \frac{1}{\sqrt{30}} \frac{\max\{\alpha, 1\}}{1-\alpha\eta}, \quad \|\kappa\|_\infty = \frac{1}{4} \frac{\max\{\alpha, 1\}}{1-\alpha\eta}.$$

The same reasoning as above leads then to the second conditions in (12.54) and (12.55).

Finally, suppose that $\alpha\eta > 1$. Then

$$\kappa(t) = \frac{\alpha}{\alpha\eta - 1} t(1-t),$$

hence

$$\|\kappa\|_2 = \frac{1}{\sqrt{30}} \frac{\alpha}{\alpha\eta - 1}, \quad \|\kappa\|_\infty = \frac{1}{4} \frac{\alpha}{\alpha\eta - 1},$$

and we arrive at the third conditions in (12.54) and (12.55). □

Proposition 12.5. *Suppose that $\alpha \neq 1$, and $f : \mathbb{R} \to \mathbb{R}$ is a Carathéodory function which satisfies the growth condition* (12.53). *Then the boundary value problem* (12.47)/(12.49) *has at least one solution provided that*

$$\|bg\|_1 < \begin{cases} 1 & \text{if } \alpha \le 0, \\ 1-\alpha & \text{if } 0 < \alpha < 1, \\ \frac{\alpha-1}{\alpha} & \text{if } \alpha > 1, \end{cases} \tag{12.56}$$

or

$$\|bg\|_2 < \begin{cases} \sqrt{3} & \text{if } \alpha \le 0, \\ \sqrt{3}(1-\alpha) & \text{if } 0 < \alpha < 1, \\ \frac{\sqrt{3}(\alpha-1)}{\alpha} & \text{if } \alpha > 1. \end{cases} \tag{12.57}$$

Proof. The proof goes exactly as that of Proposition 12.4, taking into account the fact that

$$\kappa(t) = 1 - t, \quad \|\kappa\|_2 = \frac{1}{\sqrt{3}}, \quad \|\kappa\|_\infty = 1$$

in case $\alpha \le 0$,

$$\kappa(t) = \frac{1}{1-\alpha}(1-t), \quad \|\kappa\|_2 = \frac{1}{\sqrt{3}} \frac{1}{1-\alpha}, \quad \|\kappa\|_\infty = \frac{1}{1-\alpha}$$

in case $0 < \alpha < 1$, and

$$\kappa(t) = \frac{\alpha}{\alpha-1}(1-t), \quad \|\kappa\|_2 = \frac{1}{\sqrt{3}} \frac{\alpha}{\alpha-1}, \quad \|\kappa\|_\infty = \frac{\alpha}{\alpha-1}$$

in case $\alpha > 1$. □

To conclude this section, we briefly sketch how to apply Theorem 12.4 in a very simple situation. Suppose that $f\colon [0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous and satisfies the homogeneity condition

$$f(t,cu,cv)\equiv cf(t,u,v)\quad (0\le t\le 1;\ u,v\in\mathbb{R}). \tag{12.58}$$

Consider the quasilinear three point boundary value problem

$$\left.\begin{array}{l}\ddot{x}(t)=\mu f(t,x(t),\dot{x}(t))+y(t),\\ x(0)=0,\quad x(1)=\alpha x(\eta),\end{array}\right\} \tag{12.59}$$

where $y\in C[0,1]$ is given and $\mu\neq 0$. We are interested in solutions x of (12.59) in the Sobolev space $W_1^2[0,1]$ of all absolutely continuous functions x such that $\dot{x}$ is also absolutely continuous and $\ddot{x}\in L_1[0,1]$.

Proposition 12.6. *Let $\alpha\eta\neq 1$. Then the following alternative holds.*

(a) *Either the boundary value problem* (12.59) *has a solution for $\mu=1$ and any function $y\in L_1[0,1]$;*

(b) *or there exists some $\mu\le 1$ such that the boundary value problem* (12.59) *has a nontrivial solution for $y(t)\equiv 0$.*

Proof. We apply Theorem 12.4 for the space

$$X=\{x\in W_1^2[0,1]: x(0)=0,\ x(1)=\alpha x(\eta)\},$$

equipped with the norm $\|x\|_X:=\max\{\|x\|_1,\|\dot{x}\|_1\}$, the space $Y=L_1[0,1]$, and the operators $Lx=\ddot{x}$ and

$$F(x)(t)=f(t,x(t),\dot{x}(t))\quad (x\in X). \tag{12.60}$$

From the existence of the Green's function for L one may deduce that the operator $L\colon X\to Y$ is invertible. More precisely, (12.51) shows that the inverse to L is given explicitly in the form

$$L^{-1}y(s)=\int_0^s(s-t)y(t)\,dt+\frac{\alpha s}{1-\alpha\eta}\int_0^\eta(\eta-t)y(t)\,dt-\frac{s}{1-\alpha\eta}\int_0^1(1-t)y(t)\,dt. \tag{12.61}$$

Moreover, it follows from classical imbedding theorems that the nonlinear operator (12.60) is L-compact.

Suppose that $1\in\rho_{\mathrm{F}}(L,F)$. Then the operator $L-F$ is epi on every ball, and so the problem (12.59) has a solution for $\mu=1$ and each $y\in L_1[0,1]$. On the other hand, suppose that $1\in\sigma_{\mathrm{F}}(L,F)$. Since $[K_{PQ}F]<1$, Theorem 12.4 implies then that there exists some $\mu\in(0,1]$ such that $1/\mu\in\sigma_{\mathrm{p}}(L,(I-Q)F)=\sigma_{\mathrm{p}}(L,F)$. Consequently, the equation

$$Lx(t)-\mu F(x)(t)=\ddot{x}(t)-\mu f(t,x(t),\dot{x}(t))=0$$

has some nontrivial solution $x\in W_1^2[0,1]$ which satisfies the boundary conditions $x(0)=0$ and $x(1)=\alpha x(\eta)$. □

Of course, a similar alternative holds for the three point boundary conditions (12.49) in case $\alpha \neq 1$.

The homogeneity assumption (12.58) is very special and considerably restricts the applicability of Proposition 12.6. It turns out, however, that one may prove solvability for (12.59) under mild growth condition for the function $f(t, \cdot, \cdot)$, simply by using the estimate (6.12) for the Furi–Martelli–Vignoli spectral radius. To this end, we suppose that

$$|f(t,u,v)| \leq p(t)|u| + q(t)|v| + r(t) \quad (0 \leq t \leq 1;\ u, v, \in \mathbb{R}) \tag{12.62}$$

for suitable functions $p, q, r \in L_1[0, 1]$, i.e., the nonlinearity $f(t, \cdot, \cdot)$ is of sublinear growth.

Proposition 12.7. *Let* $\alpha\eta \neq 1$. *Then the boundary value problem* (12.59) *has a solution for any function* $y \in L_1[0, 1]$, *provided that*

$$|\mu|(\|p\|_1 + \|q\|_1) < \frac{1}{c(\alpha, \eta)}, \tag{12.63}$$

where

$$c(\alpha,\eta) := 1 + \frac{\alpha\eta + 1}{|1 - \alpha\eta|} = \begin{cases} \frac{2}{1-\alpha\eta} & \text{if } \alpha\eta < 1, \\ \frac{2\alpha\eta}{\alpha\eta - 1} & \text{if } \alpha\eta > 1. \end{cases}$$

Proof. Let X, Y, L and F as in Proposition 12.6. Using the explicit representation (12.61) of L^{-1} one may prove for $x = L^{-1}y$ the estimates

$$\|x\|_1 \leq c(\alpha,\eta)\|y\|_1, \quad \|\dot{x}\|_1 \leq c(\alpha,\eta)\|y\|_1$$

which imply that $\|L^{-1}\| \leq c(\alpha, \eta)$.

Now we use Theorem 6.3 for $J = I$ and F replaced by $L^{-1}F$. Clearly, $[L^{-1}F]_{\mathrm{A}} = 0$, so we have only to calculate the quasinorm $[L^{-1}F]_{\mathrm{Q}}$ of $L^{-1}F$. But from (12.62) and (12.63) it follows that

$$\begin{aligned} [L^{-1}F]_{\mathrm{Q}} &\leq \|L^{-1}\|\,[F]_{\mathrm{Q}} \\ &\leq c(\alpha,\eta) \limsup_{\|x\|_X \to \infty} \frac{\|p\|_1\|x\|_\infty + \|q\|_1\|\dot{x}\|_\infty + \|r\|_1}{\|x\|_X} \\ &\leq c(\alpha,\eta) \limsup_{\|x\|_X \to \infty} \frac{(\|p\|_1 + \|q\|_1)\|\dot{x}\|_\infty + \|r\|_1}{\|x\|_X} \\ &\leq c(\alpha,\eta)\,(\|p\|_1 + \|q\|_1) \\ &< \frac{1}{|\mu|}. \end{aligned}$$

So $\lambda = 1/\mu \notin \sigma_{\mathrm{FMV}}(L^{-1}F)$, by (6.12), and the proof is complete. □

Let us return to the equation

$$\ddot{x}(t) = \mu f(t, x(t), \dot{x}(t)) + y(t), \tag{12.64}$$

where now $f\colon [0,1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $y\colon [0,1] \to \mathbb{R}^n$ are supposed to be continuous vector functions, and $\mu \neq 0$ as before. We consider (12.64) with either the classical boundary condition

$$x(0) = x(1) = 0, \tag{12.65}$$

or the periodic boundary condition

$$x(0) = x(1), \quad \dot{x}(0) = \dot{x}(1). \tag{12.66}$$

It is a well-known fact that these two boundary conditions lead to completely different problems: indeed, the case (12.65) may be studied by means of the Feng or FMV-spectrum, while the case (12.66) requires the semilinear spectra introduced in Sections 9.1 and 9.2.

To see this, we have to define appropriate spaces and operators. Consider first the problem (12.64)/(12.65). We put

$$X := \{x \in C^2[0,1] : x(0) = x(1) = 0\}, \quad Y := C[0,1],$$

and define operators $L, F\colon X \to Y$ by

$$Lx(t) := \ddot{x}(t), \quad F(x)(t) := f(t, x(t), \dot{x}(t)). \tag{12.67}$$

Then L is invertible on Y with inverse

$$L^{-1}y(s) = \int_0^1 k(s,t)y(t)\,dt, \tag{12.68}$$

where

$$k(s,t) = \begin{cases} s(t-1) & \text{if } 0 \le s \le t \le 1, \\ t(s-1) & \text{if } 0 \le t \le s \le 1 \end{cases}$$

is the classical Green's function of L. Consequently, in the terminology of Section 12.2 we have $X_0 = X$, $Y_0 = \{\theta\}$, $Px = Qy \equiv \theta$, and $K_{PQ} = L^{-1}$ is given by (12.68). So the solvability of equation (12.17) simply reduces to the solvability of the classical eigenvalue equation

$$\lambda x - L^{-1}F(x) = z \quad (z \in X).$$

The situation changes drastically if we consider the boundary condition (12.66) instead of (12.65). In fact, in this case the operator L is *not invertible*. Put

$$X := \{x \in C^2[0,1] : x(0) = x(1),\ \dot{x}(0) = \dot{x}(1)\}, \quad Y := C[0,1],$$

and define operators $L, F\colon X \to Y$ again by (12.67). Now we have

$$N(L) = \{x \in X : x(t) \equiv \text{const.}\} \cong \mathbb{R}^n$$

and

$$R(L) = \{y \in Y : Qy = \theta\} \cong Y/\mathbb{R}^n,$$

where

$$Qy = \int_0^1 y(t)\,dt. \tag{12.69}$$

Moreover, for the projection $P\colon X \to \mathbb{R}$ we may choose $Px = x(0)$. So we have the decomposition $X = \mathbb{R}^n \oplus X_0$ and $Y = \mathbb{R}^n \oplus Y_0$, and so $\dim N(L) = \operatorname{codim} R(L) = n$, which shows that L is a Fredholm operator of index zero.

The operator $L_P^{-1} = (L|_{X_0})^{-1}\colon R(L) \to X_0$ is now the restriction of (12.68) to the range $R(L)$ of L. The linear operators $\Pi\colon Y \to Y/R(L)$ and $\Lambda\colon Y/R(L) \to N(L)$ are given by

$$\Pi y = [y] := \{\tilde{y} \in Y : Q\tilde{y} = Qy\}, \quad \Lambda[y] = Qy.$$

So as canonical homeomorphism $h\colon Y/R(L) \to Y_0$ we may choose $h[y] = Qy$. So the linear isomorphism $L + h\Lambda^{-1}P\colon X \to Y$ mentioned in Lemma 9.1 is here

$$(L + h\Lambda^{-1}P)x(t) = \ddot{x}(t) + x(0), \tag{12.70}$$

and its inverse $\Lambda\Pi + K_{PQ} = \Lambda\Pi + L_P^{-1}(I - Q)\colon Y \to X$ is

$$(\Lambda\Pi + K_{PQ})y(s) = \int_0^1 k(s,t)y(t)\,dt + \left(1 - \int_0^1 k(s,t)\,dt\right)\int_0^1 y(t)\,dt. \tag{12.71}$$

With this choice of spaces and operators, one may then apply one of the Theorems 12.6–12.8 to problem (12.64)/(12.66).

12.4 Bifurcation and asymptotic bifurcation points

The notion of bifurcation is fundamental in nonlinear analysis; indeed, it is also one of the historical roots of nonlinear eigenvalue theory. We do not present here a general view of bifurcation theory, but restrict ourselves to some very special aspects related to nonlinear spectra.

Let X be a real Banach space. A scalar $\lambda \in \hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ is called a *zero bifurcation point* of an operator $F\colon X \to X$ if there exist sequences $(\lambda_n)_n$ in $\mathbb{R}$ and $(x_n)_n$ in $X \setminus \{\theta\}$ such that

$$F(x_n) = \lambda_n x_n, \quad \lambda_n \to \lambda, \quad \|x_n\| \to 0. \tag{12.72}$$

If the last condition in (12.72) is replaced by $\|x_n\| \to \infty$, then λ is called an *asymptotic bifurcation point*. Thus, a zero bifurcation point may be approximated by eigenvalues with eigenvectors tending to zero, while an asymptotic bifurcation point may be approximated by eigenvalues with unbounded eigenvectors. We write $B_0(F)$ for the set of all zero bifurcation points of F and $B_\infty(F)$ for the set of all asymptotic bifurcation points of F. In the following lemma we give some connections with spectral sets.

Lemma 12.1. *The sets $B_0(F)$ and $B_\infty(F)$ are closed. Moreover,*

$$B_0(F) \cap \mathbb{R} \subseteq \sigma_{\mathrm{b}}(F) \tag{12.73}$$

and

$$B_\infty(F) \cap \mathbb{R} \subseteq \sigma_{\mathrm{q}}(F). \tag{12.74}$$

Proof. The closedness of $B_0(F)$ and $B_\infty(F)$ is obvious. To prove (12.73), let $\lambda_n \to \lambda$, $\|x_n\| \to 0$, and $F(x_n) = \lambda_n x_n$. Then

$$[\lambda I - F]_{\mathrm{b}} \leq \inf_{n \in \mathbb{N}} \frac{\|\lambda x_n - F(x_n)\|}{\|x_n\|} \leq \inf_{n \in \mathbb{N}} |\lambda - \lambda_n| = 0,$$

hence $\lambda \in \sigma_{\mathrm{b}}(F)$. The proof for the inclusion (12.74) is analogous. □

It may happen that an operator F has no zero or asymptotic bifurcation points at all. For example, for the operator (3.16) in $X = \mathbb{C}^2$ we have $B_0(F) = B_\infty(F) = \emptyset$, because $F(x) = \lambda x$ implies $x = \theta$ for any λ. The next lemma gives a sufficient condition for the existence of bifurcation points.

Lemma 12.2. *Let X be an infinite dimensional real Banach space, and suppose that $F \in \mathfrak{A}(X) \cap \mathfrak{B}(X)$ satisfies*

$$[F]_{\mathrm{A}} < [F]_{\mathrm{b}}. \tag{12.75}$$

Then $B_0(F) \cap \mathbb{R} \neq \emptyset$. Similarly, suppose that $F \in \mathfrak{A}(X) \cap \mathfrak{Q}(X)$ satisfies

$$[F]_{\mathrm{A}} < [F]_{\mathrm{q}}. \tag{12.76}$$

Then $B_\infty(F) \cap \mathbb{R} \neq \emptyset$.

Proof. Suppose first that $[F]_{\mathrm{A}} < 1 < [F]_{\mathrm{b}}$, i.e., F is α-contractive, and $\|F(x)\| \geq \|x\|$ for all $x \in X$. Define $F_n \colon S_{1/n}(X) \to S_{1/n}(X)$ by

$$F_n(x) := \frac{1}{n} \frac{F(x)}{\|F(x)\|} \quad (x \in S_{1/n}(X)).$$

Then $[F_n|_{S_{1/n}(X)}]_{\mathrm{A}} = [F]_{\mathrm{A}} < 1$. By Theorem 2.5, we find $x_n \in S_{1/n}(X)$ with $F_n(x_n) = x_n$, hence

$$F(x_n) = n\|F(x_n)\| F_n(x_n) = n\|F(x_n)\| x_n = \lambda_n x_n$$

with $\lambda_n := n\|F(x_n)\|$. Obviously, $\|x_n\| \to 0$ as $n \to \infty$. Moreover, since

$$|\lambda_n| = \frac{\|F(x_n)\|}{\|x_n\|} \leq [F]_{\mathrm{B}} < \infty,$$

the sequence $(\lambda_n)_n$ is bounded. Passing to a subsequence, if necessary, we may therefore assume that $\lambda_n \to \lambda$ as $n \to \infty$. Obviously, the limit λ belongs to $B_0(F)$.

Now suppose the (12.75) holds. Fix $c \in ([F]_{\mathrm{A}}, [F]_{\mathrm{b}})$ and put $F_c(x) := \frac{1}{c}F(x)$. Then $[F_c]_{\mathrm{A}} = \frac{1}{c}[F]_{\mathrm{A}} < 1$ and $[F_c]_{\mathrm{b}} = \frac{1}{c}[F]_{\mathrm{b}} > 1$, and so we find $\lambda \in B_0(F_c)$, by what we have proved before. But then $c\lambda \in B_0(F)$, and so we are done.

The assertion (12.76) is proved analogously (and, in fact, has already been proved in Theorem 6.8). □

The example of the operator (3.16) shows that Lemma 12.2 is false in finite dimensional spaces.

To illustrate Lemma 12.2, consider the Hammerstein integral operator

$$H(x)(s) = \int_0^1 k(s,t) f(x(t))\,dt \quad (0 \leq s \leq 1), \tag{12.77}$$

where $k\colon [0,1] \times [0,1] \to [0,\infty)$ and $f\colon \mathbb{R} \to [0,\infty)$ are continuous and nonnegative, and

$$M := \max_{0 \leq s \leq 1} \int_0^1 k(s,t)\,dt > 0.$$

Concerning the function f, we collect the set of additional assumptions

$$f(0) > \frac{1}{M}, \tag{12.78}$$

$$d := \inf_{u \geq 0} f(u) > 0, \tag{12.79}$$

and

$$D := \sup_{u \geq 0} f(u) < \infty. \tag{12.80}$$

Proposition 12.8. *If either* (12.78) *or* (12.79) *is satisfied, then H has the zero bifurcation point $\lambda = \infty$. If* (12.79) *and* (12.80) *are both satisfied, then H has the asymptotic bifurcation point $\lambda = 0$.*

Proof. We study the operator (12.77) in the space $X = C[0,1]$. Suppose that f satisfies (12.78). For every $r > 0$, consider the 1-homogeneous compact operator $H_r\colon X \to X$ defined by

$$H_r(x) = \begin{cases} \|x\| H\left(\frac{r}{\|x\|}x\right) & \text{if } x \neq \theta, \\ \theta & \text{if } x = \theta. \end{cases} \tag{12.81}$$

We claim that $1 \in \sigma_F(H_r)$ for some $r > 0$. Otherwise the operator $I - H_r$ would be onto for each $r > 0$, and so we would find a function $x \in X$ such that $x(t) - H_r(x)(t) \equiv 1$. Obviously, $x(t) \geq 1$ and $\|x\| H_r(x)(s) < x(s)$ for $0 \leq s \leq 1$. Consequently,

$$\frac{x(s)}{\|x\|} > \int_0^1 k(s,t) f\left(\tfrac{r}{\|x\|} x(t)\right) dt. \tag{12.82}$$

From (12.78) and the continuity of f it follows that we can find some $\delta > 0$ such that $f(u) > 1/M$ for $|u| \leq \delta$. Put

$$r_n := \frac{\delta}{n}, \quad x_n(t) := \frac{r_n}{\|x\|} x(t).$$

Then $\|x_n\| = r_n \leq \delta$, hence

$$f(x_n(t)) > \frac{1}{M} \quad (n = 1, 2, 3, \ldots),$$

and so

$$\max_{0 \leq s \leq 1} \int_0^1 k(s,t) f(x_n(t))\, dt > \frac{1}{M} \max_{0 \leq s \leq 1} \int_0^1 k(s,t)\, dt = 1.$$

Combining this with (12.82) we obtain $x(s) > \|x\|$, an obvious contradiction. So we have proved that $1 \in \sigma_F(H_r)$ for some $r > 0$.

From Theorem 12.4 (with $\lambda = 1$, $F = H_r$, and $L = I$) it follows that there exist scalars $\lambda_n \geq 1$ and elements $y_n \in X \setminus \{\theta\}$ such that $H_{r_n}(y_n) = \lambda_n y_n$. Putting $z_n := r_n y_n / \|y_n\|$ we get

$$\lambda_n y_n = H_{r_n}(y_n) = \|y_n\| H\left(\frac{r_n}{\|y_n\|} y_n\right) = \|y_n\| H(z_n),$$

hence

$$H(z_n) = \frac{\lambda_n}{\|y_n\|} y_n = \frac{\lambda_n}{r_n} z_n.$$

Since $r_n \to 0$ we have

$$\|z_n\| = r_n \to 0, \quad \frac{\lambda_n}{r_n} \geq \frac{1}{r_n} \to \infty,$$

i.e., $\infty \in B_0(H)$.

Suppose now that f satisfies (12.79). Denote by K the cone of nonnegative functions in $X = C[0, 1]$, and consider the operator (12.81) from K into K. For $x \in K \cap S(X)$ we have $H_r(x) = H(rx)$, hence

$$\|H_r(x)\| = \max_{0 \leq s \leq 1} \int_0^1 k(s,t) f(rx(t))\, dt \geq d \max_{0 \leq s \leq 1} \int_0^1 k(s,t)\, dt = Md > 0.$$

Now we apply Theorem 10.4 with $\Omega = B^o(X)$, $F = H_r$, and $\delta = Md$. So we find some $\lambda_r \geq Md$ and $x_r \in K \cap S(X)$ such that $z_r := rx_r$ satisfies

$$H(z_r) = H_r(x_r) = \lambda_r x_r = \frac{\lambda_r}{r} z_r \tag{12.83}$$

as before. Letting $r \to 0$ in (12.83) we see that

$$\|z_r\| = r \to 0, \quad \frac{\lambda_r}{r} \geq \frac{Md}{r} \to \infty,$$

i.e., $\infty \in B_0(H)$ as before.

Finally, suppose that f satisfies both (12.79) and (12.80), and choose λ_r and z_r as in (12.83). Since

$$\lambda_r = \|H(z_r)\| = \|H(rx_r)\| = \max_{0 \leq s \leq 1} \int_0^1 k(s,t) f(rx(t))\, dt \leq MD,$$

letting $r \to \infty$ in (12.83) we conclude that

$$\|z_r\| = r \to \infty, \quad \frac{\lambda_r}{r} \leq \frac{MD}{r} \to 0,$$

and so $0 \in B_\infty(H)$. The proof is complete. □

The following theorem provides a connection between the asymptotic bifurcation points of an asymptotically linear operator F, on the one hand, and certain isolated eigenvalues of its asymptotic derivative $F'(\infty)$, on the other. As we have seen in Theorem 6.9, the FMV-spectrum $\sigma_{\mathrm{FMV}}(F)$ of a *compact* asymptotically linear operator F is at most countable, and every non-zero spectral value belongs to $\sigma_{\mathrm{p}}(F'(\infty)) \cap \sigma_{\mathrm{q}}(F)$. In the following Theorem 12.9 the operator F is *not* supposed to be compact, but only the difference $F - F'(\infty)$. We recall the definition of the various essential spectra of a linear operator in Section 1.4; in particular, the essential spectrum $\sigma_{\mathrm{eb}}(L)$ of $L \in \mathfrak{L}(X)$ in Browder's sense uses the multiplicity (1.20).

Theorem 12.9. *Let X be a real Banach space and $F\colon X \to X$ be asymptotically linear with asymptotic derivative $F'(\infty) =: T$. Assume that $F - T$ is compact. Let $\lambda_0 \in \sigma_{\mathrm{p}}(T)$ be such that $|\lambda| > r_{\mathrm{eb}}(T)$, where $r_{\mathrm{eb}}(T)$ denotes the radius of the essential spectrum* (1.74). *Moreover, suppose that the multiplicity*

$$n(\lambda_0; T) = \dim \bigcup_{k=1}^{\infty} N((\lambda_0 I - T)^k)$$

of λ_0 as an eigenvalue of T is odd. Then $\lambda_0 \in B_\infty(F)$.

Proof. As in Section 1.5 we denote by $X_{\mathbb{C}}$ the complexification of X, endowed with the projective tensor norm. Similarly, by $T_{\mathbb{C}}: X_{\mathbb{C}} \to X_{\mathbb{C}}$ we denote the complexification of T defined by $T_{\mathbb{C}}(x+iy) = Tx+iTy$. Since $\sigma_{\mathrm{p}}(T_{\mathbb{C}}) = \sigma_{\mathrm{p}}(T)$ and $r_{\mathrm{eb}}(T_{\mathbb{C}}) = r_{\mathrm{eb}}(T)$, it follows that λ_0 is a finite dimensional isolated eigenvalue of T. We may therefore choose $\varepsilon > 0$ such that $r_{\mathrm{eb}}(T) + 2\varepsilon < |\lambda_0|$, and thus $|\lambda_0 + 2\varepsilon| > r_{\mathrm{eb}}(T) + \varepsilon$.

By Leggett's theorem (see Section 1.5) we may assume, possibly after passing to an equivalent norm on $X_{\mathbb{C}}$, that $[T_{\mathbb{C}}]_{\mathrm{A}} < r_{\mathrm{eb}}(T) + \varepsilon$. So also in the space X itself we have

$$[F]_{\mathrm{A}} \leq [F - T]_{\mathrm{A}} + [T]_{\mathrm{A}} = [T]_{\mathrm{A}} < r_{\mathrm{eb}}(T) + \varepsilon. \tag{12.84}$$

Our choice of ε implies that

$$\beta := [(\lambda_0 \pm \varepsilon)I - T]_{\mathrm{b}} > 0. \tag{12.85}$$

Moreover, from $[F - T]_{\mathrm{Q}} = 0$ it follows that we can find $n_0 \in \mathbb{N}$ such that

$$\|F(x) - Tx\| \leq \frac{1}{2}\beta\|x\| \quad (\|x\| \geq n_0). \tag{12.86}$$

Combining (12.85) and (12.86) we see that, for any $n \geq n_0$,

$$\|(\lambda_0 \pm \varepsilon)x + H(x,t)\| > \frac{1}{2}\beta\|x\| \quad (x \in S_n(X),\ 0 \leq t \leq 1), \tag{12.87}$$

where we have put $H(x,t) := (1-t)Tx + tF(x)$.

For any $\lambda \in \mathbb{R}$ with $|\lambda| \geq r_{\mathrm{eb}}(T) + \varepsilon$ we have $[F/\lambda]_{\mathrm{A}} = [T/\lambda]_{\mathrm{A}} < 1$, by (12.84). So (12.87) implies that the Nussbaum–Sadovskij degree of both $I - F/\lambda$ and $I - T/\lambda$ on $B_n^o(X)$ is well-defined for each $n \geq n_0$. In particular, by the homotopy invariance of this degree we obtain

$$\begin{aligned}\deg(I - (\lambda_0 \pm \varepsilon)^{-1}F, B_n^o(X), \theta) &= \deg(I - (\lambda_0 \pm \varepsilon)^{-1}H(\,\cdot\,,1), B_n^o(X), \theta)\\ &= \deg(I - (\lambda_0 \pm \varepsilon)^{-1}H(\,\cdot\,,0), B_n^o(X), \theta)\\ &= \deg(I - (\lambda_0 \pm \varepsilon)^{-1}T, B_n^o(X), \theta).\end{aligned}$$

Moreover, a well-known formula for calculating the degree of a linear operator states that

$$\deg(I - (\lambda_0 \pm \varepsilon)^{-1}T, B_n^o(X), \theta) = (-1)^{\nu}, \tag{12.88}$$

where

$$\nu = \sum_{\mu > 1} n(\mu; (\lambda_0 \pm \varepsilon)^{-1}T)$$

is the sum of the multiplicities of all $\mu \in \sigma_{\mathrm{p}}((\lambda_0 \pm \varepsilon)^{-1}T)$ greater than 1. From (12.88) we conclude that

$$\deg(I - (\lambda_0 + \varepsilon)^{-1})F, B_n^o(X), \theta) \neq \deg(I - (\lambda_0 - \varepsilon)^{-1})F, B_n^o(X), \theta),$$

and so we can find $x_n \in S_n(X)$ and $t_n \in (0, 1)$ such that

$$t_n(\lambda_0 - \varepsilon)x_n + (1 - t_n)(\lambda_0 + \varepsilon)x_n - F(x_n) = \theta.$$

In other words, we have $F(x_n) = \lambda_n x_n$ with $\lambda_n := \lambda_0 + (1 - 2t_n)\varepsilon$. Since ε may be chosen arbitrarily small, we see that λ_0 is an asymptotic bifurcation point of F, and the proof is complete. □

12.5 The *p*-Laplace operator

In this section we briefly discuss an application of our results to nonlinear partial differential equations. As usual, we write now $u = u(x)$ for a real function of several variables. Let $2 \leq p < \infty$. Given some bounded smooth domain $G \subset \mathbb{R}^n$, we are interested in the eigenvalue problem

$$\left.\begin{array}{ll} -\Delta_p u(x) = \mu |u(x)|^{p-2} u(x) & \text{in } G, \\ u(x) \equiv 0 & \text{on } \partial G. \end{array}\right\} \tag{12.89}$$

Here

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u) = \sum_{i=1}^{n} D_i(|D_i u|^{p-2} D_i u), \tag{12.90}$$

where $D_i u = \partial u / \partial x_i$ as usual, is the so-called p-Laplace operator. Consider the reflexive Sobolev space $X := W_p^{1,0}(G)$ with norm

$$\|u\| = \left(\int_G \sum_{i=1}^{n} |D_i u(x)|^p \, dx \right)^{1/p}$$

and its dual $X^* = W_{p/(p-1)}^{-1}(G)$. As usual we pass from the problem of finding classical solutions of $-\Delta_p u = f$ to that of finding weak solutions of the generalized problem $a(u, v) = b(v)$ $(v \in X)$, where

$$a(u, v) = \int_G \Big(\sum_{i=1}^{n} |D_i u(x)|^{p-2} D_i u(x) D_i v(x) \Big) \, dx$$

and

$$b(v) = \int_G f(x) v(x) \, dx.$$

The generalized problem is equivalent to the operator equation $J(u) = b$, where $\langle J(u), v \rangle = a(u, v)$, and $J : X \to X^*$ is continuous, strictly monotone, coercive, and bounded. From Minty's fundamental theorem on monotone operators it follows that J is a homeomorphism between X and X^*.

The right-hand side of (12.89) in turn defines a Nemytskij operator

$$F(u) := |u|^{p-2}u \tag{12.91}$$

which we may also consider between X and X^*. In this way, we may rewrite the eigenvalue problem (12.89) in the form

$$F(u) = \lambda J(u) \tag{12.92}$$

with $\lambda = 1/\mu$.

We remark that the eigenvalue theory for the problem (12.89) has many features in common with the classical *linear* eigenvalue problem

$$\left.\begin{array}{ll} -\Delta u(x) = \mu u(x) & \text{in } G, \\ u(x) \equiv 0 & \text{on } \partial G. \end{array}\right\} \tag{12.93}$$

which is a special case of (12.89) for $p = 2$. For instance, the first eigenvalue μ_1 of (12.89) is always positive and simple and may be "calculated" as Rayleigh quotient

$$\mu_1 = \inf_{\substack{u\in W_p^{1,0}(G)\\ u\neq\theta}} \frac{\displaystyle\int_G |\nabla u(x)|^p\,dx}{\displaystyle\int_G |u(x)|^p\,dx}. \tag{12.94}$$

Equivalently, μ_1 may be viewed as best possible constant $C > 0$ in the Poincaré inequality

$$\int_G |\nabla u(x)|^p\,dx \le C\int_G |u(x)|^p\,dx \quad (u \in W_p^{1,0}(G)). \tag{12.95}$$

Moreover, the corresponding eigenfunction $u_1 \in W_p^{1,0}(G)$ is positive on G and simple (in the sense that any other eigenfunction is a scalar multiple of u_1). This function has the same "variational characterization" as in the linear case $p = 2$: it minimizes the functional $\Psi_p\colon W_p^{1,0}(G) \to \mathbb{R}$ defined by

$$\Psi_p(u) := \frac{1}{p}\int_G |\nabla u(x)|^p\,dx,$$

subject to the constraint

$$\frac{1}{p}\int_G |u(x)|^{p-2}u(x)\,dx = 1.$$

We also point out that there is a weak form of a nonlinear Fredholm alternative which states that, for $\mu < \mu_1$, the operator $J - \mu F = \mu(\lambda J - F)$ is surjective, but for $\mu = \mu_1$ it is not. We show now how to obtain some related results building on the spectral theory for homogeneous operators developed in Section 9.6. First of all, we state some kind of nonlinear Fredholm alternative for pairs of homogeneous operators which is just a reformulation of Theorems 9.11 and 9.12. For the definition of the characteristics occurring in this theorem we refer to (9.45), (9.54) and (9.56).

Theorem 12.10 (Nonlinear Fredholm alternative). *Suppose that $J : X \to Y$ is an odd τ-homogeneous homeomorphism with $[J]_{\mathrm{a}}^{\tau} > 0$, and $F : X \to Y$ is odd, τ-homogeneous and compact. Let $\lambda \neq 0$. Then the following assertions are equivalent.*

(a) *The eigenvalue problem* (12.92) *has only the trivial solution $u = \theta$.*

(b) *The operator $\lambda J - F$ is τ-stably solvable, $[\lambda J - F]_{\mathrm{a}}^{\tau} > 0$, and $[\lambda J - F]_{\mathrm{q}}^{\tau} > 0$.*

(c) *The operator $\lambda J - F$ is epi on $\overline{\Omega}$ for each $\Omega \in \mathfrak{OBC}(X)$, $[\lambda J - F]_{\mathrm{a}}^{\tau} > 0$, and $[\lambda J - F]_{\mathrm{b}}^{\tau} > 0$.*

(d) *The operator $\lambda J - F$ is (k, τ)-epi on $\overline{\Omega}$ for some $\Omega \in \mathfrak{OBC}(X)$ for sufficiently small $k > 0$, and*

$$\inf_{u \in \partial\Omega} \|\lambda J(u) - F(u)\| > 0.$$

To apply Theorem 12.10 we need some further properties of the operators J and F. Since

$$(|s|^{p-2}s - |t|^{p-2}t)(s - t) \geq c|s - t|^p \quad (s, t \in \mathbb{R})$$

for some $c > 0$ (it is here that we need the restriction $p \geq 2$!), we see that J satisfies the condition

$$\langle J(u) - J(v), u - v\rangle \geq c\|u - v\|^p. \tag{12.96}$$

This condition implies that

$$\|J(u) - J(v)\| \geq c\|u - v\|^{p-1}, \tag{12.97}$$

i.e., $[J]_{\mathrm{q}}^{p-1} \geq c$ and $[J]_{\mathrm{b}}^{p-1} \geq c$. Consequently, the inverse operator $J^{-1} : X^* \to X$ satisfies the Hölder-type condition

$$\|J^{-1}(f) - J^{-1}(g)\| \leq c^{-1/(p-1)}\|f - g\|^{1/(p-1)}. \tag{12.98}$$

We summarize the properties of the two operators J and F with the following

Lemma 12.3. *The operators J and F satisfy the hypotheses of Theorem* 12.10 *for $X = W_p^{1,0}(G)$, $Y = X^* = W_{p/(p-1)}^{-1}(G)$, and $\tau = p - 1$.*

Proof. We already know that J is a homeomorphism between X and Y. This implies that

$$\left([J]_{\mathrm{a}}^{\tau}\right)^{1/\tau} = \frac{1}{[J^{-1}]_{\mathrm{A}}^{1/\tau}}, \tag{12.99}$$

by Proposition 2.4 (f). On the other hand, from the Hölder condition (12.98) it follows that

$$[J^{-1}]_{\mathrm{A}}^{1/\tau} \leq c^{-1/\tau} \tag{12.100}$$

with $\tau = p-1$. Combining (12.99) and (12.100) we obtain $[J]_{\mathrm{a}}^{\tau} \geq c > 0$ as claimed.

Clearly, both J and F are odd and τ-homogeneous for $\tau = p - 1$. It remains to prove that $F : X \to X^*$ is compact. But from the classical Krasnosel'skij theorem on Nemytskij operators between Lebesgue spaces it follows that F is continuous and bounded from $L_p(G)$ into $L_p^*(G) = L_{p/(p-1)}(G)$, and hence continuous and compact between X and X^*, by the compactness of the embedding $X \hookrightarrow L_p(G)$. □

In view of Theorem 12.10, we thus obtain from Theorem 9.11 and Theorem 9.12 the following two results.

Theorem 12.11 (Discreteness theorem for Δ_p). *For the operators F and J and $\tau = p - 1$, the equalities* (9.82) *are true, i.e., all spectra defined in Section* 9.6 *coincide with the classical point spectrum $\sigma_{\mathrm{p}}(J, F)$ of problem* (12.89).

Theorem 12.12 (Fredholm alternative for Δ_p). *If $\mu \neq 0$ is not a classical eigenvalue of problem* (12.89)*, then there is some $k > 0$ such that the operator $\lambda J - F$, with $\lambda = 1/\mu$, is both (k, τ)-stably solvable and (k, τ)-epi on $\overline{\Omega}$ for every $\Omega \in \mathfrak{OBC}(W_p^{1,0}(G))$.*

In fact, a scrutiny of the proof of Theorem 12.12 exhibits that $\lambda J - F$ is (k, τ)-epi on each $\Omega \in \mathfrak{OBC}(W_p^{1,0}(G))$ for any $k < c$, where c is the constant in (12.96). Likewise, $\lambda J - F$ is (k, τ)-stably solvable for any positive $k < c$ satisfying

$$\inf_{\|u\|=1} \|\lambda J(u) - F(u)\| < \frac{1}{k}, \tag{12.101}$$

where the infimum in (12.101) is positive because $\lambda \notin \sigma_{\mathrm{b}}^{\tau}(J, F)$, see (9.61).

12.6 Notes, remarks and references

As we pointed out in the Introduction, we did not present a systematic account of applications in this chapter, but just some sample results to illustrate the usefulness of spectral methods. The necessity of replacing equations (12.2) by the more general equation (12.3) is motivated by many examples in this chapter. We remark that equation (12.3) was studied recently in [54]–[56], where the authors also define some kind of numerical range (see Chapter 11) for pairs of operators (J, F).

The assertions of Theorem 12.1 have already been proved in part before and may be found, together with some examples, in the thesis [155]. Theorem 12.2 and its applications given in Propositions 12.1 and 12.2 are taken from the recent paper [106]. One should admit, however, that some of the results of this chapter actually do not require spectral methods, but may be obtained by other methods. For instance, the first assertion of Proposition 12.1 may also be proved by Schauder's fixed point theorem. In fact, the assumption $|\lambda| > \|\kappa b\|_1$ implies that

$$\sup_{\|x\|_\infty \le R} \frac{\|H(x)\|_\infty}{|\lambda| R} \le \frac{1}{|\lambda| R}(R\|\kappa b\|_1 + \|\kappa a\|_1) < 1 + \frac{\|\kappa a\|_1}{|\lambda| R} \le 1$$

for sufficiently large $R > 0$, and so the (compact) operator H/λ maps the ball $B_R(X)$ into itself.

Theorem 12.4 which seems to be particularly useful for applications, has been proved by Feng [105] in case $L = I$ and by Feng and Webb [110] in case of a general linear Fredholm operator of index zero.

The Theorems 12.6–12.8 on the semilinar Feng spectrum are taken from [110]. They generalize corresponding solvability results by Mawhin ([187], [188], see also [126]). More precisely, Theorem 12.6 for F being L-compact and G being linear and L-compact is Corollary 1 from [187], while Theorem 12.7 is an extension of Lemma XI.3 of [126]. Moreover, if in Theorem 12.8 we take $\lambda = 1$, $[K_{PQ}F]_{\mathrm{A}} = [K_{PQ}T]_{\mathrm{A}} = 0$ and $G\colon \overline{\Omega} \to Y$ a constant map $G(x) \equiv z$ for some $z \in (L - T)(\Omega)$, then Theorem 12.8 reduces to Theorem 2.2 of [188].

Boundary value problems provide of course the most important applications of abstract solvability results for nonlinear operator equations in Banach spaces. We have sketched only a few typical results, but we think that the applicability of spectral methods goes far beyond the material presented here. Proposition 12.3 may be found in [12]. An analogous inclusion to (12.40) for the spectral set (2.30),

$$\sigma_{\mathrm{b}}(F) \subseteq \left\{\lambda \in \mathbb{R} : \lambda \exp(-M\|b\|_1/\lambda) \le \inf_{r \neq 0} \frac{\varphi(r)}{r}\right\},$$

may be proved analogously. The thesis [82] contains a similar application under slightly different hypotheses. So Dörfner assumes in [82], instead of (12.42), that

$$\lambda \exp(-M\|b\|_1/\lambda) > [U_0 L_U^{-1} LEG]_{\mathrm{Q}}.$$

If one wants to prove the surjectivity of the operator $\lambda I - F$, with F given by (12.36), and so the solvability of the problem (12.31), one may also use the characteristic (10.17). In fact, we claim that the hypothesis (12.42) implies that $\delta_r(F/\lambda) \le 1$ for some $r > 0$. To see this, assume that $\delta_r(F/\lambda) > 1$ for all $r > 0$, and so we find $\lambda_r > 1$ and $x_r \in S_r(X)$ with

$$F(x_r) = \lambda_r x_r.$$

So for every $t \in [0, 1]$ we have

$$\begin{aligned} |\lambda|\,|x_r(t)| &= \|F(x_r)(t)\| \\ &\le M\|a\|_1 + \frac{M}{\lambda}\int_0^t b(s)|x_r(s)|\,ds + M^2\|L_U^{-1}\|\,\|L\|\,|g(t, x_r(t))|. \end{aligned}$$

Taking norms and applying Gronwall's lemma implies that

$$r < \frac{1}{|\lambda|}\big(M\|a\|_1 + M^2\|L_U^{-1}\|\,\|L\|\,\mu_G(r)\big)\exp(M\|b\|_1/\lambda)$$

for all $r > 0$, and letting $r \to \infty$ yields

$$M^2\|L_U^{-1}\|\,\|L\|\,\|b\|_1 \ge |\lambda| \exp(-M\|b\|_1/\lambda),$$

contradicting (12.43). So from Proposition 10.1 (a) we conclude that the operator F/λ has a fixed point, which is of course a solution of the equation $F(x) = \lambda x$.

We point out that the idea to use the Gronwall lemma in the proof of the inequality $[\lambda I - F]_{\mathrm{q}} > 0$ is due to Conti and Iannacci [67]. In their paper [67] the authors suppose that the problem

$$\left.\begin{aligned} &\dot{x}(t) - A(t)x(t) = f(t, w(t)) \quad (0 \leq t \leq T), \\ &Lx = y \end{aligned}\right\}$$

has, for any function $w \in X$ and each $y \in \mathbb{R}^n$, a solution $x \in X$. Our proof of Proposition 12.3 shows that this assumption is superfluous.

The calculation of the growth function (12.38) of a Nemytskij operator G is by no means trivial and has been studied in [19]. Calculating this function in the space $L_p[0, T]$ is much more difficult than in the space $C[0, T]$. For example, the nonlinearity (12.46) generates in $L_1[0, T]$ the growth function

$$\mu_G(r) = \frac{(\eta - 1)^2}{2} - \frac{1}{2} + \log \frac{1}{\eta - 1},$$

where $\eta = \eta(r)$ is the unique solution of the transcendent equation $e^{\eta-1} = (\eta-1)e^{r+1}$.

The equation (12.47) arises in the study of radial solutions of *nonlinear elliptic equations* of the form

$$-\Delta u + h(\|x\|) f(u) = 0$$

on annular regions in $\mathbb{R}^n$, $n \geq 2$. Other examples are provided by the generalized *Emden–Fowler equation*, where $f(u) = u^p$ for some $p > 0$ (gas dynamics, nuclear physics, chemically reacting systems), as well as by the *Thomas–Fermi equation*, where $f(u) = u^{3/2}$ and $h(t) = t^{-1/2}$ (atomic structures).

The three point boundary value problems (12.47)/(12.48) and (12.47)/(12.72) have been studied in detail by Feng, Infante, Lan, and Webb. Thus, problem (12.47)/(12.72) has been considered for $\alpha < 0$ and $\alpha > 1$ in [154], for $\alpha = 0$ in [170], and for $0 < \alpha < 1$ in [271]. The case $\alpha = 1$ is called the *resonance case* and requires somewhat different techniques, see [109]. Similarly, $\alpha\eta = 1$ is the resonance case for the problem (12.47)/(12.48) and has also been treated in [109]. Propositions 12.4 and 12.5 are taken from the recent paper [106], Proposition 12.6 from [107], and Proposition 12.7 from [108].

In [122] it is shown that $B_\infty(F) \neq \emptyset$ for any quasibounded operator $G\colon \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$. The quasibounded operator (3.16) from Example 3.18 shows that this is not true in $\mathbb{R}^{2n}$. The application to Hammerstein operators given in Proposition 12.8 may be found in [106].

The only nontrivial result in Section 12.4 is Theorem 12.9 is due to Edmunds and Webb [103] and generalizes a classical result by Krasnosel'skij [161] in which F is required to be compact. The hypothesis in Theorem 12.9 on $F - F'(\infty)$ to be compact is much more general; for example, any operator of the form $F = L + G$, where $L \in \mathfrak{L}(X)$ and G is compact with $[G]_{\mathrm{Q}} = 0$ satisfies this hypothesis.

There has been a considerable mathematical interest in the Dirichlet problem for the p-Laplace equation (12.89). In fact, the p-Laplace operator (12.90) arises in many contexts of mechanics and physics, such as dilatant fluids ($p > 2$), elasticity and plasticity ($1 < p < 2$), reaction-diffusion problems ($p < 2$), glaceology ($p = \frac{4}{3}$), nonlinear torsional creep ($p \geq 3$), and others.

The literature on the theory of the p-Laplace operator (12.90) is vast. In particular, this operator has been studied by means of topological or variational methods, e.g., by Lindqvist [173]–[177] and Drábek et al. [41], [75], [84]–[94], see also [5], [6], [31], [40], [111], [250]. Some connections to so-called p-harmonic functions in the plane with applications to mechanical problems are described in [23]–[27]. In [42] the author investigates the eigenvalue problem (12.89) by means of Amann's approach [2] based on Ljusternik–Shnirelman theory; in particular, she proves that there exist infinitely many distinct eigenfunctions satisfying a given norm condition with the corresponding eigenvalues tending to infinity.

Minty's celebrated theorem on monotone homeomorphisms has been proved in [195], see also [287, p. 557]. The Rayleigh quotient (12.94) has been studied in [175], the Poincaré inequality (12.95) for the p-Laplacian in [167]. A generalization of (12.90) of the form

$$\Delta_m u = \operatorname{div}\left(m(|\nabla u|)\frac{\nabla u}{|\nabla u|}\right),$$

with $m\colon [0,\infty) \to [0,\infty)$ being a certain nondecreasing continuous function, is called the *m-Laplace operator* and may be studied successfully in Orlicz–Sobolev spaces [135]. Of course, the special choice $m(t) = t^{p-1}$ leads to the usual p-Laplace operator (12.90). Some authors (e.g., [147]) also consider the p-Laplace operator with varying p. Finally, an eigenvalue problem for the still more complicated operator

$$\widetilde{\Delta}_p u = \operatorname{div}\left(\frac{|\nabla u|^{2p-2}\nabla u}{\sqrt{1+|\nabla u|^{2p}}}\right)$$

which arises in the theory of so-called capillary surface equations has been studied for $p > 1$ in [117], [196]. As for the p-Laplace equation, the solutions of the corresponding differential equation may be characterized as critical point of the functional $\widetilde{\Psi}_p\colon W_p^{1,0}(G) \to \mathbb{R}$ defined by

$$\widetilde{\Psi}_p(u) := \frac{1}{p}\int_G (\sqrt{1+|\nabla u(x)|^{2p}} - 1)\,dx.$$

The classical *nonlinear Fredholm alternative* goes back to Fučik and Nečas [114], [115], [199], [200] (see also [166]) and, independently, to Pokhozhaev [221]. The non-surjectivity of the operator $J - \mu F$ for $\mu = \mu_1$ has been proved in [127], [128] by constructing explicitly an element $f \in W^{-1}_{p/(p-1)}(G)$ which is not contained in the range of the operator $J - \mu_1 F$. An illuminating discussion of the state-of-the-art of nonlinear Fredholm alternatives and their usefulness in applications to the p-Laplacian may be found in the series of papers by Drábek [84]–[90] and others.

The Theorems 12.10 and 12.11 are taken from the recent paper [17]. We remark that Theorem 12.12 applies to eigenvalue problems of type

$$\left.\begin{array}{ll} -\Delta_p u(x) = \mu|u(x)|^{p-2}u(x) + h(x, u(x)) & \text{in } G, \\ u(x) \equiv 0 & \text{on } \partial G, \end{array}\right\} \tag{12.102}$$

where $h\colon G \times \mathbb{R} \to \mathbb{R}$ is a given nonlinear Carathéodory function. In the weak formulation, the nonlinear perturbation h in (12.102) gives rise to another Nemytskij operator $H\colon X \to X^*$ defined by

$$\langle H(u), v\rangle = \int_G h(x, u(x))v(x)\,dx. \tag{12.103}$$

So applying Theorem 12.12 amounts to giving conditions for the operator (12.103) to be continuous, bounded, or compact between the Sobolev spaces $X = W_p^{1,0}(G)$ and $X^* = W_{p/(p-1)}^{-1}(G)$. Many such conditions, in terms of the generating nonlinearity h, may be found in the monograph [229].

The results of Section 12.5 may also be used to consider global bifurcation theorems of Rabinowitz type for the general equation (see [263])

$$\left\{\begin{array}{ll} -\Delta_p u(x) = \mu|u(x)|^{p-2}u(x) + h(\lambda, x, u(x), \nabla u(x)) & \text{in } G, \\ u(x) \equiv 0 & \text{on } \partial G, \end{array}\right.$$

We close with two open problems. First, the term "discreteness" in Theorem 12.11 does *not* mean that the point spectrum $\sigma_{\mathrm{p}}(J, F)$, or any other spectrum for equation (12.92), has no accumulation points. It is indeed an open problem whether or not there is only a sequence of eigenvalues μ of problem (12.89). Of course, the properties of F and J stated in Lemma 12.3 imply that $J^{-1}F\colon W_p^{1,0}(G) \to W_p^{1,0}(G)$ is compact and 1-homogeneous; moreover,

$$\sigma_{\mathrm{p}}(J, F) = \{\mu^{1/(p-1)} : \mu \in \sigma_{\mathrm{p}}(J^{-1}F, I)\},$$

and so $\sigma_{\mathrm{p}}(J, F)$, and any other spectrum for (12.89), is a homeomorphic image of the classical point spectrum $\sigma_{\mathrm{p}}(J^{-1}F)$. However, Example 10.13 shows that, even in a finite dimensional space, the point spectrum and point phantom of a compact and 1-homogeneous operator may be a continuum. The recent paper [65] gives some conditions under which the connected eigenvalues of a nonlinear operator are isolated; unfortunately, these conditions are not met for the eigenvalue problem (12.89).

Second, we have seen that, by the crucial estimate (12.96), Theorems 12.11 and 12.12 are restricted to the case $p \geq 2$. It seems very unlikely that this restriction has only technical reasons, and these theorems are true also for $p < 2$. In fact, for $p < 2$ the estimate (12.98) would be a global Hölder condition for J^{-1} with exponent $\frac{1}{p-1} > 1$. On the other hand, what we really need is not the Hölder condition (12.98), but the condition (12.100) for the τ-measure of noncompactness of J^{-1}, which is much weaker. So the problem of the validity of Theorems 12.10–12.12 in case $p < 2$ remains open.

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List of Symbols

Index